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Film draining and the Saffman-Taylor problem

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The film drained behind Saffman-Taylor fingers is studied experimentally. The film thickness is found to be in good agreement with theory. It is further shown that previous disagreement between theory and experiment in the Saffman-Taylor problem is partly due to the existence of this film.

The instability of the interface between two fluids of different viscosities flowing through a Hele-Shaw channel is one of the simplest physical situations which displays many of the phenomena observed in pattern processes. Experimentally when a viscous fluid is driven steadily by a gas through a Hele-Shaw cell, after a long transient the shape of the interface is a finger, whose size is a function of the velocity at which it moves, as shown in the early experiments of Saffman and Taylor.[1] Despite the theoretical efforts to reproduce the results of this experiment, all the numerical studies[2] of the Saffman-Taylor problem predict finger sizes significantly below those found experimentally, in a wide range of the fingers velocities. One of the purposes of the present Rapid Communication is to show experimentally that the origin of this disagreement is partly due to the fact that the existence of a film left behind the fingers was not taken into account in the theoretical models.

The Hele-Shaw cell that we have built was formed by two closely parallel glass plates, 120 cm long, 1.28 cm thick, and 10 cm wide, clamped along their sides with aluminum spacers in between. Both width w and gap b of the channel were imposed by the location and the thickness of those spacers. Most of the results reported in the present Rapid Communication have been obtained with a cell for which $b=0.0795\pm0.001$ cm and $w=5.27\pm0.01$ cm (hereafter called cell I). The uniformity of the gap was determined by interferometric measurements. The gap in the central part of the cell was found to be remarkably homogeneous, the value of b being constant within 3 μm over more than 30 cm of the length of the channel. Air was used as the less viscous fluid and silicone oil as the more viscous one. Most of the experiments were performed with Rhodorsil oil 47V10, whose viscosity μ, surface tension T, and mass density ρ are 0.093 P, 20.1 dyn/cm, and 0.93 g/cm^3, respectively. Additional data were obtained with Rhodorsil oil 47V500($\mu=4.87$ P, $T=21.1$ dyn/cm, and $\rho=0.973$ g/cm^3). All these oils wet the glass.

The experiments were performed with the cell lying horizontally. The procedure consisted in partially filling the cell with oil and then pulling oil back by means of gravitational forces. Velocities U of the steady fingers were found to be constant in the major part of the cell within a good accuracy (2×10^{-2}). Finger sizes were precisely measured by means of a traveling microscope.

In Fig. 1, following Treyggvason and Aref,[3] we have plotted the relative size λ of steady fingers (normalized by the channel width) as a function of the dimensionless number $1/B=12\mu Uw^2/Tb^2$, for cell I, using oil 47V10 as the viscous fluid. Additional experimental data were obtained in a similar cell but with oil 47V500. $1/B$ physically represents the ratio of the viscous forces over the capillary forces. Figure 1 shows that all the experimental points lie on a single curve; λ first linearly decreases at very small values of $1/B$ and then tends to saturate at larger value of this parameter. In the very low velocity regime, i.e., $1/B<50$, the finger shapes are close to half circles in a large region around the tip while for larger values of $1/B$ the fingers shapes are very similar to those observed in the early experiments of Saffman and Taylor; for such fingers, the empirical relation derived by Pitts[4] is found to describe the shape of the interface with reasonable accuracy. The results which we obtained with two oils of different viscosities show that, for the aspect ratio $\Gamma=w/b=66.5$ used, $1/B$ is the control parameter of the system. However, when we compared our data with other experiments, carried with different aspect ratios, it did not appear that all curves were reduced to a single one in the plane $(1/B,\lambda)$. For example, (as shown in Fig. 1), the experimental results of Saffman and Taylor, obtained in a cell with aspect ratio $\Gamma=32$, lie significantly above our data. Experimental uncertainty could hardly explain such a discrepancy. Therefore, in contrast to a recent claim,[5] one could be somewhat skeptical about the existence of a single control parameter for the Saffman-Taylor problem. We have also plotted the theoretical predictions of McLean and Saffman[2] on Fig. 1. They are

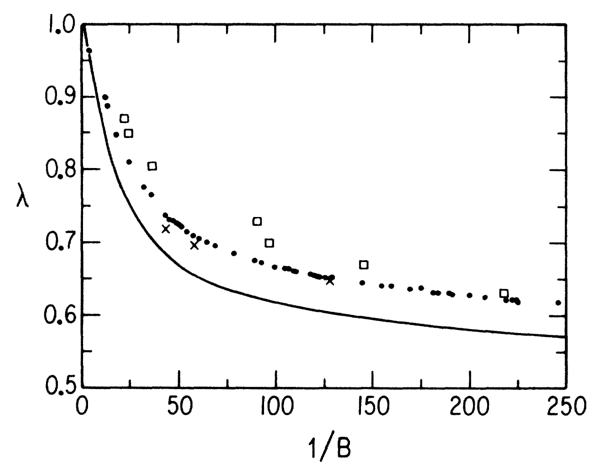

FIG. 1. Relative size λ of the finger as a function of the parameter $1/B$. □, Saffman-Taylor experiment (aspect ratio $w/b=31.8$). ●, experimental results obtained in cell I, with Rhodorsil oil 47V10. ×, experimental results obtained in a cell of aspect ratio $w/b=62.3$ with Rhodorsil oil 47V50. Theoretical results of McLean and Saffman are presented as a continuous curve.

based upon the resolution of equations derived in the paper of Saffman and Taylor, for which actually $1/B$ is the single control parameter. As shown in Fig. 1, theoretical results disagree with our experiments by a factor larger than 1.5 (concerning $1/B$ values). Such a discepancy is actually larger for the experiment of Saffman and Taylor; therefore one could suspect that some additional effect, not taken into account in the theory, might be involved in the experiments. We will show that the film drained by the finger plays an important role in the determination of the finger size.

The method used for measuring the thickness of the film left behind the finger consisted in bringing a laser beam approximately normally to the glass plates and looking at the reflection (Fig. 2). The two films of oil, lying above the lower plate and below the upper one provide two reflecting surfaces. One can easily show that the system of fringes observed on the reflected image is related to the spatial nonuniformity of the sum of the thickness of the two films. By use of this method, we could measure locally the variations of the mean thickness $t=(t_1+t_2)/2$ of the oil films left behind the fingers.

The structure of the films left behind a finger advancing at constant velocity appeared to be surprisingly simple; the mean thickness is essentially uniform in the direction parallel to the finger velocity and varies only in the direction normal to it. The films are symmetric with respect to $0X$ axes (Fig. 2). The experiment shows that the mean height is maximum at the center and minimum at the edges of the finger.[6]

Figure 3 shows the variation of the mean thickness t as a function of the coordinate $2y/\lambda w$, far behind the tip (i.e., where the finger edges are parallel to the sides of the channel). As shown in Fig. 3, the film profiles, obtained for different velocities U, accurately fit a law of the form

$$t=t_{max}[\cos(y\pi/\lambda w)]^{2/3} ,$$

in which t_{max} depends on the finger velocity. With use of Pitts law for the finger shape it is equivalent to $t=t_{max}(U_n/U)^{2/3}$. Thus the film thickness is determined at each point by the normal velocity of the finger at that point. The variations of maximum film thickness t_{max} with capillary number $C=\mu U/T$ are represented in Fig. 4. We obtain[7] a power law with an exponent close to $\frac{2}{3}$ in the range $6\,10^{-4}<C<3\,10^{-3}$.

It is reasonable to compare these results with the calculations of Bretherton[8] for a flat interface. The predictions of Bretherton are represented in Fig. 4. We find a good agree-

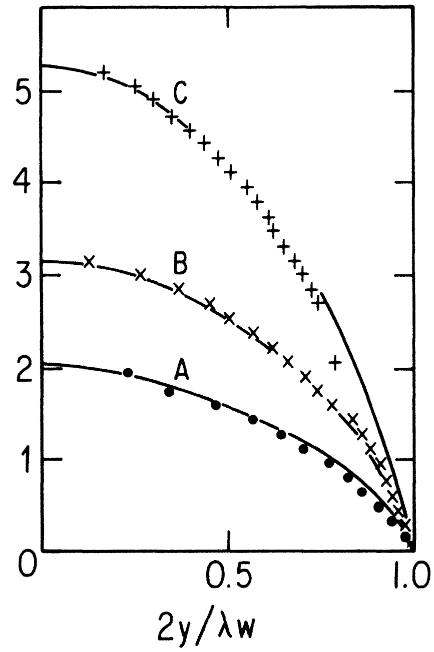

FIG. 3. Plot of the mean thickness of the film as a function of reduced coordinate $2y/\lambda w$. A corresponds to $U=0.10$ cm/s, B to 0.13 cm/s, and C to 0.265 cm/s.

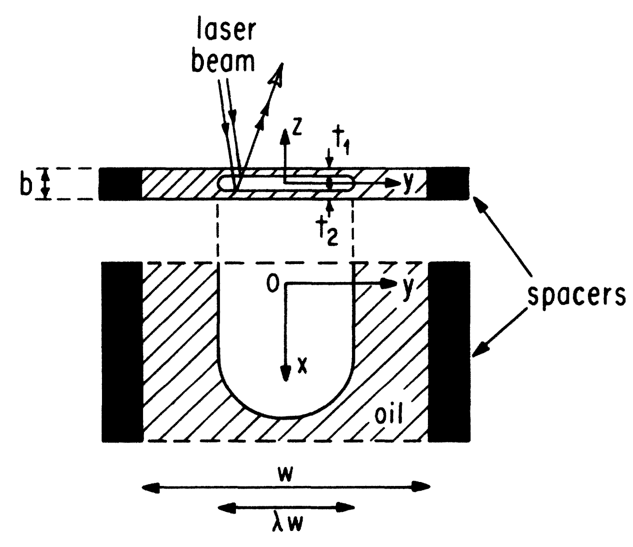

FIG. 2. Schematic view of the finger and the oil film. The film thickness is blown up. The optical beams for the interference study are also shown.

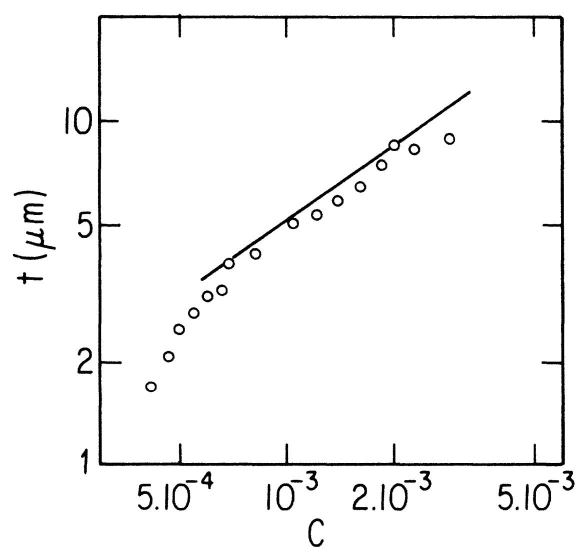

FIG. 4. Plot of the maximum mean thickness of the films as a function of the capillary number. The straight line corresponds to Bretherton calculation.

ment between theory and experiment which indirectly shows that gravity does not much affect the thickness of the films drained by the finger.[9]

Let us now estimate the effect of the film on the finger size. Park and Homsy[5] have recently derived an expression for the pressure drop across the interface, taking into account the effect of the film. Applying their results to our experiments, we find for the pressure drop ΔP,

$$\Delta P = \frac{T}{R}\left(\frac{\pi}{4} + 7.4C^{2/3}\cos^{2/3}\theta\frac{R}{b}\right) , \tag{1}$$

in which θ is the angle of the normal of the interface with the symmetry axis $0X$ (Fig. 2), and R is the radius of curvature of the finger in the xy plane.

Taking into account the shape of the interface found experimentally, we now introduce a kind of effective surface tension T^* defined by

$$T^* = T[\pi/4 + \alpha\lambda(w/b)(\mu U/T)^{2/3}] , \tag{2}$$

in which α is a number related to the average of the $(\cos\theta)^{2/3}R$ term along the interface. Reasonable values of α, deduced from (1), lie between 1.5 and 4.

With this T^* one can recalculate $1/B$. The experimental values of $1/B$ have been calculated by use of T^*, with $\alpha = 1.7$ (Fig. 5). We now find a better agreement between theory and experiments in a range of velocities where Bretherton law applies. This indicates that the film is partly responsible for the previous discrepancies between the theoretical model and the experimental data. For larger values of $1/B$, there is still a disagreement between theory and experiments.

Thus, as theoretically suggested, the effect of the film has an important influence on the experimental results. By the introduction of an effective surface tension T^* a better agreement is obtained. This does not preclude other three-dimensional effects not taken into account here.

The main conclusion of this paper is that, with a proper account of the three-dimensional phenomena, a better agreement is achieved between the experiments and the McLean-Saffman solutions for small values of the capillary number. This conclusion is not true for larger values of t. At least the $\frac{2}{3}$ power law for the evolution of the film thickness with velocity has to break down, otherwise the film would fill up the whole finger. We will now expand the experiment to higher capillary numbers and according to recent theoretical developments,[2,10] test some of the predictions concerning finger stability.

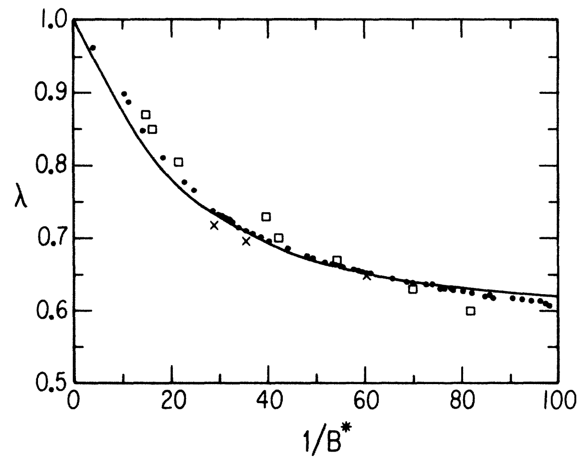

FIG. 5. Relative size λ of the finger as a function of the renormalized parameter $1/B^* = 12\mu(U/T^*)(w/b)^2$, calculated from Eq. (2) with $\alpha = 1.7$. All the symbols are as those in Fig. 1. The difference of horizontal scale, as compared to Fig. 1, comes from a contraction introduced by the renormalized surface tension.

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[2]J. W. McLean and P. G. Saffman, J. Fluid Mech. **102**, 455 (1981); J-M. Vanden-Broeck, Phys. Fluids **26**, 2033 (1983); A. J. DeGregoria and L. W. Schwartz (unpublished); D. Bensimon (unpublished).

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[5]C. W. Park and G. M. Homsy, Phys. Fluids **28**, 1583 (1985); J. Fluid Mech. **139**, 291 (1984).

[6]Such a structure further evolves very slowly with time. The time scales involved in the films dynamics were found to be much larger than those related to the finger. In this slow process, small droplets appear and further grow until they reach a limiting size.

[7]Quantitative measurements at larger velocities are more difficult because the number of fringes increase with velocity. On the other hand, the measurements of very thin film ($< 5\ \mu$m) are significantly affected by the nonuniformity of the cell. In the future we will expand those measurements up to $C = 1$.

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[9]It has been recently shown by Pelce and Jensen (private communication) that the mean thickness of the films drained by fingers in the presence of gravity is $t = t_0/1 - G^2$, where $G = gb^2/4T$ and t_0 is the value of t for $G = 0$. Since G is about 0.08 in our experiment, we find that the effect of gravity on the film is small.

[10]D. A. Kessler and H. Levine (unpublished); L. P. Kadanoff (unpublished).