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Superimposed renewal processes: A new method of superimposing a Poisson distribution with periodic pulses

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In this paper a new kind of superposition which mixes a dead-time-distorted Poisson process with periodic pulses is extensively described. The two original sequences are assumed distorted separately by insertion of proper dead times and are then subsequently superimposed: this method allows one an exact theoretical treatment valid in general for arbitrary values of the parameters. This technique and the relative formulas have been subjected to experimental check. The excellent agreement between theory and experiment supports the reliability of the method when it is applied to dead-time measurements.

I. INTRODUCTION

For a Poisson statistical process developing in time, the most direct consequence of the presence of dead time is in general the modification of the distribution of time intervals between successive pulses. A vast literature can be found on problems related to dead-time-distorted Poisson processes involving both one channel (e.g., counting rate and losses[1,2] experimental distribution[3–6]) or two channels (e.g., coincidence measurements, pulse-height converters[7–10]). As is well known, the theory of renewal processes[11] has played the predominant role in explaining these problems. However, the superposition of renewal processes having the simple Markov property does not produce in general a new simple Markov renewal process: for this reason the superposition of a Poisson sequence on periodic pulses, when the whole train of pulses is dead-time distorted[12] (i.e., distorted after the superposition), is not susceptible to an exact theoretical analysis. In this case only approximate formulas or Monte Carlo simulation may give the appropriate solution.[13]

In the present paper we propose a new method of superimposing a Poisson distribution on periodic pulses: the two original sequences are assumed distorted separately by a nonextended dead time and then superimposed; this fact can cause only a lengthening of some pulses but no change in the relative arrival times between distorted random pulses. Exact theoretical calculations have been performed, as we shall discuss in Sec. II. In order to test the resulting formulas, measurements have been done for different values of the parameters, obtaining an excellent agreement between theory and experiment, as shown in Sec. III. Finally we suggest a possible application of our method for dead-time measurements.

II. THEORY AND CALCULATIONS

First of all we recall that a Poisson sequence of pulses developing in time is a simple Markov renewal process: in fact the random variables representing the waiting times between successive pulses are mutually independent with a common exponential distribution. Moreover, as for any exponential distribution, this distribution has the simple Markov property: the residual lifetime is unaffected by the past and has the same distribution as the lifetime itself.[11]

When a pure Poisson process, with a rate α, is distorted by a dead time τ of nonextended type,[8] pulses with width τ are distributed with a mutual relative arrival time represented by a stochastic variable U_k whose density can be written as

$$\alpha e^{-\alpha(u_k-\tau)} \quad u_k \geq \tau, \quad k \geq 1 .$$

The entire sequence is a convolution with a probability density:

$$u = u_0 + u_1 + \cdots + u_k + \cdots + u_m \geq m\tau, \quad m \geq 0$$

$$g_{m+1}(u - m\tau) = \alpha^{m+1} \frac{(u - m\tau)^m}{m!} e^{-\alpha(u - m\tau)} ,$$

where u_0 is the waiting time of the first pulse from an arbitrary time origin.

Let us consider now a train of periodic pulses with period T. Its density can be written as

$$p_d = \sum_{n=0}^{\infty} \delta(\Theta - t_0 - nT)$$

i.e., a sum of δ functions starting at the time $\Theta = t_0$ at which the pulser is switched on. If these pulses are processed by a device introducing a width τ_1 and afterwards are superimposed on the distorted Poisson train previously described, it is clear that there is no interference between the two distorted processes. In fact this kind of superposition has the following effect: when two or more pulse of the two sequences overlap there is a change of the height in the overlapping region and a possible increase of the length of some pulse (see Fig. 1) when a pulser (random) signal arrives within and extends beyond the width τ (τ_1) of a random (pulser) pulse. The change in the height is irrelevant for our purposes: the important feature being that no pulse has been canceled by this superposition so

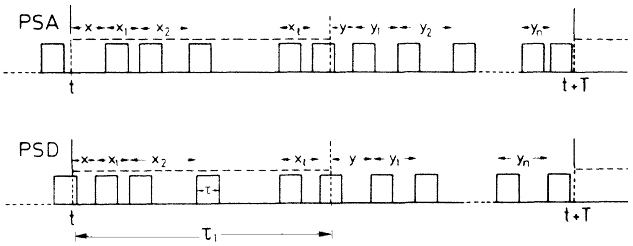

FIG. 1. Sketch of possible pulse sequences showing typical situation in which the pulser signal (dashed) is alive (PSA) or dead (PSD). For the sake of clarity, the PS (width τ_1) has been drawn higher than the random pulses (width τ).

that the random relative arrival times remain unchanged. However, from an experimental point of view any lengthened pulse will be detected by a scaler as a single count. So we shall calculate the total experimental rate under the previous conditions which can be done exactly.

With the obvious condition $T>\tau_1$, let us consider an interval T between two pulser signals (PS's) arriving at t and $t+T$, respectively (Fig. 1). The PS at t will be considered "alive" if it can be registered by a scaler: in this case no random pulse arrives in an interval τ before t. Otherwise the PS will be "dead," i.e., it has prolonged a previous random pulse, and cannot be registered. We discuss separately these two cases: pulser signal alive (PSA) and pulser signal dead (PSD).

A. PSA

As previously outlined, in this case the PS arriving at t is alive; this means that (i) there is no random pulse between $t-\tau$ and t and (ii) in the interval T the counts registered will be due to the random pulses which do not overlap the PS plus one single count due to the pulser signal.

We shall calculate now the mean counting rate. This will be done in the following three steps. The first step is the determination of the probability density of the residual waiting time y of the first random pulse not overlapping the end τ_1 of the PS. This in turn requires, of course, the knowledge of the random sequence within τ_1. The second step shall be the convolution of this density with the density of the successive random sequence up to $t+T$; the third is the integration of the final density between 0 and $T-\tau_1$ and the consequent sum to get the mean value.

We explain now in more detail the previous points. The residual waiting-time density for the first random pulse after T is[11,14]

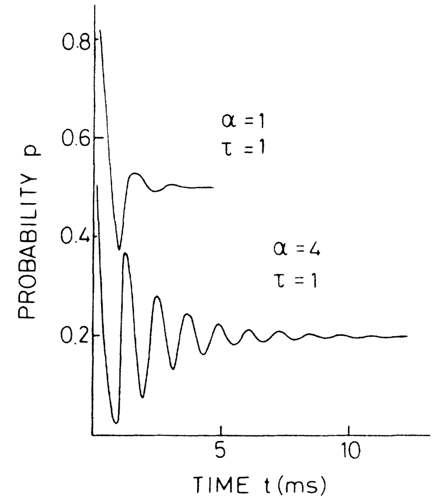

FIG. 2. Plot of the probability p that a PS arriving at time t will be alive if the random sequence is switched on at the instant $t=0$. As can be seen the stationary state is reached faster when the rate α is smaller. Its asymptotic value is $(1+\alpha\tau)^{-1}$. Values of the rate α are in ms^{-1}, of the width τ in ms.

$$\rho_{\mathrm{PSA}}(x)=e^{-\alpha x}\sum_{m=0}^{M} g_{m+1}(t-m\tau)\,,$$

M the integral part of t/τ ,

i.e., the product of the probability

$$p=\frac{1}{\alpha}\sum_{m=0}^{M} g_{m+1}(t-m\tau)$$

that a PS be alive times the density $\alpha e^{-\alpha x}$ of the following random pulse.

If l pulses of width τ enter the PS time width τ_1, the density of the variable[11,14] $z=x+x_1+\cdots+x_l$ is

$$f_{l+1}^{\mathrm{PSA}}(z)=p g_{l+1}(z-l\tau),\quad z\geq l\tau,\quad 0\leq l\leq L\,,$$

L the integral part of τ_1/τ .

We call this sequence of pulses "sequence z." The probability distribution of the residual waiting time y can be now evaluated:

$$P(W_t\leq y)=p\int_{\tau_1}^{\tau_1+y} g_1(s)ds+p\sum_{l=0}^{L}\int_{l\tau}^{\tau_1-\tau} g_{l+1}(z-l\tau)dz\int_{\tau_1-\tau-z}^{\tau_1-\tau-z+y} g_1(s)ds$$
$$+p\sum_{l=0}^{L}\int_{\tau_1-\tau}^{\tau_1-\tau+y^*} g_{l+1}(z-l\tau)dz\int_{0}^{\tau_1-\tau-z+y} g_1(s)ds,\quad y^*=\begin{cases}\tau & y\geq\tau\\ y & y<\tau\end{cases} \tag{1}$$

where the first integral represents the contribution with no random pulse within τ_1, the second integral is the contribution due to pulses of the sequence z completely inside τ_1, and the third is the contribution with a pulse of the sequence z crossing the end of the PS.

The derivative with respect to y of the integrals (1) gives the probability density of y which, convoluted with (see Fig. 1)

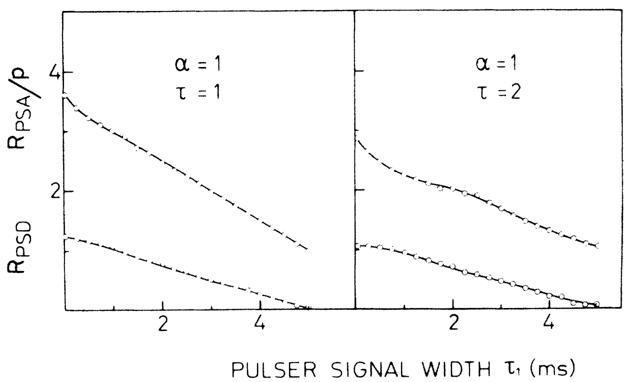

FIG. 3. Upper curves: plot of the theoretical mean counting rate in the stationary state in an interval $T = 5$ ms, divided by the parameter p (see Table I), i.e., R_{PSA}/p [pulser signal alive (PSA)], together with the experimental points for the indicated values of the parameters. Values of the rate α in ms^{-1}, of the width in ms. Lower curves: plot, as before of R_{PSD} (PS dead) against τ_1.

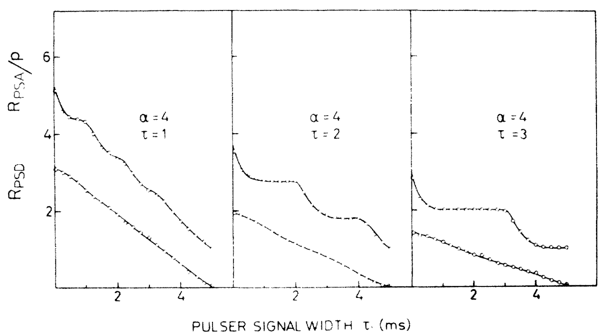

FIG. 4. Same as Fig. 3 for the indicated values of the parameters.

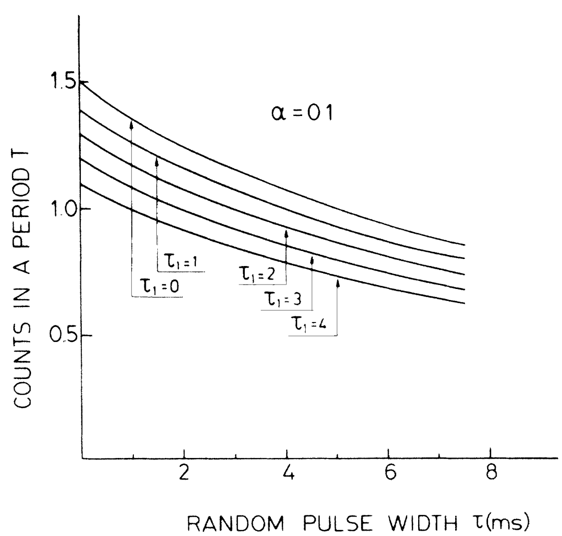

FIG. 5. Plot of the calculated total rate $R = R_{\mathrm{PSA}} + R_{\mathrm{PSD}}$, for an interval $T = 5$ ms, versus τ for different values of the pulser signal width τ_1 (ms); rate in ms^{-1}.

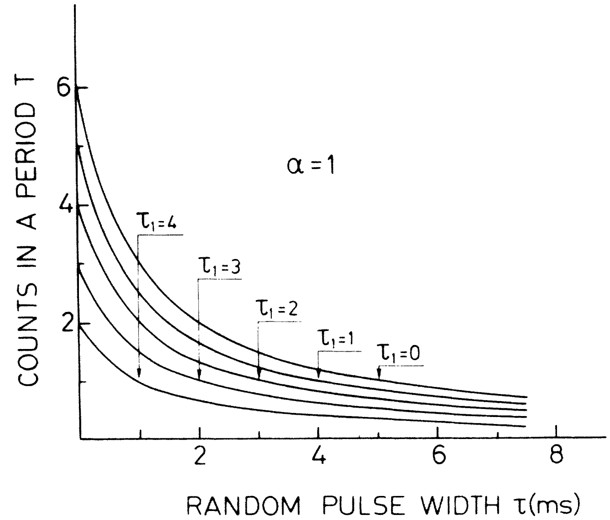

FIG. 6. Same as Fig. 5.

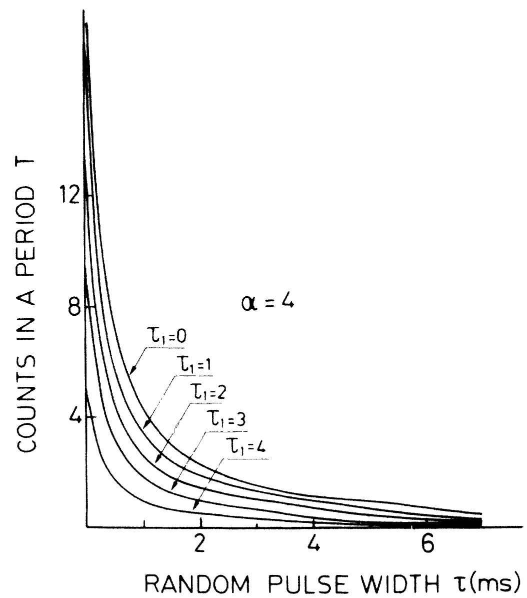

FIG. 7. Same as Fig. 5.

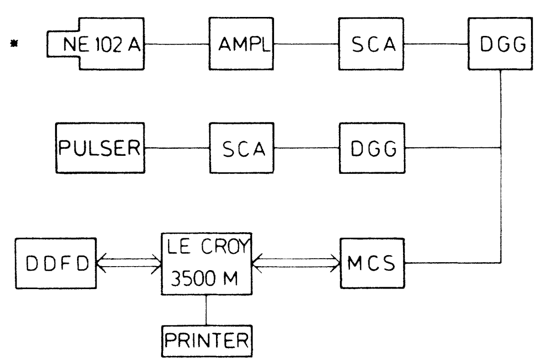

FIG. 8. Block diagram of the experimental setup: S, ^{22}Na source; AMPL, Canberra amplifier 2035 A; DGG, dual gate and delay generator Le Croy 222; MCS, Le Croy 3521 multichannel scaler; DDFD, dual-drive floppy disk computer.

TABLE I. Theoretical and experimental values of the parameter p, in the stationary state, for the indicated rate α (ms^{-1}) and width τ (ms).

	$\alpha=1$ $\tau=1$	$\alpha=1$ $\tau=2$	$\alpha=4$ $\tau=1$	$\alpha=4$ $\tau=2$	$\alpha=4$ $\tau=3$
p_{theor}	0.50	0.33	0.20	0.11	0.077
p_{expt}	0.50 ± 0.002	0.33 ± 0.002	0.20 ± 0.001	0.11 ± 0.001	0.076 ± 0.001

$$g_n(w-n\tau),\quad w=y_1+y_2+\cdots+y_n$$

integrated in the interval 0 to $T-\tau_1$ and averaged, gives after easy calculation the following mean rate in the interval T:

$$R_{\text{PSA}}=p(e^{-\alpha T}+\phi)+p\sum_{n=1}^{N+1}\frac{n+1}{\alpha}F_n(b_{n-1})\sum_{l=1}^{L+1}g_l(a_{l-1})$$
$$+p\sum_{n=1}^{N+1}(n+1)\left[F_{n+L+1}(a_L+b_n)+\frac{1}{\alpha}\sum_{l=1}^{L+1}\sum_{v=1}^{l}[g_{l-v+1}(a_l)F_{n+v}(b_{n-1})-g_{l-v+1}(a_{l-1})F_{n+v}(b_n)]\right],\qquad(2)$$

where

$$a_l\equiv\tau_1-l\tau,\quad b_n\equiv T-\tau_1-n\tau,\quad \phi=\begin{cases}\dfrac{e^{-\alpha b_1}}{\alpha}\displaystyle\sum_{l=1}^{L+1}g_{l+1}(a_{l-1}) & \text{if } T\geq\tau_1+\tau\\ \displaystyle\sum_{l=1}^{L+1}[G_l(a_{l-1})-G_{l+1}(a_l+b_0)] & \text{if } T<\tau_1+\tau\end{cases}$$

with

N the integral part of $(T-\tau_1+\tau)/\tau$,

L the integral part of τ_1/τ,

$F_n(s)=G_n(s)-G_{n+1}(s-\tau)$,

and

$$G_n(s)=1-e^{-\alpha s}\left[1+\frac{\alpha s}{1!}+\cdots+\frac{(\alpha s)^{n-1}}{(n-1)!}\right]$$

the modified incomplete Γ function.[15] $G_n(s)\equiv0$ if $s<0$.

$$g_{k+1}(x)=\alpha^{k+1}\frac{x^k}{k!}e^{-\alpha x},\quad x\geq0,\quad k\geq0$$

$$g_{k+1}(x)\equiv0\quad\text{if}\quad x<0\,.$$

The previous formula (2) gives the mean number of counts (PS included) in the interval t to $t+T$. It is valid also for $\tau_1<\tau$ and does depend on t through the factor p. If the process is old enough, i.e., in the stationary state, the factor p has already reached its saturation value.[16]

In Fig. 2 we show a plot of p against t to give an idea of the dependence of this saturation value on the rate α.

B. PSD

In this case the PS at t is dead, i.e., it cannot be counted because of a random pulse arriving between $t-\tau$ and t, which covers the leading edge of the PS constituting its long tail. On the other hand no contribution of course can be due to the sequence z falling within τ_1, the only counts detectable being those of the random pulses (not overlapping τ_1) which fall between $t+\tau_1$ and $t+T$.

We shall distinguish the following two subcases: (i) $\tau_1\leq\tau$, (ii) $\tau_1>\tau$.

In the first one, (i) $\tau_1\leq\tau$, a procedure analogous to that of the previous section gives, in the interval T, the following rate:

$$R_{\text{PSD}}^{\leq}=\sum_{n=1}^{N+1}\left[\sum_{m=1}^{M+1}\frac{n}{\alpha}\left[[g_{m+1}(t_m+\tau_1)-e^{-\alpha\tau_1}g_{m+1}(t_m)][F_n(b_{n-1})-F_{n+1}(b_n)]\right.\right.$$
$$+\sum_{v=1}^{m}[g_{m-v+1}(t_m+\tau_1)F_{n+v}(b_{n-1})-g_{m-v+1}(t_{m-1})F_{n+v}(b_n+\tau_1)]$$
$$\left.+\sum_{v=1}^{m}[g_{m-v+1}(t_m)F_{n+v+1}(b_n+\tau_1)-g_{m-v+1}(t_m+\tau_1)F_{n+v+1}(b_n)]\right]$$
$$\left.+nF_{n+M+1}(b_n+t_M+\tau_1)\overline{U}(t_{M+1}+\tau_1)+nF_{n+M+2}(b_n+t_{M+1}+\tau_1)U(t_{M+1}+\tau_1)\right],$$

where

$$t_m \equiv t - m\tau ,$$

$$U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

$$\bar{U}(x) = 1 - U(x) .$$

(ii) $\tau_1 > \tau$. Following the same three-step procedure of Sec. II A we get for the mean rate in the time T:

$$\begin{aligned}
R^{>}_{\rm PSD} = & \sum_{n=1}^{N+1} \sum_{m=1}^{M+1} \sum_{l=1}^{L+1} \sum_{v=1}^{m} \frac{n}{\alpha^2} [g_{m-v+1}(t_m) g_{l+v}(a_{l-1}) - g_{m-v+1}(t_{m-1}) g_{l+v}(a_l)] F_n(b_{n-1}) \\
& + \sum_{n,m,l,v} \sum_{i=1}^{l+v} \frac{n}{\alpha^2} [g_{m-v+1}(t_m) g_{l+v-i+1}(a_l) F_{n+i}(b_{n-1}) \\
& \qquad - g_{m-v+1}(t_m) g_{l+v-i+1}(a_{l-1}) F_{n+i}(b_n) \\
& \qquad - g_{m-v+1}(t_{m-1}) g_{l+v-i+1}(a_{l+1}) F_{n+i}(b_{n-1}) \\
& \qquad + g_{m-v+1}(t_{m-1}) g_{l+v-i+1}(a_l) F_{n+i}(b_n)] \\
& + \sum_{n,l} \frac{n}{\alpha} \left\{ g_{l+M+1}(t_{M+l} + \tau_1) F_n(b_{n-1}) + \sum_{i=1}^{l+M+1} [g_{l+M-i+2}(t_{M+1} + a_l) F_{n+i}(b_{n-1}) \right. \\
& \qquad \left. - g_{l+M-i+2}(t_M + a_l) F_{n+i}(b_n)] \right\} \\
& + \sum_{n,m,v} \frac{n}{\alpha} [g_{m-v+1}(t_m) F_{n+v+L+1}(a_L + b_n) - g_{m-v+1}(t_{m-1}) F_{n+v+L}(a_L + b_n)] \\
& + \sum_{n} n [F_{L+M+n+1}(t_M + a_L + b_n) \bar{U}(t_M + a_{L+1}) + F_{L+M+n+2}(t_M + a_{L+1} + b_n) U(t_M + a_{L+1})]
\end{aligned}$$

with the previous meaning of symbols.

Notwithstanding the apparent complexity, the final result can be easily calculated in any computer, being a simple combination of g functions.

We observe that, in this second case, the final formula depends on t in a more complicated form than before. However the final expression can be computed in the stationary state, which from an experimental point of view is the most meaningful one. In Figs. 3 and 4 are shown the plots of the contributions $R_{\rm PSA}/p$ and $R_{\rm PSD}$ against τ_1 for some values of the parameters α, τ. In Table I values for the parameter p are reported for the stationary state. Finally we show in Figs. 5–7 the total rate in the interval T, i.e., $R = R_{\rm PSA} + R_{\rm PSD}$ against τ for some values of α and τ_1.

III. EXPERIMENTAL

In order to check the previous calculations we have performed several measurements with the apparatus shown in Fig. 8. We have been able to verify separately the PSA contribution and PSD using a new experimental method described in Ref. 16. We recall briefly the principle of operation of such a method. Pulses from a photomultiplier coupled with a plastic scintillator detecting the γ radiation of a ^{22}Na source are sent through a single-channel analyzer (Canberra 2035A) to a dual-gate generator (DGG) (Le Croy 422) which introduces the required width τ (dead time of nonextended type). Another DGG processes the periodic pulses of a pulser set at a given period T, introducing a dead time $\tau_1 < T$. Afterwards the two sequences of pulses are mixed together and the final sequence feeds a Camac multichannel scaler (MCS) (Le Croy 3521) with a presettable dwell time per channel as short as 1 μs, interfaced with a programmable Le Croy system 3500 M. The MCS provided a 100-MHz input rate capability of less than 5 ns interchannel dead time and its dwell time was controlled by the computer of the Le Croy 3500 M system. Facilities for data acquisition, display, storage, analysis by software and input and output allowed automatic data handling. To discriminate the two contributions, PSA and PSD, obtained theoretically, it was enough to choose the dwell time d of the MCS in such a way as to satisfy the relations $\alpha d \ll 1$, $T = kd$, $\tau = k'd$ with k, k' integers. Under the previous conditions the MCS can accumulate only one count into the channel corresponding to the leading edge of a pulse and zero in the other channels covered by the pulse length. Identifying first, with the pulser alone, the MCS channels corresponding to any PS we are able to determine by software if, in the experimental spectra, a pulser signal in the mixed sequence will remain surviving or if it will be covered by a preceding pulse of the random sequence.

Automatic analysis of the spectra by the computer of Le Croy 3500 M allowed the separations of the intervals and the counting of the experimental rates.

In Figs. 3 and 4 are shown the experimental results obtained for a few values of the parameters α and τ against τ_1. Values of the parameter p are reported in Table I together with the experimental data.

The present method of superposition can be applied to dead-time measurements. In fact looking at the nomograms shown in Figs. 5–7 where the total rate of counting is plotted against τ, it is clear how a precision measurement of dead time can be made. It is enough to mix a random sequence of known rate α processed by an apparatus of unknown dead time τ with periodic pulses of given frequency and width. The final total rate measured with a scaler will give the required τ by the corresponding crossing with the proper curve. Several checks can be done, of course, by varying for instance α and/or τ_1. The excellent agreement between theory and experiment supports the reliability of the present method and also affords direct evidence of the distorted Poisson distribution.

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