Exact solution of collective spontaneous emission from an assembly of N atoms in the case of single-atom excitation

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An exact solution for the collective spontaneous emission from an assembly of N identical two-level atoms, placed into a resonant damped cavity and excited to a symmetrical Dicke state with only one atom in the excited state, is derived.

The recent experimental observations[1] of spontaneous emission on Rydberg atoms in cavities make the testing of simple quantum-mechanical models of radiation-matter interaction possible. In this paper we treat an exact soluble case of collective spontaneous emission which has not drawn enough attention in the literature, but whose experimental realization could be possible. The model consists of a system of N identical two-level atoms being initially in a symmetrical Dicke state[2] with only one atom excited. This case arises if the atoms in a cavity are excited by only one photon, but the excitation is distributed symmetrically among all the atoms, so that it is not possible to determine which atom was excited. Then, from Dicke's calculations,[2] it follows that the radiation rate of the atomic system is N times the single-atom rate.

In the present paper we solve this model exactly for the case of N atoms placed into a resonant damped cavity. Since this model is a generalization of the single-atom spontaneous emission in a resonant cavity, which was experimentally observed,[1] its exact solution could be very significant.

In contrast to Dicke's calculations,[2] which are obtained in the first-order perturbation theory and therefore have a very limited applicability, we solve the problem exactly by using a new technique.[3]

The Liouvillian for N atoms interacting with a resonant single-mode radiation field in a cavity in the rotating-wave approximation reads as[4]

$$L=L_0+L_{AR}+i\Lambda_R \tag{1}$$

$$L_0=[H_0,\ \cdots],\quad L_{AR}=[H_{AR},\ \cdots] , \tag{2}$$

with corresponding Hamiltonians ($\hbar=1$)

$$H_0=H_A+H_R=\omega(R^z+a^\dagger a) , \tag{3}$$

$$H_{AR}=g(a\otimes R^+ + a^\dagger\otimes R^-) , \tag{4}$$

$$R^\pm=\sum_{l=1}^{N} R_l^\pm e^{\pm i\mathbf{k}\cdot\mathbf{r}_l},\quad R^z=\sum_{l=1}^{N} R_l^z , \tag{5}$$

and the field-damping Liouvillian

$$\Lambda_R(\ \cdots\ )=\kappa\{[a(\ \cdots\ ),a^\dagger]+[a,(\ \cdots\ )a^\dagger]\} , \tag{6}$$

where $R_l^z, R_l^\pm$ are the population inversion and dipole-moment operators of the lth atom, respectively, ω is the frequency of the atomic transition and the resonant field mode, $a^\dagger,a$ are the photon creation and annihilation operators for the resonant field mode with the wave vector $\mathbf{k}$, g is the atom-field coupling constant, $\mathbf{r}_l$ is the position vector of the lth atom, and κ is the cavity damping factor.

At the initial time $t=0$ the system A of N atoms is excited to a symmetrical Dicke state[2] with only one atom excited:

$$\rho_A(0)=|r=N/2,\ m=-N/2+1\rangle\langle r=N/2,\ m=-N/2+1| . \tag{7}$$

The radiation field R is initially in a vacuum state:

$$\rho_R(0)=|0\rangle\langle 0| , \tag{8}$$

and systems A and R are initially decoupled:

$$\rho(0)=\rho_A(0)\otimes\rho_R(0) . \tag{9}$$

The statistical density operator for the total system $A+R$ takes the following form:

$$\rho(t)=\exp(-itL_0)\exp[-it(L_{AR}+i\Lambda_R)]\rho(0) , \tag{10}$$

where we used the commutation relations

$$[H_0,H_{AR}]=0,\quad L_0\Lambda_R(\ \cdots\ )=\Lambda_R L_0(\ \cdots\ ) . \tag{11}$$

As a consequence of the special initial condition (7)–(9) and special form of the Liouvillian [cf. Eqs. (1)–(6)], the time evolution of the statistical density operator $\rho(t)$ is restricted to the subspace spanned by the state vectors:

$$\begin{aligned}&|1\rangle=|r,-r+1\rangle\otimes|0\rangle,\quad |2\rangle=|r,-r\rangle\otimes|1\rangle ,\\ &|3\rangle=|r,-r\rangle\otimes|0\rangle,\quad r=\frac{N}{2} ,\end{aligned} \tag{12}$$

and it holds that

$$\begin{aligned}&\rho(t)=\sum_{i,i'=1}^{3}|i\rangle\langle i'|\rho_{ii'}(t) ,\\ &\rho_{ii'}(t)=\langle i|\rho(t)|i'\rangle ,\end{aligned} \tag{13}$$

$$\mathrm{Tr}\rho(t)=\rho_{11}(t)+\rho_{22}(t)+\rho_{33}(t)=1 . \tag{14}$$

The expectation value of the atomic population inversion operator is then

$$\langle R^z\rangle_t=\mathrm{Tr}[R^z\rho(t)]=(-r+1)\rho_{11}-r(\rho_{22}+\rho_{33}),\quad r=\frac{N}{2} , \tag{15}$$

where we used $R^z|rm\rangle=m|rm\rangle$, ($\hbar=1$) . Further from Eqs. (14) and (15) we obtain for the expectation value of the squared population inversion

$$\langle R^z R^z\rangle_t = -r\langle R^z\rangle_t + (-r+1)\rho_{11} = (-2r+1)\langle R^z\rangle_t + r(-r+1), \qquad r=\frac{N}{2} . \tag{16}$$

By successive differentiations of equation:

$$\langle R^z\rangle_t = \mathrm{Tr}\{R^z \exp[-it(L_{AR}+i\Lambda_R)]\rho(0)\} , \tag{17}$$

we obtain[3]

$$\frac{d\langle R^z\rangle_t}{dt} = ig\,\mathrm{Tr}\{(-a\otimes R^+ + a^\dagger \otimes R^-)\exp[-it(L_{AR}+i\Lambda_R)]\rho(0)\} , \tag{18}$$

$$\frac{d^2\langle R^z\rangle_t}{dt^2} = -4g^2 r(\langle R^z\rangle_t + r) - \kappa\frac{d\langle R^z\rangle_t}{dt} - 4g^2\,\mathrm{Tr}\{a^\dagger a\otimes R^z \exp[-it(L_{AR}+i\Lambda_R)]\rho(0)\}, \quad r=\frac{N}{2} , \tag{19}$$

where we used Eq. (16) and the following relations:

$$R^2|rm\rangle = r(r+1)|rm\rangle, \quad R^2 = R^+R^- + (R^z)^2 - R^z , \tag{20}$$

$$[a,a^\dagger]=1, \quad [R^+,R^-]=2R^z, \quad [R^z,R^\pm] = \pm R^z . \tag{21}$$

A further differentiation gives an exact closed linear differential equation of third order:

$$\frac{d^3\langle R^z\rangle_t}{dt^3} = -3\kappa\frac{d^2\langle R_z\rangle_t}{dt^2} - [2\kappa^2+8g^2r]\frac{d\langle R^z\rangle_t}{dt} - 8\kappa g^2 r\langle R^z\rangle_t - 8\kappa g^2 r^2, \quad r=\frac{N}{2} , \tag{22}$$

where we have taken into account that

$$\mathrm{Tr}\{a^\dagger aa\otimes R^+\exp[-it(L_{AR}+i\Lambda_R)]\rho(0)\}=0 , \tag{23}$$

$$\mathrm{Tr}\{a^\dagger a^\dagger a\otimes R^-\exp[-it(L_{AR}+i\Lambda_R)]\rho(0)\}=0 , \tag{24}$$

which follows immediately if the expression for $\rho(t)$ (13) is inserted into the left-hand sides of Eqs. (23) and (24).

By using the time transformation $\bar{t}=g\sqrt{N}t$, the quotient $a=\kappa/(g\sqrt{N})$, and the abbreviation $Z(t)=\langle R^z\rangle_t$, the exact Eq. (22) reduces to

$$\frac{d^3Z(\bar{t})}{d\bar{t}^3} + 3a\frac{d^2Z(\bar{t})}{d\bar{t}^2} + 2(a^2+2)\frac{dZ(\bar{t})}{d\bar{t}} + 4aZ(\bar{t}) = -2aN , \tag{25}$$

$$Z(0) = -\frac{N}{2}+1, \quad \left[\frac{dZ(\bar{t})}{d\bar{t}}\right]_{\bar{t}=0}=0, \quad \left[\frac{d^2Z(\bar{t})}{d\bar{t}^2}\right]_{\bar{t}=0} = -2 . \tag{26}$$

The exact analytic solution of Eq. (25) reads as follows.

(i) For $a>2$:

$$Z(t) = e^{-\kappa t}[A_1^+\exp(\sqrt{\kappa^2-4g^2N}t) + A_1^-\exp(-\sqrt{\kappa^2-4g^2N}t) + A_2] - \frac{N}{2} , \tag{27a}$$

with

$$A_1^\pm = \frac{1}{2} + \frac{g^2N}{\kappa^2-4g^2N} \pm \frac{1}{2}\frac{\kappa}{\sqrt{\kappa^2-4g^2N}} , \qquad A_2 = -\frac{2g^2N}{\kappa^2-4g^2N} . \tag{27b}$$

(ii) For $a=2$:

$$Z(t) = 2\exp(-2g\sqrt{N}t)\cdot(\tfrac{1}{2}+g\sqrt{N}t+g^2Nt^2) - \frac{N}{2} . \tag{28}$$

(iii) For $a<2$:

$$Z(t) = e^{-\kappa t}[B_1\cos(\sqrt{4g^2N-\kappa^2}t) + B_2\sin(\sqrt{4g^2N-\kappa^2}t) + B_3] - \frac{N}{2} \tag{29a}$$

with

$$B_1 = \frac{2g^2N}{4g^2N-\kappa^2}, \quad B_2 = \frac{\kappa}{\sqrt{4g^2N-\kappa^2}}, \quad B_3 = 1-\frac{2g^2N}{4g^2N-\kappa^2} . \tag{29b}$$

In Fig. 1 we have plotted the numerical solutions for different values of a.

From the exact analytic solutions (27)–(29) and numerical results in Fig. 1, it can be concluded that an assembly of N atoms, which is initially in a symmetrical Dicke state with only one atom excited, shows collective effects in the spontaneous emission. We will discuss these collective effects in two cases.

1. For high-Q cavities $a \ll 1$, i.e., $\kappa \ll g\sqrt{N}$, Eq. (29) reduces to

$$Z(t) \approx e^{-\kappa t}\cos^2(g\sqrt{N}t) - \frac{N}{2} \tag{30}$$

and the radiation rate reads

$$I(t) = -\frac{dZ(t)}{dt} \approx e^{-\kappa t}g\sqrt{N}\sin(2g\sqrt{N}t) . \tag{31}$$

In this case the atomic system is able to reabsorb the emitted photon and the time evolution of the atomic population

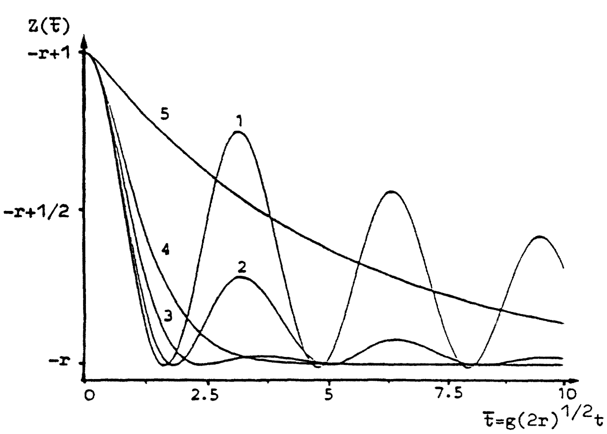

FIG. 1. The time evolution of the atomic population inversion $Z(\bar{t})$ for $r=N/2$ and $\kappa/(g\sqrt{N})=0.1, 0.4, 1, 2, 10$ (curves denoted by 1–5).

inversion $Z(t)$ undergoes damped squared cosine oscillations. The collective spontaneous emission becomes apparent if we compare the above results with the single-atom case ($N=1$). The oscillation frequency as well as the amplitudes of the radiation rate increase by a factor of $\sqrt{N}$. Moreover, in contrast to the single-atom case, it is evident that the condition of a high-Q cavity ($\kappa << g\sqrt{N}$) can easily be fulfilled for large number N of atoms even in cases where the cavity damping factor κ is large ($\kappa > g$). This fact simplifies significantly the realization of high-Q cavities for the present model.

2. For low-Q cavities $a >> 1$, i.e., $\kappa >> g\sqrt{N}$, Eq. (27) reduces to

$$Z(t) \approx \exp(-2g^2Nt/\kappa) - \frac{N}{2} \ , \tag{32}$$

$$I(t) \approx -\frac{2}{\kappa} g^2 N \exp(-2g^2Nt/\kappa) \ . \tag{33}$$

In this case the emitted photon is not stored long enough in the cavity and cannot be reabsorbed by the atom. The atomic population inversion as well as the radiation rate decay exponentially (increased by a factor of N in the exponent as compared to the single-atom case). This result coincides with the result obtained from the Bonifacio-Schwendimann-Haake master equation (derived in the Born and Markov approximations),[4] and for $t=0$ it yields the Dicke result:[2]

$$I(0) = NI_0, \quad I_0 = I(N=1) \ . \tag{34}$$

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