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High-frequency expansion of the plasma dielectric tensor

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We derive high-frequency sum-rule expansion for the transverse elements of the plasma dielectric tensor. The correlation contribution to the ω^{-4} sum-rule coefficient has a sign opposite to that of the longitudinal element. In addition, photon contributions add to the coefficient.

I. INTRODUCTION

High-frequency exact "sum-rule" expansions of linear-response functions, originally formulated by de Gennes,[1] have proven to provide an extremely powerful avenue both for developing approximation methods and for checking their consistencies. de Gennes's original expansion, valid to order ω^{-4} for classical equilibrium systems has been extended up to order ω^{-6}, both for neutral[2] and charged[3] systems. At the same time, high-frequency expansion methods have been formulated for a variety of systems: the degenerate electron gas,[4] spin systems,[5] two-component plasmas,[6] and the two-dimensional classical electron gas.[7]

As far as classical plasmas are concerned, the existing results pertain to the longitudinal density-density response functions. This latter, however, is only a part of the total linear-response function which is represented by the dielectric tensor. In an isotropic system the dielectric tensor has two independent elements, the longitudinal and transverse (with respect to the wave vector $\mathbf{k}$) elements. Thus, in order to assess the high-frequency behavior of the complete dielectric tensor, one has to complement the existing information with the analysis of the transverse element.

In an anisotropic system, in the presence of an external magnetic field, the situation is more complicated. The dielectric tensor now has six independent elements. Also, the relationship between the elements of the external and current-current response function elements (for which the analysis can be directly performed) and the elements of the dielectric tensor become quite involved. Finally, the appearance of an additional characteristic frequency, the cyclotron frequency, renders the structure more complex.

The present series of papers, in which this paper is the first, is devoted to the study of the high-frequency behavior of the full dielectric tensor. This paper deals with the isotropic situation. The high-frequency expansion is carried out to $O(\omega^{-4})$ and it reveals a drastic difference between the correlational contributions of the longitudinal and transverse elements. The method of derivation is similar to the standard approach[1] and relies heavily on the Hamiltonian formalism. In order to describe, however, the transverse interaction, the particle Hamiltonian has to be enlarged to include the photon degrees of freedom. Thus, inevitably, a further important new feature emerges: In addition to the particle contribution to the sum-rule coefficients, the photon-gas, coexistent with the high-temperature plasma, generates its own contribution. The precise evaluation of this contribution is hampered by two circumstances. The first is the well-known classical ultraviolet divergence which requires that even within the framework of a classical theory one describe the photons via the quantum Bose-Einstein distribution. The second difficulty arises from the fact that the equilibrium description implies the existence of one single temperature for the combined particle-photon system. Such an equilibrium, however, seldom prevails in any but astrophysical situations. Thus an *ad hoc,* although reasonable, approximation—described in Sec. II—is used to decouple the photons from the particle system.

In Sec. II of this paper we present the general relationships (valid for anisotropic situations as well) between the external or current-current response function sum-rule coefficients and those of the dielectric tensor. Then we calculate the exact ω^{-4} sum-rule coefficient for the transverse element. In Sec. III the long wavelength limit of the result is calculated and the possible implications for the dispersion relation of transverse plasma modes is discussed. In the Appendix the details of the rather lengthy algebra leading to the results of Sec. II are presented.

II. TRANSVERSE SUM RULES

The complete dielectric tensor $\epsilon_{\mu\nu}(\mathbf{k},\omega)$ or the complete polarizability tensor $\alpha_{\mu\nu}(\mathbf{k},\omega)$, are expressible in terms of the corresponding "external" quantity $\hat{\alpha}_{\mu\nu}(\mathbf{k},\omega)$ as

$$\begin{aligned} \underline{\alpha} &= \underline{\hat{\alpha}}(\underline{\Delta}-\underline{\hat{\alpha}})^{-1}\underline{\Delta}\ , \\ \underline{\Delta} &= \underline{1}-n^2\underline{T}\ , \\ n &= kc/\omega\ , \\ \underline{T} &= \underline{1}-\frac{\mathbf{kk}}{k}\ . \end{aligned} \tag{1}$$

The significance of expressing $\underline{\alpha}$ through $\underline{\hat{\alpha}}$ in (1) lies in the fact that $\underline{\hat{\alpha}}(\mathbf{k},\omega)$ possesses the well-known high-frequency sum-rule expansion

$$\underline{\hat{\alpha}}^{H'}(\mathbf{k},\omega) = -\sum_{\substack{l=1\\ l\ \mathrm{odd}}}^{\infty} \frac{\underline{\hat{\Omega}}_{l+1}(\mathbf{k})}{\omega^{l+1}}\ , \tag{2}$$

$$\underline{\hat{\alpha}}^{H''}(\mathbf{k},\omega) = -\sum_{\substack{l=2\\ l\ \mathrm{even}}}^{\infty} \frac{\underline{\hat{\Omega}}_{l+1}(\mathbf{k})}{\omega^{l+1}}\ , \tag{3}$$

where the superscript H stands for "Hermitian part of" and prime and double prime designate "real part of" or "the imaginary part of," respectively. The $\hat{\Omega}$ coefficients are calculated from the relation

$$\hat{\Omega}_l^{\mu\nu}(\mathbf{k})=\frac{1}{V}4\pi e\beta\left[\frac{id}{d\tau}\right]^l\langle j_{\mathbf{k}}^{\mu}(\tau)j_{\mathbf{k}}^{\nu}(0)\rangle_{t=0} \tag{4}$$

obtained through the routine derivation from the fluctuation-dissipation relations. Thus the high-frequency expansion of $\underline{\alpha}(\mathbf{k},\omega)$ becomes similar to that of $\hat{\alpha}(\mathbf{k},\omega)$ as given by (2) and (3), with $\underline{\Omega}_{l+1}(\mathbf{k})$ replacing the corresponding $\hat{\underline{\Omega}}_{l+1}(\mathbf{k})-s$. The relationships between the two sets of coefficients up to $l=6$ are given by

$$\begin{aligned}
&\underline{\Omega}_2=\hat{\underline{\Omega}}_2\,,\\
&\underline{\Omega}_4=\hat{\underline{\Omega}}_4-\hat{\underline{\Omega}}_2\cdot\hat{\underline{\Omega}}_2\,,\\
&\underline{\Omega}_3=\hat{\underline{\Omega}}_3\,,\\
&\underline{\Omega}_5=\hat{\underline{\Omega}}_5-\hat{\underline{\Omega}}_3\cdot\hat{\underline{\Omega}}_2-\hat{\underline{\Omega}}_2\cdot\hat{\underline{\Omega}}_3\,,\\
&\underline{\Omega}_6=\hat{\underline{\Omega}}_6-\hat{\underline{\Omega}}_2\cdot\hat{\underline{\Omega}}_4-\hat{\underline{\Omega}}_4\cdot\hat{\underline{\Omega}}_2-\hat{\underline{\Omega}}_2\cdot\underline{T}\cdot\hat{\underline{\Omega}}_2 n^2\,,\\
&\underline{\Omega}_7=\hat{\underline{\Omega}}_7-\hat{\underline{\Omega}}_3\cdot\hat{\underline{\Omega}}_4-\hat{\underline{\Omega}}_4\cdot\hat{\underline{\Omega}}_3-\hat{\underline{\Omega}}_3\cdot\underline{T}\cdot\hat{\underline{\Omega}}_2 n^2\\
&\qquad-\hat{\underline{\Omega}}_2\cdot\underline{T}\cdot\hat{\underline{\Omega}}_3 n^2+\hat{\underline{\Omega}}_2\cdot\hat{\underline{\Omega}}_3\cdot\hat{\underline{\Omega}}_3+\hat{\underline{\Omega}}_2\cdot\hat{\underline{\Omega}}_3\cdot\hat{\underline{\Omega}}_2\\
&\qquad+\hat{\underline{\Omega}}_3\cdot\hat{\underline{\Omega}}_2\cdot\hat{\underline{\Omega}}_2\,.
\end{aligned}\tag{5}$$

As discussed in the Introduction, the Hamiltonian appropriate for the description of the interaction of the plasma with the transverse electromagnetic field must include the photon degrees of freedom. Thus we have

$$\begin{aligned}H=&\frac{m}{2}\sum_i v_i^2+\tfrac{1}{2}\sum_{\substack{i,j\\ i\neq j}}V(\mathbf{x}_i-\mathbf{x}_j)\\&+\tfrac{1}{2}\sum_{\mathbf{q}}(\mathbf{e}_{\mathbf{q}}\cdot\mathbf{e}_{-\mathbf{q}}+q^2c^2\mathbf{a}_{\mathbf{q}}\cdot\mathbf{a}_{-\mathbf{q}})\end{aligned}\tag{6}$$

with

$$\mathbf{v}_i=\frac{1}{m}\mathbf{p}_i-\frac{e}{m}\sqrt{4\pi/V}\sum_{\mathbf{q}}\mathbf{a}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}_i}\,. \tag{7}$$

We now turn to the calculation of the frequency moments (up to $l=4$). Without external magnetic field the system is isotropic and $\hat{\underline{\alpha}}$ is diagonal. Consequently only $\hat{\underline{\Omega}}_2$ and $\hat{\underline{\Omega}}_4$ survive; moreover, in a coordinate system in which $\mathbf{k}=(00k)$ one is left only with the $\hat{\Omega}_n^{11}=\hat{\Omega}_n^{22}$ and $\hat{\Omega}_n^{33}$ elements.

The first moment is trivial,

$$\begin{aligned}\hat{\Omega}_2^{\mu\nu}(\mathbf{k})&=\frac{4\pi e^2\beta_p}{V}\langle j_{\mathbf{k}}^{\mu}(0)j_{-\mathbf{k}}^{\nu}(0)\rangle\\&=\frac{4\pi e^2\beta}{V}N\frac{m}{\beta_p}\delta^{\mu\nu}\\&=\omega_0^2\delta^{\mu\nu}\,.\end{aligned}\tag{8}$$

The third moment is given by

$$\begin{aligned}\hat{\Omega}_4^{\mu\nu}(\mathbf{k})&=\frac{4\pi e^2\beta}{V}\langle \dot{j}_{\mathbf{k}}^{\mu}(0)\dot{j}_{-\mathbf{k}}^{\nu}(0)\rangle\\&=\frac{4\pi e^2\beta}{V}\sum_{i,j}\langle(\dot{v}_i^{\mu}\dot{v}_j^{\nu}+k^{\alpha}k^{\beta}v_i^{\alpha}v_i^{\mu}v_j^{\beta}v_j^{\nu}\\&\qquad-ik^{\alpha}v_i^{\alpha}v_i^{\mu}\dot{v}_j^{\nu}+ik^{\beta}\dot{v}_i^{\mu}v_j^{\beta}v_j^{\nu})\\&\qquad\times e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle\,.\end{aligned}\tag{9}$$

The four terms in the parentheses can be reduced by a series of manipulations based on the canonical equations. Details of the calculation are given in the Appendix. The presence of the photon degrees of freedom $a_{\mathbf{q}}^{\mu},e_{\mathbf{q}}^{\mu}$ now leads to the appearance of averages of field coordinates of the type $\langle a_{\mathbf{q}}^{\mu}a_{-\mathbf{q}}^{\mu}\rangle$ and $\langle e_{\mathbf{q}}^{\mu}e_{-\mathbf{q}}^{\mu}\rangle$. In strict thermal equilibrium such terms are easily expressible in terms of the inverse temperature β. Nevertheless, the attainment of strict thermal equilibrium in any but astrophysical situations is not to be expected: the photon mean free path is simply too long for photons to become thermalized. In order to describe such a situation without abandoning the framework of equilibrium statistical mechanics, we will introduce two distinct, particle and radiation, temperatures represented by β_p and β_r, respectively, for the system. A further problem that arises from the evaluation of field coordinate averages of the type quoted is that when such averages are evaluated classically, a summation over the possible $\mathbf{q}$ modes of the electromagnetic field generates divergent integrals, in agreement with the classical ultraviolet divergence of the electromagnetic field energy. Thus such averages have to be evaluated quantum mechanically even in the framework of a classical theory, such as the present one. Introducing

$$\begin{aligned}c_{\mathbf{q}}^{i}&=\frac{1}{\sqrt{2}}\left[\sqrt{\omega_{\mathbf{q}}}a_{\mathbf{q}}^{\mu}+\frac{i}{\sqrt{\omega_{\mathbf{q}}}}e_{\mathbf{q}}^{\mu}\right]\varepsilon_{\mathbf{q}}^{\mu i}\,,\\ \omega_{\mathbf{q}}&=qc\end{aligned}\tag{10}$$

as a new set of coordinates with the polarization vectors $\varepsilon_{\mathbf{q}}^{\mu i}$, and identifying

$$n_{\mathbf{q}}^{i}=c_{\mathbf{q}}^{i*}c_{\mathbf{q}}^{i} \tag{11}$$

as the equivalent of the photon number operator, one evaluates averages by setting

$$\langle n_{\mathbf{q}}^{i}\rangle=\frac{1}{e^{\beta_r\hbar\omega_{\mathbf{q}}}-1}\,. \tag{12}$$

The results of the lengthy algebra displayed in the Appendix can be summarized as

$$\hat{\Omega}_4^{\mu\nu}(\mathbf{k})=\omega_0^4\left\{\frac{1}{N}\sum_{\mathbf{q}}\frac{q_\mu q_\nu}{q^2}(S_{\mathbf{k}-\mathbf{q}}-S_{\mathbf{q}})+\frac{k^\mu k^\nu}{k^2}+\frac{\beta_p}{\beta_r}\left[\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right]n(\omega_{\mathbf{k}})\hbar\omega_{\mathbf{k}}\beta_r\right.$$
$$+\frac{k^2}{\kappa^2}\left[3\frac{k^\mu k^\nu}{k^2}+\left[\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right]\right]+\frac{\beta_p}{\beta_r}\frac{1}{N}\sum_{\mathbf{q}}\left[\delta^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]S_{\mathbf{k}-\mathbf{q}}n(\omega_{\mathbf{q}})\hbar\omega_{\mathbf{q}}\beta_r$$
$$\left.+4\frac{1}{\beta_r mc^2}\frac{1}{N}\sum_{\mathbf{q}}\frac{q_\mu q_\nu}{q^2}n(\omega_{\mathbf{q}})\hbar\omega_{\mathbf{q}}\beta_r\right\}, \quad (13)$$

(13) above, as it stands, is not acceptable: in the $k\to 0$ limit no difference should exist between the longitudinal and transverse elements of $\hat{\Omega}_4^{\mu\nu}$. The offending term, violating this requirement is

$$\omega_0^4\frac{k_\mu k_\nu}{k^2}+\frac{\beta_p}{\beta_r}\left[\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right].$$

We argue that in this limit the distinction between particle (longitudinal) and radiation (transverse) temperatures is meaningless: thus we treat β_r as a k-dependent quantity, such that β_r $(k\to 0)=\beta_p$, while β_r $(k\neq 0)$ is not affected by this condition. We also drop the last term, which is manifestly of the same order as relativistic corrections not considered in the present derivation. Equation (13) so modified, now becomes

$$\hat{\Omega}_4^{\mu\nu}(\mathbf{k})=\omega_0^4\left[\delta^{\mu\nu}+\frac{\beta_p}{\beta_r}T_{\mathbf{k}}^{\mu\nu}[f(x_{\mathbf{k}})-1]+\frac{k^2}{\kappa^2}(3L_{\mathbf{k}}^{\mu\nu}+T_{\mathbf{k}}^{\mu\nu})+\frac{1}{N}\sum_{\mathbf{q}}L_{\mathbf{q}}^{\mu\nu}(S_{\mathbf{k}-\mathbf{q}}-S_{\mathbf{q}})+\frac{\beta_p}{\beta_r}\frac{1}{N}\sum_{\mathbf{q}}T_q^{\mu\nu}S_{\mathbf{k}-\mathbf{q}}f(x_{\mathbf{q}})\right],$$
$$L_{\mathbf{k}}^{\mu\nu}=\frac{k^\mu k^\nu}{k^2},$$
$$T_{\mathbf{k}}^{\mu\nu}=\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}, \quad (14)$$
$$x_{\mathbf{k}}=\hbar\omega_{\mathbf{k}}\beta_r,$$
$$f(x)=\frac{x}{e^x-1}.$$

III. LONG-WAVELENGTH LIMIT

The long-wavelength $(k\to 0)$ limit of (13) illuminates the substantial difference between longitudinal and traverse sum rules. First we consider the photon independent terms, i.e., the ones that survive in the $\beta_r\to\infty$ limit. The correlational term

$$\frac{\omega_0^4}{N}\sum_{\mathbf{q}}L_{\mathbf{q}}^{\mu\nu}(S_{\mathbf{k}-\mathbf{q}}-S_{\mathbf{q}})\equiv\frac{\omega_0^4}{V}\sum_{\mathbf{q}}(L_{\mathbf{k}-\mathbf{q}}^{\mu\nu}-L_{\mathbf{q}}^{\mu\nu})g_{\mathbf{q}} \quad (15)$$

yields the well-known longitudinal

$$\tfrac{4}{15}\omega_0^4\frac{k^2}{\kappa^2}\beta_p E_{\text{corr}} \quad (16)$$

and the new transverse

$$-\frac{2}{15}\omega_0^4\frac{k^2}{\kappa^2}\beta_p E_{\text{corr}} \quad (17)$$

expressions, where E_{corr} is the (negative) correlation energy per particle. There are, for obvious reasons, no off-diagonal elements. The difference of sign between (16) and (17) should be noted.

Combining (16) and (17) with the photon-related terms, one obtains the complete $k\to 0$ expressions,

$$\hat{\Omega}_4^{33}(\mathbf{k})=\omega_0^4\left[1+\frac{k^2}{\kappa^2}(3+\tfrac{4}{15}\beta_p E_{\text{corr}})+\frac{\beta_r}{\beta_p^4 n\hbar^3c^3}\left[\frac{\pi^2}{45}+\frac{1}{3\pi^2}G_0+\frac{k^2}{30\pi^2}(4G_1+G_2)\right]\right], \quad (18a)$$

$$\hat{\Omega}_4^{11}(\mathbf{k})=\omega_0^4\left[1+\frac{k^2}{\kappa^2}(1-\tfrac{2}{15}\beta_p E_{\text{corr}})+\frac{\beta_p}{\beta_r^4 n\hbar^3c^3}\left[\frac{\pi^2}{15}+\frac{1}{3\pi^2}G_0+\frac{k^2}{15\pi^2}(G_1+2G_2)\right]\right], \quad (18b)$$

with

$$G_0=\int dx\,x^2f(x)ng_{\mathbf{q}},$$
$$G_1=\int dx\,x^2f(x)\frac{1}{q}\frac{\partial}{\partial q}ng_{\mathbf{q}}, \quad (19)$$
$$G_2=\int dx\,x^2f(x)\frac{\partial^2}{\partial q^2}ng_{\mathbf{q}}.$$

The high-frequency sum-rule expression, apart from providing an exact relationship which can serve as a standard against which approximations can be evaluated, can also be regarded as an approximate expression for response functions in the frequency domain where using the expansion is not well justified. In particular, it can be used for the analysis of collective modes in the long-wavelength limit. Even though there is no *a priori* reason for the expansion to be convergent in that frequency domain (and we have argued elsewhere[8] that the expansion is indeed not convergent), there is empirical evidence[9,10] indicating that terminating the expansion of the $\hat{\Omega}_4$ term and using the resulting expression to represent $\hat{\alpha}$, yields a qualitatively correct and quantitatively reasonable result for plasmon dispersion. Adopting this philosophy, we can now estimate what novel physical effects are described by the expressions (18a) and (18b).

Using (18a) in conjunction with (5), the dispersion relation

$$\epsilon_{33}(\mathbf{k},\omega)\equiv 1+\alpha_{33}(\mathbf{k},\omega)=0 \qquad (20)$$

determines the behavior of longitudinal plasmons. Writing the ensuing expression for the plasmon frequency as

$$\omega^2(\mathbf{k})=\omega_0^2\left[1+C(\gamma,\beta_r)+A_L(\gamma,\beta_r)\frac{k^2}{\kappa^2}\right] \qquad (21)$$

one finds

$$A_L(\gamma,\beta_r)=3+\tfrac{4}{15}\beta_p E_{\rm corr}+\frac{1}{30\pi^2}\frac{\beta_p\kappa^2}{\beta_r^4 n\hbar^3 c^3}(4G_1+G_2) \qquad (22)$$

and

$$C(\gamma,\beta_r)=\frac{\beta_p}{\beta_r^4 n\hbar^3 c^3}\left[\frac{\pi^4}{15}+\frac{1}{3\pi^2}G_0\right]. \qquad (23)$$

For $\beta_r^{-1}\to 0$ A_L is known[8] to change from positive to negative values for $\gamma>\gamma_{\rm crit}$. While according to (22) $\gamma_{\rm crit}>\frac{45}{2}$, both molecular dynamics computer results[11] and recent more sophisticated theoretical results[10] indicate an actual $\gamma_{\rm crit}\simeq 45$ value. The effect of finite radiation temperature is manifested through G_0, G_1, and G_2; for all situations but the combined occurrence of strong enough coupling to induce oscillations in $g_{\mathbf{q}}$ *and* extremely high radiation temperature causing the photon thermal wavelength to become shorter than the interparticle spacing, it is expected that $G_0<0$, while $G_1>0$, $G_2>0$; however, even for $G_0<0$, the $\pi^2/15$ term is expected to dominate. Thus finite radiation temperature results (i) in an upward renormalization of the plasma frequency from ω_0 to $\omega_0(1+\frac{1}{2}C)$, and (ii) in a reduction of the negative correlational effect on plasmon dispersion for finite k. It should be kept in mind, however, that under normal conditions both of these effects are very small; some numerical values will be given in a separate publication.

Turning now to (18b), in conjunction with (5) it determines the behavior of transverse photons through the dispersion relation

$$\epsilon_{11}(\mathbf{k},\omega)\equiv 1+\alpha_{11}(\mathbf{k},\omega)=n^2\,. \qquad (24)$$

The ensuing photon frequency can be written as

$$\omega^2(\mathbf{k})=\omega_0^2\left[1+C(\gamma,\beta_r)+A_T(\gamma,\beta_r)\frac{k^2}{\kappa^2}\right]+k^2c^2 \qquad (25)$$

with

$$A_T(\gamma,\beta_r)=1-\tfrac{2}{15}\beta_p E_{\rm corr}+\frac{\kappa^2}{15\pi^2}(G_1+2G_2)\,. \qquad (26)$$

In contrast to the effect of correlations on longitudinal plasmon dispersion, here the correlations are seen to further enhance, rather than to reduce, the positive thermal dispersion. If one argues that the explanation of correlational effects on dispersion can be sought in the system for strong coupling trying to emulate the mode structure of a Wigner lattice, this result is not surprising, since the high-frequency transverse phonons, in contrast to the longitudinal ones, do exhibit a positive dispersion.

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APPENDIX

In this appendix, we give details of the derivation leading to (13). Consider the four terms in (9):

$$(\hat{\Omega}_4^{\mu\nu})_{\rm I}=\frac{4\pi e^2\beta_p}{V}\sum_{i,j}\langle \dot v_i^\mu \dot v_j^\nu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle\,,$$

$$(\hat{\Omega}_4^{\mu\nu})_{\rm II}=\frac{4\pi e^2\beta_p}{V}\sum_{i,j}k^2k^\beta\langle v_i^\alpha v_i^\mu v_j^\beta v_j^\nu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle\,,$$

$$(\hat{\Omega}_4^{\mu\nu})_{\rm III}=-i\frac{4\pi e^2\beta_p}{V}\sum_{i,j}k^\alpha\langle v_i^\alpha v_i^\mu \dot v_j^\nu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle\,, \qquad (A1)$$

$$(\hat{\Omega}_4^{\mu\nu})_{\rm IV}=-i\frac{4\pi e^2\beta_p}{V}\sum_{i,j}k^\beta\langle \dot v_i^\mu v_j^\beta v_j^\nu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle\,.$$

Since $\dot v_i^\mu$ is

$$\dot v_i^\mu=\frac{1}{m}\left[-\frac{\partial\Phi}{\partial x_i^\mu}+\lambda\sum_{\mathbf{q}}e_{\mathbf{q}}^\mu e^{i\mathbf{q}\cdot\mathbf{x}_i}+i\lambda v_i^\alpha\sum_{\mathbf{q}}q^\mu a_{\mathbf{q}}^\alpha e^{i\mathbf{q}\cdot\mathbf{x}_i}-i\lambda v_i^\alpha\sum_{\mathbf{q}}q^\alpha a_{\mathbf{q}}^\alpha e^{i\mathbf{q}\cdot\mathbf{x}_i}\right] \qquad (A2)$$

with

$$\Phi=\tfrac{1}{2}\sum_{\substack{m,n\\ m\neq n}}V(\mathbf{x}_m-\mathbf{x}_n)\,,\qquad \lambda=\sqrt{4\pi e^2/V}\,, \qquad (A3)$$

$(\hat{\Omega}_4^{\mu\nu})_{\rm I}$ consists of 4×4 terms which we label as $(\hat{\Omega}_4^{\mu\nu})_{\rm I,11}$, $(\hat{\Omega}_4^{\mu\nu})_{\rm I,12},\ldots,(\hat{\Omega}_4^{\mu\nu})_{\rm I,44}$. Then

$$(\hat{\Omega}_4^{\mu\nu})_{\rm I,11}=\frac{\kappa^2}{m^2}\frac{1}{N}\sum_{i,j}\left\langle\frac{\partial\Phi}{\partial x_i^\mu}\frac{\partial\Phi}{\partial x_i^\nu}e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\right\rangle\,,\qquad \kappa^2=4\pi e^2 n\beta\,. \qquad (A4)$$

But

$$\frac{\partial\Phi}{\partial x_j^\nu}=\frac{\partial H}{\partial x_j^\nu}-\frac{\partial K}{\partial x_j^\nu}\,, \qquad K=\frac{m}{2}\sum_m v_m^2 \tag{A5}$$

which further splits $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11}$. The usual procedure[1,12] allows one to evaluate averages of the type $\langle A(\partial H/\partial x_j^\nu)\rangle$ as

$$\left\langle A\frac{\partial H}{\partial x_j^\nu}\right\rangle=\frac{1}{\beta_p}\left\langle\frac{\partial A}{\partial x_j^\nu}\right\rangle\,. \tag{A6}$$

Introducing the notation

$$E_{ij}=e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)} \tag{A7}$$

the repeated application of (A5) and (A6) transforms $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11}$ into

$$(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11}=\frac{\kappa^2}{m^2}\frac{1}{N}\sum_{i,j}\left[\frac{1}{\beta_p}\left\langle\frac{\partial^2\Phi}{\partial x_i^\mu\partial x_j^\nu}E_{ij}\right\rangle+\frac{1}{\beta_p^2}\left\langle\frac{\partial^2 E_{ij}}{\partial x_i^\mu\partial x_j^\nu}\right\rangle-\frac{2}{\beta_p}\left\langle\frac{\partial K}{\partial x_j^\nu}\frac{\partial E_{ij}}{\partial x_i^\mu}\right\rangle-\frac{1}{\beta_p}\left\langle\frac{\partial^2 K}{\partial x_i^\mu\partial x_j^\nu}E_{ij}\right\rangle+\left\langle\frac{\partial K}{\partial x_i^\mu}\frac{\partial K}{\partial x_j^\nu}E_{ij}\right\rangle\right]. \tag{A8}$$

The derivatives of K can be evaluated with the aid of

$$v_m^\alpha=\frac{1}{m}p_m^\alpha-\lambda\sum_{\mathbf{q}}a_{\mathbf{q}}^\alpha e^{i\mathbf{q}\cdot\mathbf{x}_i}\,. \tag{A9}$$

As a result, only the first two terms in (A8) survive: the third term averages out to zero, since $\mathbf{v}_m$ and $\mathbf{a}_\mathbf{q}$ are uncorrelated (note, however, that $\mathbf{p}_m$ and $\mathbf{a}_\mathbf{q}$ are not) and the last two terms cancel each other. The remaining terms [say $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11a}$ and $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11b}$] can be evaluated in the same way as in the case of the conventional longitudinal sum-rule calculations:[12]

$$(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11a}=\frac{\kappa^2}{m^2\beta_p^2}\frac{1}{N}\sum_{i,j}\left\langle\frac{\partial^2\Phi}{\partial x_i^\mu\partial x_j^\nu}E_{ij}\right\rangle=\omega_0^4\frac{1}{N}\sum_{\mathbf{q}}\frac{q^\mu q^\nu}{q^2}(\delta_{\mathbf{k}-\mathbf{q}}-\delta_\mathbf{q})+\omega_0^4\frac{k^\mu k^\nu}{k^2}\,, \tag{A10}$$

$$(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},11b}=\frac{\kappa^2}{m^2\beta_p^2}\frac{1}{N}\sum_{i,j}\left\langle\frac{\partial^2 E_{ij}}{\partial x_i^\mu\partial x_j^\nu}\right\rangle=\frac{\omega_0^4}{k^2}k^\mu k^\nu n g_\mathbf{k}=\frac{\omega_0^4}{mc^2}\frac{2}{N}\sum_\mathbf{q}\frac{q^\mu q^\nu}{q^2}n(\hbar\omega_\mathbf{q})\hbar\omega_\mathbf{q}\,. \tag{A11}$$

Next we consider $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},22}$:

$$(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},22}=\frac{\kappa^2\lambda^2}{Nm^2}\sum_{i,j}\sum_\mathbf{q}\langle e_\mathbf{q}^\mu e_{-\mathbf{p}}^\nu e^{i\mathbf{q}\cdot\mathbf{x}_i}e^{-i\mathbf{p}\cdot\mathbf{x}_j}e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle$$
$$=\omega_0^4\beta_p\frac{1}{N}\sum_\mathbf{q}\langle e_\mathbf{q}^\mu e_{-\mathbf{q}}^\nu\rangle(S_{\mathbf{k}-\mathbf{q}}+N\delta_{\mathbf{k}-\mathbf{q}})$$
$$=\omega_0^4\beta_p\frac{1}{N}\sum_\mathbf{q}\left[\delta^{\mu\nu}-\frac{q^\mu a^\nu}{q^2}\right]\delta_{\mathbf{k}-\mathbf{q}}n(\hbar\omega_\mathbf{q})\hbar\omega_\mathbf{q}+\omega_0^4\beta_p\left[\delta^{\mu\nu}-\frac{k^\mu k^\nu}{k}\right]n(\hbar\omega_\mathbf{k})\hbar\omega_\mathbf{k}\,. \tag{A12}$$

Furthermore, $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},33}$ becomes

$$(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},33}=\frac{\kappa^2\lambda^2}{Nm^2}\sum_{i,j}\sum_\mathbf{q}\langle v_i^\alpha v_j^\beta q^\mu p^\nu a_\mathbf{q}^\alpha a_{-\mathbf{p}}^\beta e^{i\mathbf{q}\cdot\mathbf{x}_i}e^{-i\mathbf{p}\cdot\mathbf{x}_j}e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle$$
$$=\frac{\omega_0^4}{Nm}\sum_\mathbf{q}q^\mu q^\nu\langle a_\mathbf{q}^\alpha a_{-\mathbf{q}}^\alpha\rangle\,. \tag{A13}$$

Similarly, $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},44}$ yields

$$(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},44}=\frac{\omega_0^4}{mc^2}\frac{1}{N}\sum_\mathbf{q}\left[\delta^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]n(\hbar\omega_\mathbf{q})\hbar\omega_\mathbf{q}\,. \tag{A14}$$

As to the cross terms, $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},34}=(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},43}=0$, because of the appearance of $T_\mathbf{q}L_\mathbf{q}$ products. $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},13}$ and $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},14}$ also vanish, as can be seen writing, e.g., $(\hat{\Omega}_4^{\mu\nu})_{\mathrm{I},13}$, in the form, where $B_{ij}^{\alpha\nu}$ and D below are independent of the velocities:

$$(\hat{\Omega}_4^{\mu\nu})_{\text{I},13}=\sum_{i,j}\left\langle \frac{\partial\Phi}{\partial x_i^\nu}v_j^\alpha B_{ij}^{\alpha\nu}\right\rangle=\sum_{i,j}\left\langle\left[\frac{\partial H}{\partial x_i^\mu}-\frac{\partial K}{\partial x_i^\mu}\right]v_j^\alpha B_{ij}^{\alpha\nu}\right\rangle$$
$$=\frac{1}{\beta_p}\sum_{i,j}\left\langle \delta_{ij}\delta^{\alpha\mu}D-m\beta_p\sum_m \delta_{im}\delta^{\beta\mu}v_m^\beta v_j^\alpha DB_{ij}^{\alpha\nu}\right\rangle$$
$$=\frac{1}{\beta_p}\sum_i[\delta^{\alpha\mu}-m\beta_p\langle v_i^\mu v_i^\alpha\rangle]\langle DB_{ii}^{\alpha\nu}\rangle$$
$$=0\;. \tag{A15}$$

$(\hat{\Omega}_4^{\mu\nu})_{\text{I},14}$ vanishes for the same reason. Finally the cross terms $(\hat{\Omega}_4^{\mu\nu})_{\text{I},21}$, $(\hat{\Omega}_4^{\mu\nu})_{\text{I},23}$, and $(\hat{\Omega}_4^{\mu\nu})_{\text{I},24}$ are all equal to zero because of the presence of the uncorrelated $e_{\mathbf{q}}$ contributions.

The calculation of II is quite simple:

$$(\hat{\Omega}_4^{\mu\nu})_{\text{II}}=\frac{\kappa^2}{N}\sum_{i,j}k^\alpha k^\beta\langle v_i^\alpha v_j^\beta v_i^\mu v_j^\nu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle$$
$$=\frac{\kappa^2}{N}\sum_{i,j}(k^\alpha k^\beta[(1-\delta_{ij})\delta^{\alpha\mu}\delta^{\beta\nu}\langle v^{\mu 2}\rangle^2]\langle e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle$$
$$+\delta_{ij}\{[\delta^{\mu\nu}\delta^{\alpha\beta}(1-\delta^{\alpha\mu})+(1-\delta^{\mu\nu})(\delta^{\alpha\mu}\delta^{\beta\nu}+\delta^{\alpha\nu}\delta^{\beta\mu})]$$
$$\times\langle v^{\mu 2}\rangle^2+\delta^{\mu\nu}\delta^{\alpha\beta}\delta^{\alpha\mu}\langle v^{\mu 4}\rangle\}) \text{ (no summation over } \mu,\nu)$$
$$=\frac{\omega_0^4}{\kappa^2}(k^\mu k^\nu n g_{\mathbf{k}}+k^2\delta^{\mu\nu}+2k^\mu k^\nu)\;. \tag{A16}$$

Finally we turn to III and IV:

$$(\hat{\Omega}_4^{\mu\nu})_{\text{III}}=-i\frac{\kappa^2}{N}\sum_{i,j}k^2\langle v_i^\alpha v_i^\mu \dot{v}_j^\nu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle$$
$$=i\frac{\kappa^2}{Nm}\sum_{i,j}k^2\left\langle v_i^\alpha v_i^\mu\frac{\partial\Phi}{\partial x_j^\nu}-\lambda\sum_{\mathbf{q}}e_{\mathbf{q}}^\nu e^{i\mathbf{q}\cdot\mathbf{x}_i}\right.$$
$$-\frac{i\lambda}{m}v_j^\beta\sum_{\mathbf{q}}q^\nu a_{\mathbf{q}}^\beta e^{i\mathbf{q}\cdot\mathbf{x}_j}$$
$$\left.+\frac{i\lambda}{m}v_j^\beta\sum_{\mathbf{q}}q^\beta a_{\mathbf{q}}^\nu e^{i\mathbf{q}\cdot\mathbf{x}_j}e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\right\rangle\;. \tag{A17}$$

The four terms in III we label as $(\hat{\Omega}_4^{\mu\nu})_{\text{III},1}$, $(\hat{\Omega}_4^{\mu\nu})_{\text{III},2}$, $(\hat{\Omega}_4^{\mu\nu})_{\text{III},3}$, and $(\hat{\Omega}_4^{\mu\nu})_{\text{III},4}$, respectively. It is not difficult to see that only $(\hat{\Omega}_4^{\mu\nu})_{\text{III},1}$ survives the averaging. $(\hat{\Omega}_4^{\mu\nu})_{\text{III},2}$ vanishes because of the presence of the uncorrelated $\mathbf{e}_{\mathbf{q}}$ contribution, while $(\hat{\Omega}_4^{\mu\nu})_{\text{III},3}$ and $(\hat{\Omega}_4^{\mu\nu})_{\text{III},4}$ vanish due to the odd velocity moments. Rewriting now $(\hat{\Omega}_4^{\mu\nu})_{\text{III},1}$ as

$$(\hat{\Omega}_4^{\mu\nu})_{\text{III},1}=i\frac{\kappa^2}{Nm}\sum_{i,j}k^\alpha\left\langle v_i^\alpha v_i^\mu\left[\frac{\partial H}{\partial x_j^\nu}-\frac{\partial K}{\partial x_j^\nu}\right]e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\right\rangle \tag{A18}$$

one can easily convince oneself that the second term in $(\hat{\Omega}_4^{\mu\nu})_{\text{III},1}$ also vanishes because of the odd velocity moments. The first term also leads to odd velocity moments, except for one contribution which yields

$$(\hat{\Omega}_4^{\mu\nu})_{\text{III},1}=-\frac{\kappa^2}{Nm\beta_p}\sum_{i,j}k^2k^\nu(1-\delta_{ij})\langle v_i^\alpha v_i^\mu e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\rangle$$
$$=\frac{\omega_0^4}{\kappa^2}k^\mu k^\nu n g_{\mathbf{k}}\;. \tag{A19}$$

IV provides an identical contribution. Combining now (A10)–(A14), (A16), and (A19), one obtains the result quoted in (13).

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