Quantum and classical Liouville dynamics of the anharmonic oscillator

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We consider the dynamics of a quantum joint phase-space probability density in an exactly solvable model. The density is defined to be a true (i.e., positive) probability distribution for approximate (and thus simultaneously measurable) position and momentum variables. The dynamics of the quantum density is governed by a second-order partial-differential equation with non-positive-definite second-order coefficients. The quantum dynamics is contrasted with the dynamics of a similar joint density in a classical description. The non-positive-definite second-order terms in the quantum evolution equation, not present in the classical case, are responsible for quantum recurrences and prevent the appearance of fine-scale-structure "whorls" predicted in a classical description. The generation of "squeezing" in the model is also discussed.

I. INTRODUCTION

What is the quantum analogue of a classical phase-space trajectory? This is an important and difficult question of relevance to many inquiries. It arises when we seek to understand the quantum dynamics of systems whose classical counterpart exhibits chaos. It also arises in discussions on semiclassical dynamics. There have been many studies related to this question over recent years.[1-5] Consideration of this question will be one of the two guiding themes of this paper.

The central difficulty any reply to this question must face is the inherent probabalistic character of quantum descriptions. Of course, probabilistic concepts are essential for a complete classical description as well. However, despite the fact that the probabilities which arise in the quantum theory are (usually) interpreted in the same sense as in the classical theory (that is, as limits of relative frequencies), quantum probabilities exhibit properties which have no counterpart in a classical theory. Investigation of these differences has been the subject of active research from the earliest days of quantum theory (see Ref. 6) and will comprise the second major theme of this paper. Of particular relevance to this discussion will be the different time development of probability distributions in classical and quantum dynamics.

The most familiar departure of quantum statistics from classical is expressed in the Heisenberg uncertainty relation. This, as is well known, places a nonzero lower bound on the product of the variances in the canonically conjugate position and momentum for every state. In the classical theory this product can be zero. This has many consequences. For example, any attempt to produce a state with fluctuations reducing to zero in one canonical variable, necessarily has divergent variance in the conjugate variable. (Assuming of course, the operators are unbounded.) It is, of course, possible to produce states which simultaneously minimize the fluctuations in the two conjugate variables so as to be at the lower bound in the uncertainty product. These are the minimum uncertainty states a subclass of which, the "squeezed states," have been the subject of a great deal of attention recently.[7]

In this paper we will consider the concept of squeezing as a particular manifestation of nonclassical statistical behavior. However, as we shall show, one must exercise caution here, as the concept of squeezing is really only meaningful for quantum systems. Classically, the fluctuations in the canonical variables can simultaneously be reduced to zero. This point is further discussed in Sec. III.

One may also view the uncertainty relations as reflecting the difficulty of defining a true joint phase-space probability density in quantum mechanics.[6] Were it possible to define such an object, a comparison of classical and quantum dynamics would be straightforward; one would simply compare the dynamics of the relevant phase-space distributions. This suggests that the question in the opening paragraph of this section should perhaps be put differently: What is the quantum analogue of a classical joint phase-space probability density and how does its dynamics differ from its classical counterpart? That this change of emphasis in comparison of quantum and classical dynamics leads to a more direct and conceptually simpler formulation of the problem has only recently been recognized.[4]

There have been many attempts to define a quantum joint probability density in various contexts. Perhaps the most famous of these is the Wigner function.[8] Unfortunately, the Wigner function for some states can become negative (e.g., energy eigenstates of the harmonic oscillator) and thus cannot have any direct interpretation as a probability density. Despite this difficulty the Wigner function has been extremely useful and has been used to contrast classical and quantum dynamics via a phase-space picture.[9,10]

In this paper we will not use the Wigner function as the appropriate quantum analogue of a classical phase-space density. Instead we will use a true probability density defined in terms of the oscillator coherent states as[11]

$$Q(\alpha,\alpha^*,t)=\mathrm{Tr}[\rho(t)\,|\,\alpha\rangle\langle\alpha\,|\,]\,, \tag{1}$$

where $\rho(t)$ is the density operator of the system and $|\alpha\rangle$ are the oscillator coherent states. In quantum optics this function is known as the Q function and is directly associated with anti-normally-ordered moments of the annihilation and creation operators. It has also been used to discuss quantum maps.[2]

A more revealing insight into the Q function is provided when we note that it is indeed a true joint probability density whose marginal variables are approximate position and momentum operators $\hat{q}_\alpha$ and $\hat{p}_\alpha$.[12] These variables are defined by their spectral decomposition as

$$\hat{q}_\alpha = \int_R x\, d\hat{E}_\alpha(x) , \tag{2}$$

$$\hat{p}_\alpha = \int_R k\, d[F^{-1}\hat{E}_\alpha(x)F] , \tag{3}$$

where

$$d\hat{E}_\alpha(x) = \left[\int_{-\infty}^{\infty} dy\, |\psi_\alpha(y)|^2 |x-y\rangle\langle x-y|\right] dx . \tag{4}$$

F is the Fourier-transform operator, and $\psi_\alpha(y)$ is the coordinate-space wave function corresponding to the coherent state $|\alpha\rangle$ [that is, $\psi_\alpha(y)=\langle y|\alpha\rangle$]. As pointed out by Davies, $|\alpha\rangle$ describes some measuring apparatus which is designed to measure $\hat{q}$ and $\hat{p}$ simultaneously to an equally high degree of accuracy, as is permitted by the uncertainty relations.

It seems that these measurements are the appropriate ones to consider to determine a phase-space dynamic picture in the quantum theory. It is for this reason we regard the Q function as the appropriate quantum analogue of a classical joint phase-space density.

Intuitively, we can regard the Q function as a restriction of the class of joint probability densities to a certain "minimum width" subset. To see this we note that the Q function must satisfy two relations. Firstly, if it is to be a true joint probability density, it must be normalizable. Thus,

$$\int Q(\alpha,\alpha^*,t)\frac{d^2\alpha}{\pi} = 1 , \tag{5}$$

which follows from $\mathrm{Tr}(\rho)=1$ and the completeness relation for coherent states.

Furthermore, we have

$$0 \le Q(\alpha,\alpha^*,t) \le 1 . \tag{6}$$

This may be proved as follows. One may define a norm on the space of bounded operators by $||AB||=\mathrm{Tr}(AB)$ and $||A||=\mathrm{Tr}(A)$. Then it follows that $\mathrm{Tr}(AB) < \mathrm{Tr}(A)\mathrm{Tr}(B)$. From Eq. (1), this implies

$$Q(\alpha,\alpha^*,t) \le \mathrm{Tr}[\rho(t)]\,\mathrm{Tr}(|\alpha\rangle\langle\alpha|) .$$

But $\mathrm{Tr}(\rho)=1$, as ρ is a density operator and $\mathrm{Tr}(|\alpha\rangle\langle\alpha|)=1$ as $|\alpha\rangle$ is a normalized state. Thus $Q(\alpha,\alpha^*,t)\le 1$. The equality is excluded by Eq. (5). The second part of the inequality in Eq. (6) follows, as both ρ and $|\alpha\rangle\langle\alpha|$ are positive operators. Taking conditions (5), we see that the more narrowly we try to concentrate Q, the higher it must become. However, the growth is limited by condition (6) and, thus, the "narrowness" of allowed Q functions is restricted. The delta functions are excluded from the class of Q functions.

The object of this paper is to compare and contrast the dynamics of the Q function, and its associated moments, for an exactly solvable system, with the corresponding behavior of a suitably chosen classical density.

It should be noted that the stochastic behavior in the quantum case arises wholly through the intrinsic quantum fluctuations in the initial (pure) state and in the classical case by uncertainty in the initial conditions. There are no additional stochastic influences acting on the system.

II. ANHARMONIC OSCILLATOR

The anharmonic oscillator we wish to discuss has the Hamiltonian

$$H = H_0 + \frac{\mu}{\hbar\omega_0} H_0^2 , \tag{7}$$

where H_0 is the free Hamiltonian of the simple harmonic oscillator. The anharmonicity parameter μ is positive. The semiclassical dynamics of systems such as this has been analyzed extensively in Ref. 13. The appearance of $\hbar$ in Eq. (7) is purely to provide a convenient scaling for the energies.

We first discuss the classical description of the dynamics for this system.

A. Classical description

In terms of the position and momentum, the free Hamiltonian H_0, is

$$H_0 = \tfrac{1}{2}(p^2+\omega^2 q^2) \tag{8}$$

(we choose the mass to be unity). To obtain a direct comparison with the quantum description, we introduce the complex variable by

$$\alpha = (\omega/2\hbar)^{1/2} q + i(\tfrac{1}{2}\hbar\omega)^{1/2} p . \tag{9}$$

Then, $H_0=\hbar\omega_0|\alpha|^2$, and

$$H = \hbar\omega_0(|\alpha|^2+\mu|\alpha|^4) . \tag{10}$$

The equations of motion may be written as

$$\frac{d\alpha}{d\tau} = -i(1+2\mu|\alpha|^2)\alpha , \tag{11}$$

where we define the dimensionless time $\tau=\omega_0 t$. We will find it convenient to work in a frame rotating at frequency ω_0. The solution in this rotating frame is

$$\alpha(\tau) = \alpha_0 \exp(-2i\mu\tau|\alpha|^2) . \tag{12}$$

Following Mackey,[14] we define a classical "state" to be a probability measure on phase space of the form $Q(\alpha,\alpha^*)d^2\alpha$, where $Q(\alpha,\alpha^*)$ is the joint probability density. The density then obeys the time-evolution equation

$$\frac{\partial Q}{\partial\tau} = -\{H,Q\} , \tag{13}$$

where $\{\,,\}$ is the usual Poisson bracket. In the rotating frame our model gives

$$\frac{\partial Q}{\partial \tau} = 2i\mu |\alpha|^2 \alpha \frac{\partial Q}{\partial \alpha} - 2i\mu |\alpha|^2 \alpha^* \frac{\partial Q}{\partial \alpha^*} . \tag{14}$$

This may be solved by the method of characteristics, the characteristics being the trajectories in Eq. (12). Let us choose the initial state to be

$$Q(\alpha,\alpha^*,0) = \exp(-|\alpha-\alpha_0|^2) . \tag{15}$$

This is a Gaussian centered on α_0 with the covariance matrix

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \tag{16}$$

and $\langle |\alpha|^2 \rangle = 1 + |\alpha_0|^2$, which is invariant. As α_0 becomes large, the relative width becomes small. With this initial condition the solution of Eq. (14) is

$$Q(\alpha,\alpha^*,\tau) = \exp(-\bar{n}|ze^{i\nu|z|^2/\bar{n}} - 1|^2) , \tag{17}$$

where

$$z = \alpha/\alpha_0, \quad \nu = 2\mu\tau , \quad \bar{n} = |\alpha_0|^2 .$$

The simplest way to represent the time evolution of the density is to consider the behavior of a particular initial contour. In a time τ each point on the initial contour will move according to Eq. (12). Clearly, this is simply a rotational sheer. This is evident in Figs. 1 and 2, where we have followed the evolution of the contour $|\alpha-\alpha_0| = \frac{1}{2}$ over $0 \le \nu \le 2$. The initial circular contour develops the phase-space structure known as a "whorl."[10,13] This is expected, as the invariant curves in this system are circles centered on the elliptic fixed point at the origin. (In fact, whorls are generic behavior in one-dimensional systems.) As $t \to \infty$, the initial contour becomes increasingly more convoluted on a finer and finer scale.

The behavior of the density is reflected in the moments. For example, the mean amplitude is given by

$$\langle \alpha(\tau) \rangle = \alpha_0 (1+i\nu)^{-2} \exp\left[-\bar{n} \left[1 - \frac{1}{1+i\nu} \right] \right] , \tag{18}$$

which decays to zero as $t \to \infty$.

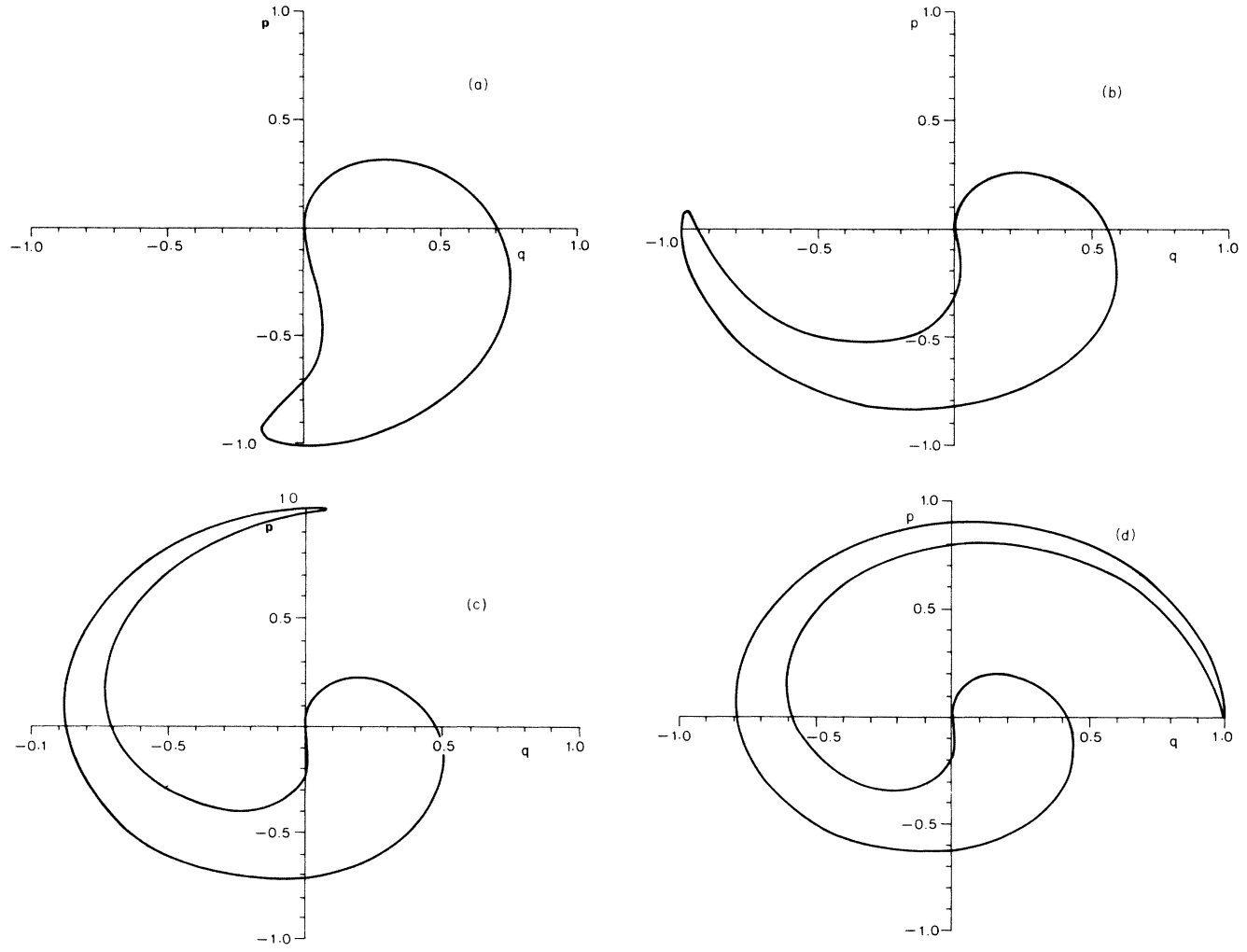

FIG. 1. Phase-space evolution of an initial Gaussian contour centered on $\alpha_0 = 0.5$ in a classical description of the anharmonic oscillator. (a) $\nu = \pi/2$, (b) $\nu = \pi$, (c) $\nu = 3\pi/2$, and (d) $\nu = 2\pi$.

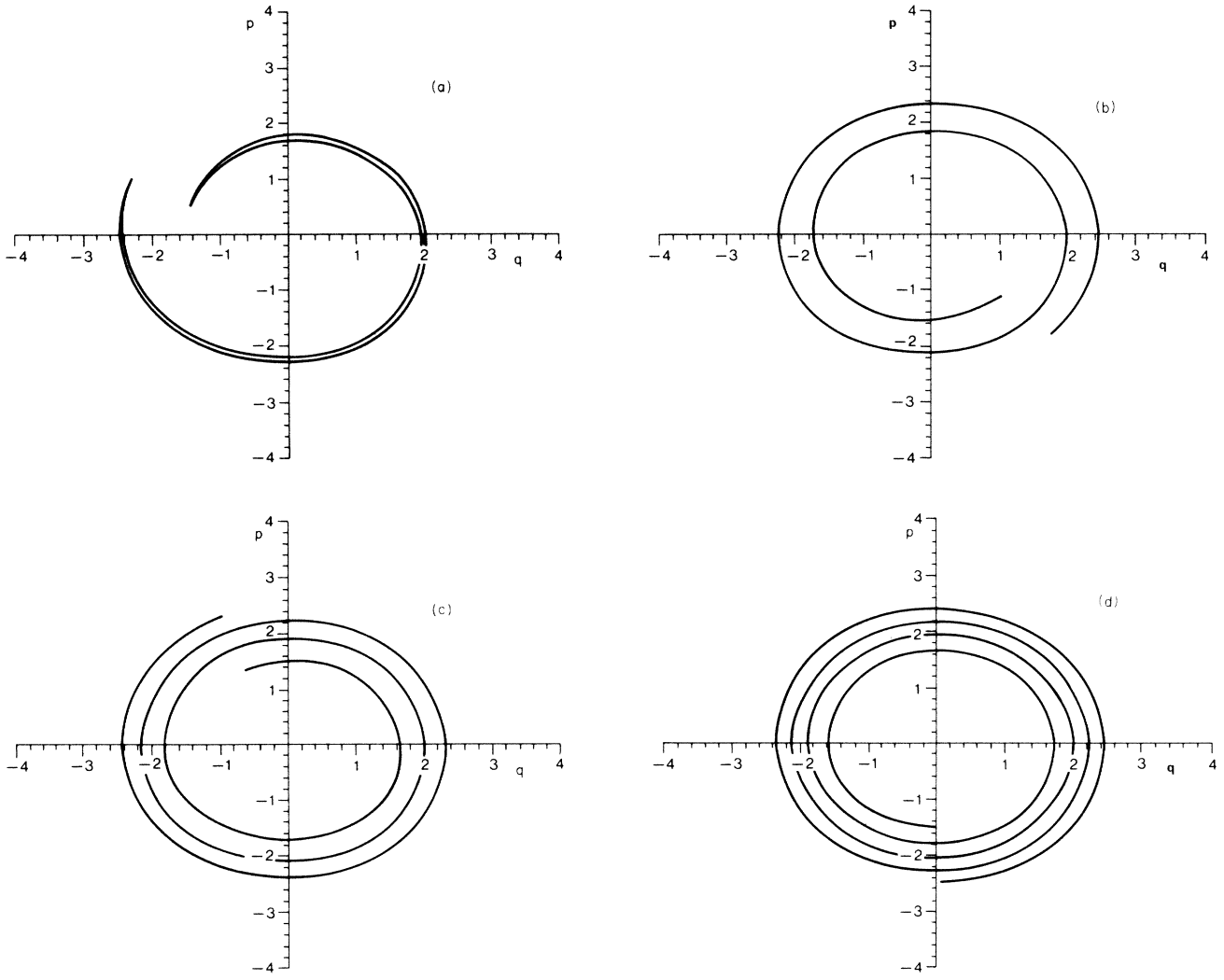

FIG. 2. As for Fig. 1, with an initial Gaussian contour centered on $\alpha_0=2.0$.

A convenient route to the phase-dependent properties is to define the quadrature phase amplitudes

$$X_1=\frac{1}{2}(\alpha+\alpha^*),\quad X_2=\frac{1}{2i}(\alpha-\alpha^*)\ . \tag{19}$$

With α_0 real, the variances in each of these quantities is then found to be

$$\mathrm{var}(X_{1,2})=\tfrac{1}{2}[1+\bar{n}-\bar{n}f_1(\nu)\pm\bar{n}f_2(\nu)]\ , \tag{20}$$

where

$$f_1(\nu)=(1+\nu^2)^{-2}\exp\left[-2\bar{n}\frac{\nu^2}{1+\nu^2}\right]\ , \tag{21}$$

$$f_2(\nu)=\mathrm{Re}\left[(1+2i\nu)^{-3}\exp\left[-2i\bar{n}\frac{\nu}{1+2i\nu}\right]-(1+i\nu)^{-4}\exp\left[-2i\bar{n}\frac{\nu}{1+i\nu}\right]\right]\ . \tag{22}$$

In Figs. 3(a) and 3(b) we have plotted the phase-space trajectory of $\langle\alpha(t)\rangle$ over one cycle $(0\le\nu\le2)$ for two values of α_0. The decay of the mean amplitude due to the development of the whorl is clearly evident. In Figs. 4(a) and 4(b) we have plotted $\mathrm{var}(X_{1,2})$ versus $\nu/2\pi$ for two values of α_0. Note that, initially, the fluctuations in X_1 are reduced as those in X_2 increase, but eventually the whorl develops and the fluctuations saturate at

$$\mathrm{var}(X_{1,2})=\tfrac{1}{2}(1+\bar{n})\ . \tag{23}$$

B. Quantum description

In the quantum description of the model, the canonical coordinate and momentum become self-adjoint operators on Hilbert space while the states are trace class operators, the density operators on Hilbert space. The free Hamiltonian may then be written in terms of amplitude operators (the annihilation and creation operators) $a,a^\dagger$, where

$$a=(\omega/2\hbar)^{1/2}\hat{q}+i(\tfrac{1}{2}\hbar\omega)^{1/2}\hat{p}\ , \tag{24}$$

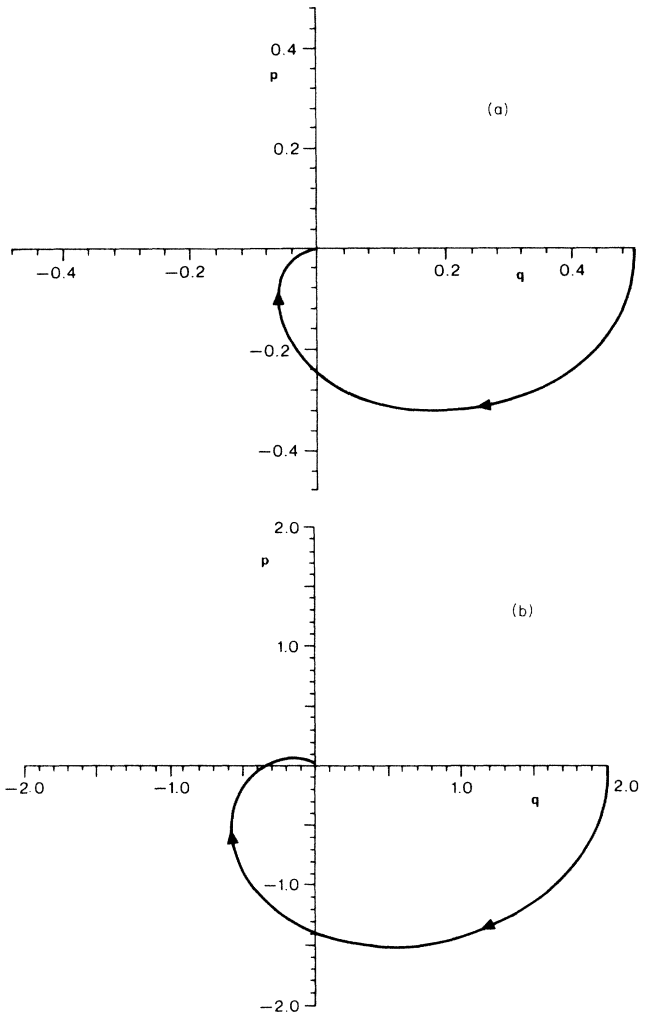

FIG. 3. Phase-space plot of the trajectory $\langle \alpha(\tau) \rangle$ for (a) $\alpha_0=0.5$ and (b) $\alpha_0=2.0$ over the range $0 \leq \nu \leq 2\pi$.

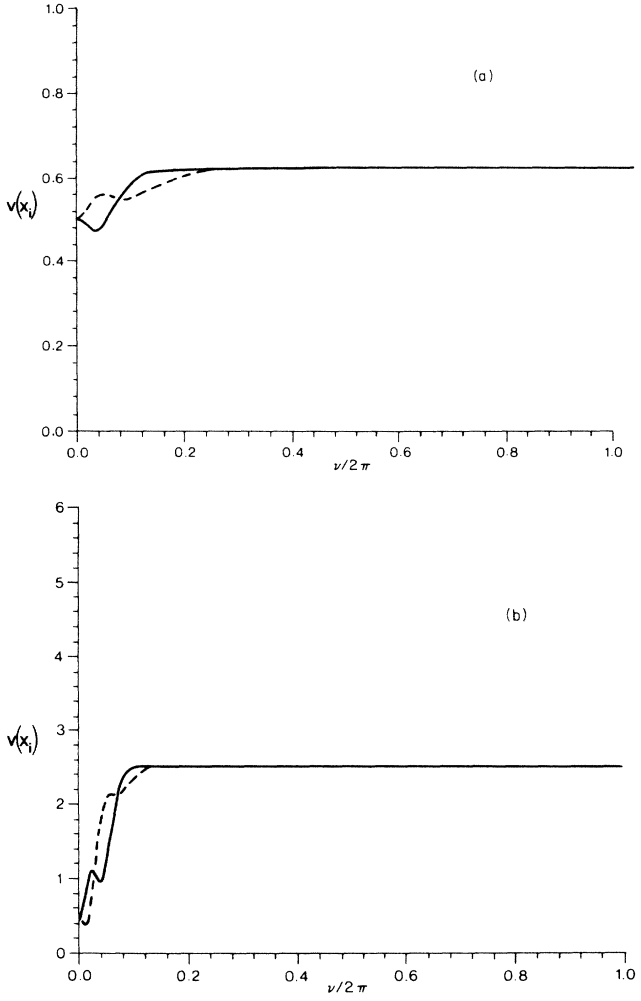

FIG. 4. Plot of classical quadrature phase variances versus ν for (a) $\alpha_0=0.5$ and (b) $\alpha_0=2.0$. Solid line, $V(X_2)$; dashed line, $V(X_1)$.

and $[a,a^\dagger]=1$. As is well known, $H_0=\hbar\omega_0(a^\dagger a+\frac{1}{2})$, and thus

$$H=\hbar\omega_0[(1+\mu)a^\dagger a+\mu(a^\dagger a)^2] \tag{25}$$

(we have neglected constant-energy shifts). We now transform to an interaction picture via

$$U(\tau)=\exp[-i(1+\mu)a^\dagger a\tau] .$$

This is equivalent to the transformation to the rotating frame used in the classical picture. The quantum description requires a slight modification of the rotating-frame frequency.

The Heisenberg equations of motion in the interaction picture are

$$\frac{da(t)}{d\tau}=-2i\mu(a^\dagger a+\tfrac{1}{2})a(\tau) . \tag{26}$$

As $a^\dagger a$ is a constant of motion, the solution is

$$a(\tau)=\exp[-i\nu(a^\dagger a+\tfrac{1}{2})]a(0) , \tag{27}$$

where $\nu=2\mu t$.

We now assume the initial state to be given by $\rho(0)=|\alpha_0\rangle\langle\alpha_0|$ that is a pure coherent state of complex amplitude α_0. The corresponding Q function is

$$Q(\alpha,\alpha^*,0)=\exp(-|\alpha-\alpha_0|^2) , \tag{28}$$

which has the same form as the classical density considered in Sec. III A.

Of course, all moments for this problem may be calculated in the Heisenberg picture using Eq. (27) and $\rho(0)$. However, in order to make more direct comparisons between the quantum and classical descriptions, we will firstly consider the time evolution of the Q function.

The equation of motion for the density operator in the interaction picture is

$$\frac{\partial\rho}{\partial\tau}=-i\mu[(a^\dagger a)^2,\rho] . \tag{29}$$

This may be converted to an equation of motion for the Q function using standard techniques.[15]

$$\frac{\partial Q}{\partial \tau}=i\mu\alpha(1+2|\alpha|^2)\frac{\partial Q}{\partial \alpha}-i\mu\alpha^*(1+2|\alpha|^2)\frac{\partial Q}{\partial \alpha^*}+i\mu\alpha^2\frac{\partial^2 Q}{\partial \alpha^2}-i\mu\alpha^{*2}\frac{\partial^2 Q}{\partial \alpha^{*2}}\ . \tag{30}$$

If we compare this with the classical equation [Eq. (14)], we immediately see that the quantum dynamics of the density will be quite different due to the appearance of the second-order derivatives. (There is also a slight modification of the first-order coefficients.) If higher than quartic terms had been included in the Hamiltonian, higher-order derivatives would occur in this equation. For deterministic systems such as that considered here, the classical density evolution is necessarily a first-order partial differential equation. The quantum-density-evolution equation, however, is not so restricted, as the particular example of Eq. (30) demonstrates. How can this be reconciled with the statement that Q is always a true positive probability density? The answer is that the quantum description requires us to restrict the class of initial conditions so that Eqs. (5) and (6) hold. This immediately excludes certain functional forms, such as delta functions.

In this paper we simply acknowledge the fact that the equation of motion for the quantum analogue of a phase space density will not necessarily be a first-order partial-differential equation. As we shall see, this greater flexibility is essential in order that the characteristic recurrences of the quantum description result.

In the more general case of true stochastic evolution, such as arises when the system of interest is coupled to a heat bath, the classical evolution equation may contain second-order derivatives. This is the Fokker-Planck equation. However, the coefficients of the second-order terms determine a diffusion matrix which must be positive definite. Other authors[16] consider it desirable to define a quantum density in such a way that its evolution equation has Fokker-Planck form for both deterministic and stochastic dynamics. We will not consider these in this paper, as it is not clear how they permit direct comparisons of quantum and classical dynamics of phase-space densities.

Equation (30) may be solved subject to the initial condition [Eq. (28)] to yield

$$Q(\alpha,\alpha^*,\tau)=\exp(-|\alpha|^2-|\alpha_0|^2)|S|^2\ , \tag{31}$$

with

$$S=\sum_{n=0}^{\infty}\frac{(\alpha_0\alpha^*)^n}{n!}e^{-i\mu\tau n^2}$$

(the details may be found in Appendix A). That this is indeed correct may be verified more directly as follows. The density operator at time τ, $\rho(\tau)$, is related to the initial density operator by

$$\rho(\tau)=e^{-iH_I\tau/\hbar}\rho(0)e^{iH_I\tau/\hbar}\ . \tag{32}$$

Thus for $\rho(0)=|\alpha_0\rangle\langle\alpha_0|$,

$$Q(\alpha,\alpha^*\tau)=|\langle\alpha|e^{-iH_I\tau/\hbar}|\alpha_0\rangle|^2\ .$$

Expanding the coherent states in number states,

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\ ,$$

we obtain Eq. (31).

It is clear from the form of Eq. (31) that the quantum probability density will exhibit periodic recurrences of its initial form. This is in marked contrast to the evolution of the initial density in the absence of the second-order terms, and, of course, is a reflection of the "discreteness" of the quantum description. It is the appearance of non-positive-definite second-order terms in the quantum equation which is directly responsible for these recurrences. We thus expect non-positive-definite "diffusion" terms to be characteristic of the evolution equations for quantum Q-type phase-space densities.

In Sec. III we shall show that in the semiclassical limit $\hbar\to 0$ these second-order terms are unimportant for small times and semiclassical dynamics results.

We now consider the time-dependent moments in the quantum case. As the Q function directly gives moments of the form

$$\langle a^m a^{\dagger n}\rangle=\int\alpha^m\alpha^{*n}Q(\alpha,\alpha^*,\tau)d^2\alpha/\pi\ , \tag{33}$$

we may determine the time-dependent moments using Eq. (31). Alternatively, they are directly obtained via the trace operation from the initial density operator and the Heisenberg operators. We find

$$\langle a(\tau)\rangle=\alpha_0 e^{-i\nu/2}\exp[-\bar{n}(1-e^{-i\nu})]\ , \tag{34}$$

$$\langle a^2(\tau)\rangle=\alpha_0^2 e^{-2i\nu}\exp[-\bar{n}(1-e^{-2i\nu})]\ . \tag{35}$$

Comparing Eq. (34) with its classical analogue in Eq. (18), we see how the quantum recurrences are manifested in the moments. We also note that for ν small the quantum and classical results are quite similar. In Sec. III we show that this is indeed the correct semiclassical limit.

To calculate the quadrature phase statistics as described by the Q function, we must proceed a little more carefully. As noted in the Introduction, the Q function is a true joint probability density for approximate canonical variables. Let $\tilde{X}_1,\tilde{X}_2$ denote these approximate variables. However, the true, nonapproximate canonical quadrature phase amplitudes are defined by

$$\hat{X}_1=\tfrac{1}{2}(a+a^\dagger),\quad \hat{X}_2=\frac{1}{2i}(a-a^\dagger)\ . \tag{36}$$

How are the variances in these quantities related to the variances in the approximate variables $\tilde{X}_1$ and $\tilde{X}_2$ as determined by the Q function? Using the commutation relations between a and $a^\dagger$ and the fact that Q directly gives anti-normally-ordered expectation values of a and $a^\dagger$ [Eq. (33)], we find

$$\mathrm{var}(\tilde{X}_{1,2})=\mathrm{var}(\hat{X}_{1,2})+\tfrac{1}{4}\ . \tag{37}$$

The additional term of $\frac{1}{4}$ is, of course, a reflection of the "built-in" impossibility for the measuring instrument to measure both $\hat{X}_1$ and $\hat{X}_2$ simultaneously to an accuracy inconsistent with the uncertainty relations.

One easily shows that

$$\mathrm{var}(\tilde{X}_{1,2})=\tfrac{1}{2}[(\langle aa^{\dagger}\rangle-\langle a\rangle\langle a^{\dagger}\rangle)\pm\mathrm{Re}(\langle a^2\rangle-\langle a\rangle^2)]\ . \tag{38}$$

Noting that $a^{\dagger}a$ is a constant of the motion, we have $\langle aa^{\dagger}\rangle=1+|\alpha_0|^2$, then using Eqs. (34) and (35) the variances in the approximate canonical quadrature phase amplitudes may be easily determined. These, of course, are analogues to the classical quadrature phase variances calculated in Sec. II A.

The mean amplitude $\langle a(t)\rangle$ for two values of α_0 are shown in Figs. 5(a) and 5(b) for $0\leq\nu\leq 2\pi$. It is clear that for $|\alpha_0|^2$ sufficiently large and sufficiently small times the classical and quantum trajectories are quite similar. However, for later times the amplitude $\langle a(t)\rangle$ shows a complete recurrence of its initial value at $\nu=2\pi$ (up to a phase of $e^{i\pi}$).

In Figs. 6(a) and 6(b) are plotted the variances in $\tilde{X}_1$ and $\tilde{X}_2$. Once again, for large $|\alpha_0|^2$ and small times we see a close correspondence between the classical and quantum result. However, for later times there is an important divergence related to the concept of "squeezing."

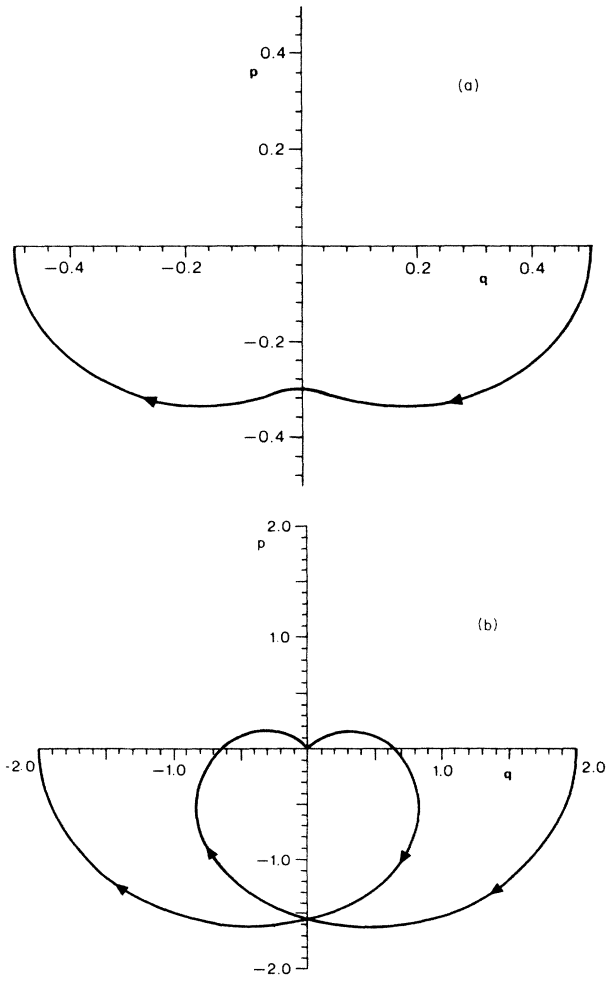

FIG. 5. Phase-space plot of the quantum trajectory $\langle a(\tau)\rangle$ over $0\leq\nu\leq 2\pi$: (a) $\alpha_0=0.5$ and (b) $\alpha_0=2.0$.

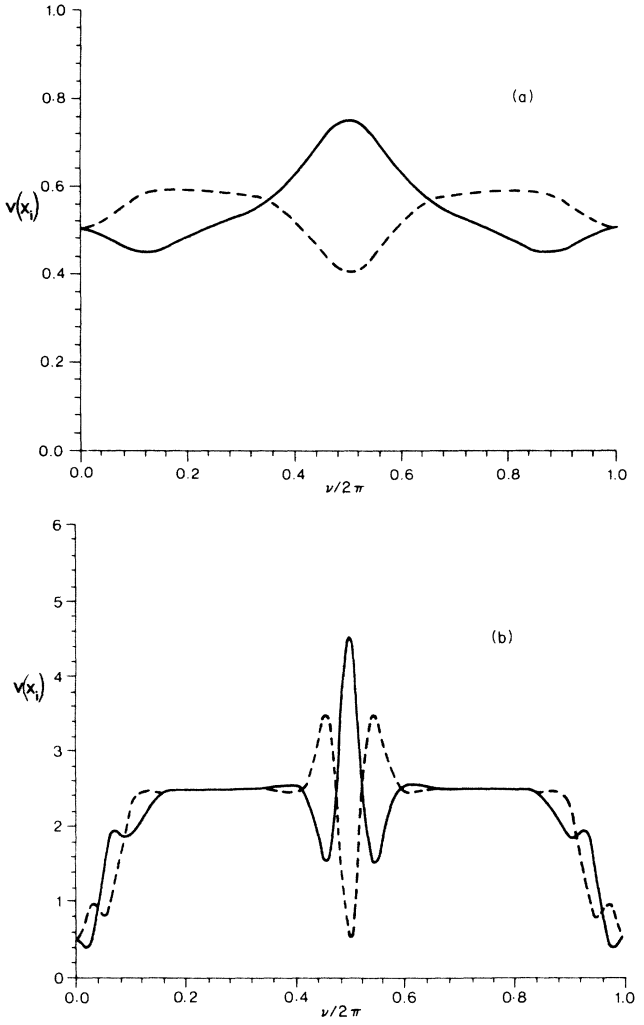

FIG. 6. Plot of quantum quadrature phase variances versus (a) $\alpha_0=0.5$ and (b) $\alpha_0=2$. Solid line, $V(\tilde{X}_2)$; dashed line, $V(\tilde{X}_1)$.

Squeezing is said to occur when[7]

$$\mathrm{var}(\hat{X}_i)<\tfrac{1}{4} \tag{39}$$

for $i=1$ or 2. In terms of the approximate variables $\tilde{X}_1$ and $\tilde{X}_2$, squeezing will be reflected in the variances in $\tilde{X}_1$ or $\tilde{X}_2$ falling below their initial value of $\frac{1}{2}$. Of course, this can occur classically as we have seen. What is characteristic of the quantum case is that this cannot occur simultaneously in both $\tilde{X}_1$ and $\tilde{X}_2$. From Figs. 6(a) and 6(b) we see that the variances differ most dramatically from the classical case at $\nu=\pi$. At this point the fluctuations in $\tilde{X}_2$ suddenly drop below the classical saturation level while, correspondingly, the fluctuations in $\tilde{X}_1$ increase. In fact, the greatest reduction in $\mathrm{var}(\tilde{X}_2)$ occurs for $\alpha_0=0.5$ when squeezing is present and

$$\mathrm{var}(\hat{X}_1)=\tfrac{1}{4}(1-1/e)\ . \tag{40}$$

This is the greatest squeezing possible in the model.

It is clear that the second-order derivatives in Eq. (30)

play a rather complicated role from the point of view of squeezing. They are not entirely responsible for squeezing, for if they are neglected, we still expect some squeezing for short times. (We must then interpret the resulting equation as an approximate quantum equation, not a classical equation. The concept of squeezing is meaningless in classical mechanics.) However, these terms are essential for the occurrence of maximum squeezing at midcycle. We conclude that the existence of non-positive-definite "diffusion" terms is not essential for squeezing; however, it does lead to more squeezing than would otherwise be expected.

We turn now to a study of the dynamics of the time-dependent Q function [Eq. (31)] itself. In Figs. 7 and 8 we have plotted contours of the quantum Q function for two initial conditions over the range $0 \leq \nu < 2\pi$. It has not been possible to follow the evolution of the same contour; however, the contours plotted do give a clear indication of the Q-function dynamics.

There is a clear distinction between the dynamic behavior for α_0 above and below $\alpha_0 = 1.0$. This is to be expected as the state $|\alpha_0 = 0.5\rangle$ has a mean occupation number of 0.25 and is closely related to the invariant ground state. In Fig. 7 we see the characteristic distortion of the initial circular contour. This is reflected in the squeezing in the quadrature phase variables. As expected, there is a complete recurrence (up to a phase of $e^{i\pi}$) of the initial circular Gaussian contour at $\nu = 2\pi$. However, for $\alpha_0 = 2.0$ some rather surprising features emerge. For short times the initial circular contour rotates and stretches in a way similar to the classical result. However, as the distribution is smeared around the origin, additional structure in the form of separate peaks form on a fairly flat background. These peaks smear, flatten, and reemerge as the cycle proceeds, eventually yielding two identical Gaussian peaks on opposite sides of the origin. The entire process then repeats in the opposite direction, leading to the expected recurrence at $\nu = 2\pi$ centered on $\alpha_0 = -2.0$. This behavior is remarkably different from the expected classical dynamics for this value of α_0.

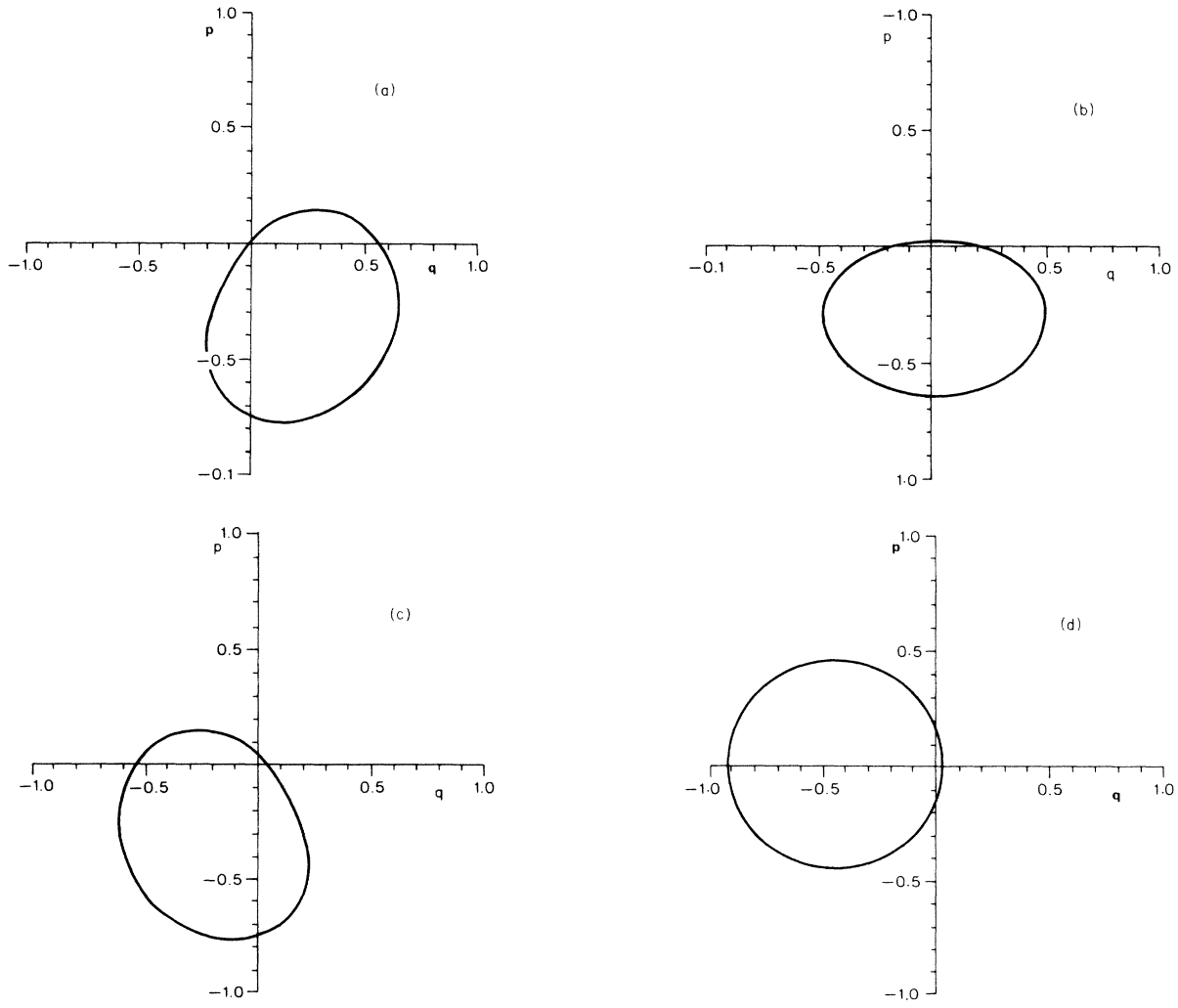

FIG. 7. Phase-space plot of the quantum Q function for initial circular Gaussian contour centered on $\alpha_0 = 0.5$: (a) $\nu = \pi/2$, (b) $\nu = \pi$, (c) $\nu = 3\pi/2$, (d) $\nu = 2\pi$.

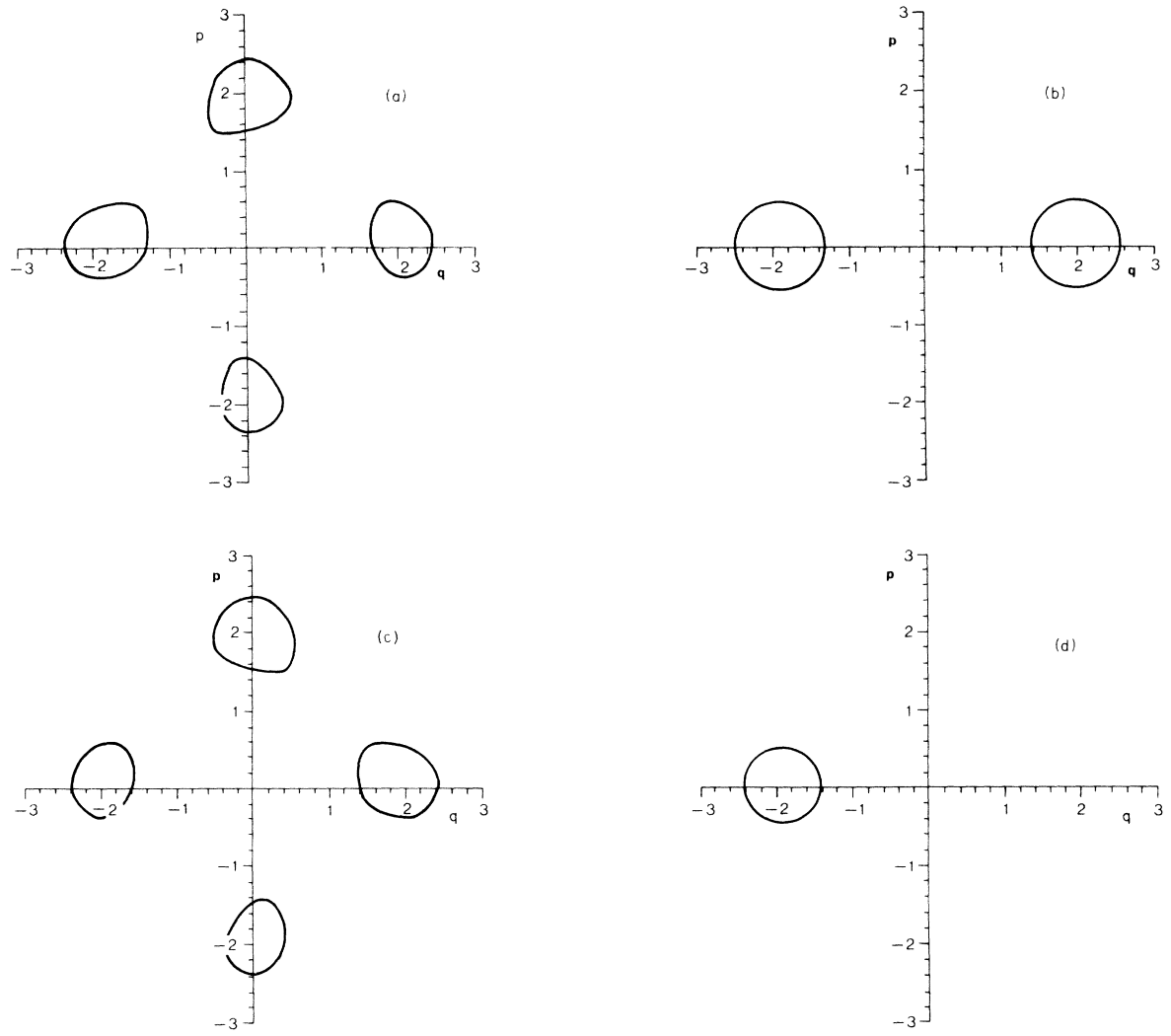

FIG. 8. As for Fig. 7, for an initial Gaussian contour centered on $\alpha_0=2.0$.

III. SEMICLASSICAL DYNAMICS

We have shown in the preceding discussion that the dynamics of phase-space densities in the quantum theory differ quite markedly from classical behavior. The quantum dynamics exhibits the typical recurrence not present in a classical description and also generates a greater reduction in the quadrature phase fluctuations. This behavior, it has been suggested, is intimately connected with the appearance of non-positive-definite second-order differential terms in the evolution equation for the phase-space density. We might expect, however, that the quantum dynamics approaches arbitrarily closely the classical dynamics in the semiclassical limit of $\hbar\rightarrow 0$ and for short times). We now investigate this in some detail.

In view of the scaling used in this paper, the semiclassical limit $\hbar\rightarrow 0$ cannot be taken directly. However, as we now show, this limit is equivalent to choosing the initial condition such that $|\alpha_0|\rightarrow\infty$. We write Eq. (30) in terms of the variable $\tilde{\alpha}=\alpha/\alpha_0$, and then

$$\frac{\partial Q(\tilde{\alpha},\tau)}{\partial\tau}=\left[i(\mu\bar{n})\tilde{\alpha}\left[\frac{1}{\bar{n}}+2|\tilde{\alpha}|^2\right]\frac{\partial Q}{\partial\tilde{\alpha}}+\text{c.c.}\right]+\left[i(\mu\bar{n})\frac{1}{\bar{n}}\tilde{\alpha}^2\frac{\partial^2 Q}{\partial\tilde{\alpha}^2}+\text{c.c.}\right], \tag{41}$$

where $\bar{n}\equiv|\alpha_0|^2$. The semiclassical limit is then obtained by keeping $\mu\bar{n}$ constant while $\bar{n}\rightarrow\infty$. The evolution equation then approaches the classical equation (14). To see that this is equivalent to letting $\hbar\rightarrow 0$, we evaluate the average energy in the initial state. For $\bar{n}$ large this is given by

$$\langle H\rangle=\hbar\omega_0\bar{n}(1+\mu\bar{n})\,.$$

Then for a chosen value of the initial energy, $\hbar\rightarrow 0$ requires $\bar{n}\rightarrow\infty$ with $\mu\bar{n}$ held constant. This ensures that we are comparing the classical and quantum dynamics on the same energy surface.

We first consider the semiclassical limits of the quan-

tum moments. These are functions of $v=2\theta/\bar{n}$, where $\theta=\mu\bar{n}\tau$. The large-$\bar{n}$ limit is taken with $\mu\bar{n}$ fixed and thus $\theta=\text{const}\times\tau$. For sufficiently large τ, there will always be a departure of the quantum and classical results. For example, there will always be a recurrence at $v=2\pi$ no matter how large $\bar{n}$. This suggests that in taking the semiclassical limit we restrict $\tau\ll\bar{n}/(\bar{n}\mu)$, that is, $v\ll 1$. In this limit the mean quantum amplitude is given by

$$\langle a(\tau)\rangle\simeq\alpha_0(1-iv/2)\exp\left[-\bar{n}\left[1-\frac{1}{1+iv}\right]\right], \tag{42}$$

which is close to the mean classical amplitude given in Eq. (18).

The quantum quadrature phase variances also become closer to their classical counterparts for large $\bar{n}$. The reduction in fluctuations of $\tilde{X}_2$ at mid-cycle occurs for larger times and is diminished.

To take the semiclassical limit of the quantum Q function directly is a rather more difficult task. The details are relegated to Appendix B. Essentially, this involves converting the sum S, in Eq. (31), to an integral, which is evaluated by the saddle-point method. This is quite similar to the procedure used in Ref. 17 to determine the semiclassical limit of collapses and revivals in the Jaynes-Cummings model for the interaction of a two-level atom with a single-mode field.

To consider the semiclassical limit of the quantum Q function, we first note that for large $\bar{n}$ the initial distribution retains its shape but is displaced a long way from the origin. Thus the contours in which we are interested correspond to those points α, such that $\alpha/\alpha_0=(1+\epsilon)e^{i\phi}$, where ϵ and ϕ are small. For small v the quantum Q function evaluated at these points is given approximately by

$$Q(\alpha,\tau)\simeq\exp[\bar{n}\epsilon+4\bar{n}\theta\phi(1+2\epsilon)], \tag{43}$$

where $\theta=\hbar\mu\tau$. The classical Q function at these points is given approximately as

$$Q(\alpha,\tau)\simeq\exp[\bar{n}\epsilon+4\bar{n}\theta\phi(1+3\epsilon)], \tag{44}$$

which is quite close to the quantum expression.

IV. CONCLUSION

In this paper we have sought to contrast classical and quantum dynamics by considering the evolution of joint probability densities in phase space. We have suggested that the appropriate quantum analogue of the classical phase-space density, in this context, is the Q function. This is a true probability density restricted to a certain "minimum-width" class.

Using a particular exactly solvable model, we have shown how the well-known recurrences of quantum dynamics and squeezing are related to the appearance of second-order differential terms with nonpositive coefficients, in the evolution equation for the density. It is expected that this is characteristic of quantum systems. These second-order terms prevent the fine-scale classical structure, known as a "whorl," from developing. It is an interesting mathematical question to ask for the possible general forms of the partial-differential equations for the evolution of Q-type functions. Work on answering this question is in progress.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix the solution of Eq. (30) is presented. Consider

$$\frac{\partial Q}{\partial\tau}=i\mu\alpha(1+2|\alpha|^2)\frac{\partial Q}{\partial\alpha}+i\mu\alpha^2\frac{\partial^2 Q}{\partial\alpha^2}+\text{c.c.}, \tag{A1}$$

set

$$Q(\alpha,\alpha^*,\tau)=\exp(-|\alpha|^2-|\alpha_0|^2)P(\alpha,\alpha^*,\tau), \tag{A2}$$

and then

$$\frac{\partial P}{\partial\tau'}=i\left[\left[\alpha\frac{\partial P}{\partial\alpha}-\alpha^*\frac{\partial P}{\partial\alpha^*}\right]+\alpha^2\frac{\partial^2 P}{\partial\alpha^2}-\alpha^{*2}\frac{\partial^2 P}{\partial\alpha^{*2}}\right], \tag{A3}$$

where $\tau'=\mu\tau$.

Assume

$$P(\alpha,\alpha^*,\tau')=P_1(\alpha,\tau')P_2(\alpha^*,\tau'), \tag{A4}$$

and then

$$\frac{\partial P_1}{\partial\tau'}+g(\tau')P_1=i\alpha\frac{\partial P_1}{\partial\alpha}+i\alpha^2\frac{\partial^2 P_1}{\partial\alpha^2}, \tag{A5}$$

$$\frac{\partial P_2}{\partial\tau'}-g(\tau')P_2=-i\alpha^*\frac{\partial P_2}{\partial\alpha^*}-i\alpha^2\frac{\partial^2 P_2}{\partial\alpha^{*2}}, \tag{A6}$$

where $g(\tau')$ is an arbitrary function of τ' alone. With no loss of generality, we may set $g(\tau)$ to zero. Then, defining $\beta=\ln\alpha$, Eq. (A5) becomes

$$\frac{\partial P_1}{\partial\tau}=i\frac{\partial^2 P_1}{\partial\beta^2}. \tag{A7}$$

We define

$$\tilde{P}_1(z,\tau)=\int_{\mathscr{C}}e^{-i\beta z}P_1(\beta,\tau')d\beta, \tag{A8}$$

where $\mathscr{C}$ is a suitably chosen contour. Then (A7) becomes

$$\frac{d\tilde{P}_1}{d\tau'}=-iz^2\tilde{P}_1. \tag{A9}$$

The initial condition

$$Q(\alpha,0)=\exp(-|\alpha-\alpha_0|^2) \tag{A10}$$

implies

$$P_1(\alpha,0)=e^{\alpha_0^*\alpha}\ , \tag{A11}$$

and thus

$$\tilde{P}_1(z,0)=\sum_{n=0}^{\infty}\frac{(\alpha_0^*)^n}{n!}\delta(z-in)\ . \tag{A12}$$

Solving Eq. (A9) subject to (A12), we have

$$\tilde{P}_1(z,\tau')=\sum_{n=0}^{\infty}\frac{(\alpha_0^*)^n}{n!}e^{-iz^2\tau'}\delta(z-in)\ . \tag{A13}$$

Thus,

$$P_1(\alpha,\tau)=\sum_{n=0}^{\infty}\frac{(\alpha_0^*\alpha)^n}{n!}\exp(in^2\mu\tau) \tag{A14}$$

and

$$Q(\alpha,\tau)=\exp(-|\alpha|^2-|\alpha_0|^2)|S|^2\ , \tag{A15}$$

with

$$S=\sum_{n=0}^{\infty}\frac{(\alpha_0\alpha^*)^n}{n!}e^{-i\mu\tau n^2}\ .$$

APPENDIX B

Consider the quantum Q function given in Eq. (31). The semiclassical limit essentially involves finding the asymptotic form of

$$S=\sum_{n=0}^{\infty}\frac{(\alpha_0\alpha^*)^n}{n!}e^{-i\theta n^2/\bar{n}}\ , \tag{B1}$$

where $\theta=\mu\bar{n}\tau$ and $\bar{n}=|\alpha_0|^2$, for $\bar{n}$ large but $\mu\bar{n}$ fixed.

Consider

$$\tilde{S}=\sum_{n=0}^{\infty}e^{-\bar{n}}\frac{\bar{n}^n}{n!}\nu^n e^{-i\theta n^2/\bar{n}}\ , \tag{B2}$$

where

$$\nu=\alpha^*/\alpha_0^*=Re^{i\phi}\ . \tag{B3}$$

Then,

$$S=e^{\bar{n}}\tilde{S}\ . \tag{B4}$$

We may write $\tilde{S}$ as

$$\tilde{S}=\sum_{n=0}^{\infty}P(n)\nu^n e^{-i\theta n^2/\bar{n}}\ , \tag{B5}$$

where $P(n)$ is a Poisson distribution, which for $\bar{n}$ large is sharply peaked at $n=\bar{n}$.

When $\bar{n}\gg 1$ we may approximate $P(n)$ by the continuous function

$$P(x)=(2\pi\bar{n})^{-1/2}x^{-1/2}\exp[\bar{n}f(x)]\ , \tag{B6}$$

where $x=n/\bar{n}$ and

$$f(x)=x(1-\ln x)-1\ .$$

The sum in Eq. (B5) may be written as an integral,

$$\tilde{S}\simeq(\bar{n}/2\pi)^{1/2}\int_0^{\infty}dx\,x^{-1/2}e^{\bar{n}h(x)}\ , \tag{B7}$$

where

$$h(x)=f(x)+ix(\phi-\theta x)\ . \tag{B8}$$

We now evaluate this integral by the saddle-point method. Thus we consider

$$I=\int_{\mathscr{C}}dz\,z^{-1/2}e^{\bar{n}h(z)}\ , \tag{B9}$$

where $\mathscr{C}$ is a contour along the real axis from $z=0$ to $z\to\infty$. The saddle points are given by

$$z_0=R\exp[i(\phi-2\theta z_0)]\ . \tag{B10}$$

Writing $z_0=r_0e^{i\xi_0}$, Eq. (B10) may be separated into two equations,

$$r_0=R\exp[(\phi-\xi_0)\tan\xi_0]\ , \tag{B11}$$

$$r_0=\frac{\phi-\xi_0}{2\theta}\sec\xi_0\ . \tag{B12}$$

The saddle points are determined by the intersection of these curves. There are many solutions to these equations; however, as we are primarily interested in the small time limit $\theta\ll 1$, and points $\nu\simeq(1+\epsilon)e^{i\phi}$ where ϕ and ϵ are small, we need only consider solutions in the range $-\pi/2\le\xi_0\le\pi/2$.

In Fig. 9 we have plotted

$$f_1(x)=R\exp[(\phi-x)\tan x]$$

and

$$f_2(x)=\frac{\phi-x}{2\theta}\sec x$$

on the interval $|x|<\pi/2$. As $\theta\to 0$ the intersection A comes in from $r_0\simeq 0, x\simeq-\pi/2$ to $x=\phi$ (from below) and $r_0=R$ (from above). We thus expect that for small θ, $\xi_0\simeq\phi+\delta$, and we expand f_1 and f_2 around $x=\phi$. In this approximation,

$$\delta\simeq-2\theta R\cos\phi\ , \tag{B13}$$

$$r_0\simeq R(1+2\theta R\sin\phi)\ . \tag{B14}$$

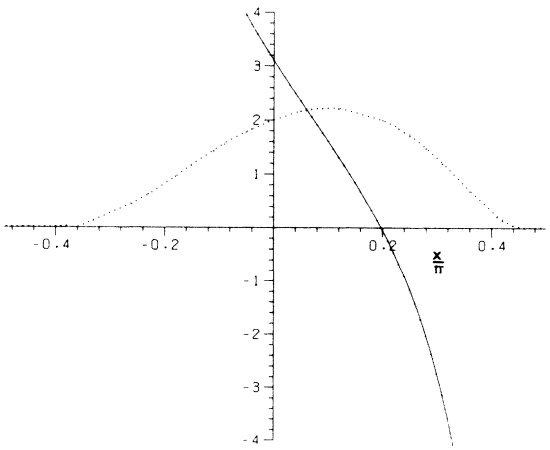

FIG. 9. Plot of $f_1(x)$ and $f_2(x)$ (see Appendix B) versus x over $|x|<\pi/2$. Solid line, $f_2(x)$; dashed line, $f_1(x)$.

The saddle point is given approximately by

$$z_0 = Re^{i\phi}(1+2\theta R\sin\phi)e^{-2i\theta R\cos\phi} . \tag{B15}$$

When $\theta=0$, $z_0=Re^{i\phi}$, as expected, and for small times the saddle point moves away from this point in a clockwise direction.

For convenience we may write

$$z_0 = Re^{i\rho} , \tag{B16}$$

where $\rho=\phi-2\theta Re^{i\phi}$, being careful to take only linear terms in θ at the end. The integral in Eq. (B9) then becomes

$$I = \left[\frac{2\pi}{\bar{n}}\right]^{1/2} |h''(z_0)|^{-1/2} z_0^{-1/2} \times \exp\left[\bar{n}h(z_0)+\frac{i}{2}(\pi-\alpha)\right] , \tag{B17}$$

where

$$\alpha = \arg[h''(z_0)] .$$

To linear order in θ and small ϕ,

$$|h''(z_0)|^{-1/2} \simeq R^{1/2} ,$$

$$\alpha \simeq \phi(1+2\theta R\phi) ,$$

$$h(z_0) \simeq R[1+i(\phi-\theta R)]+2\theta R^2\phi-1 .$$

Then, from (B7) and (B4),

$$|S|^2 \simeq \exp(2\bar{n}R+4\bar{n}\theta R^2\phi) . \tag{B18}$$

Thus,

$$Q(\alpha,\alpha^*,\tau) \simeq \exp[-\bar{n}(R+1)+2\bar{n}R+4\bar{n}\theta R^2\phi] \tag{B19}$$

is the asymptotic form for the quantum Q function when $\bar{n}\gg 1$, $\mu\bar{n}$ is constant, and τ is small, evaluated at points α, such that $\alpha/\alpha_0=(1+\epsilon)e^{-i\phi}$, with ϕ and ϵ small.

It is expected that were one to account fully for the multiplicity of saddle points as in Ref. 16, a greater understanding of the recurrences in the semiclassical Q-function dynamics may be obtained.

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