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Randomness of a true coin toss

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A discussion is given of what it means for a coin toss to be random. To aid in that discussion, we produce and solve numerically a physically realistic model of such a toss. The ideas we develop should apply in a general way to other commonly used mechanical randomizers. The coin's randomness is determined by the nature of the basins of attraction of heads and tails. Although mechanical systems are known which are intrinsically random, we conclude that the coin flip is not among them. Rather, the effective randomness in practice is based on the magnitude of the scale of the variation of the basins of attraction *relative* to the precision of the flipping mechanism.

I. INTRODUCTION

It is commonly taken for granted, even by sophisticated writers,[1–3] that the toss of a "true coin" is "random." The technical and mathematical definitions of "randomness" are of great interest. The explication of this subject is in fact the aim of the semipopular[1,2] expositions referred to above. In contrast, the definition and analysis of the term true coin are completely lacking. In fact, we know of just one paper, so far unpublished,[4] which has any relevance to the subject. Our main conclusion is that true coins have *no intrinsic randomness* and that what is relevant is the relationship of the (initial) parameters of the coin to the precision of the coin tosser. Further, there is nothing *critical* about this relationship, so that the toss of a coin, by a continuous, realizable change of parameters, could become as obviously nonrandom as switching on or off a light.

Since no one has defined or even discussed what is meant by a true coin, we shall make several remarks. If the term is to be more than a tautology, it will have to mean a system (or class of systems) whose generic features are well approximated by physical coins. We cannot expect the detailed characteristics to be fundamental, since many sizes and shapes of physical coins can be tossed in a variety of conditions and all will be for practical purposes random. Indeed the motion of other common mechanical "randomizers," such as dice, roulette wheels, spins of the tennis racket, etc. will be subject to essentially the same analysis.

By "true" we mean that the coin itself is completely symmetric between heads and tails, in other words, that the *equations of motion* are independent of these labels. Further, we assume that the equations of motion are Newton's equations, with no external source of random fluctuations included. We thus assume that the fluctuations of the air, and the thermodynamic and/or quantum fluctuations of the coin and the surface on which it falls are absent or negligible. In any case, if these factors were essential for a coin toss to be random, it would be inaccurate to call the *coin* random. The equations of motion are in this approximation *completely deterministic.*

The equations of motion connect the *initial conditions* to a final observed *output configuration.* This mapping is, as a rule, strongly nonlinear. In the case of a coin (if it is treated as a rigid body) the initial conditions are the position, configuration, momentum, and angular momentum of the coin just after it leaves the "thumb" of the flipper and begins its fall onto the surface. There are three possible final configurations: the coin resting flat on the surface with its head side up, its tail side up, or the coin balanced on its edge. All initial configurations are mapped into one of these three final configurations. The first two final configurations, heads and tails, are *stable point attractors.*[5] Regions of initial condition space mapped onto these two attractors are called the *basins of attraction* (BA's) for the coin toss. The boundary which separates BA's for heads and tails consists of initial conditions mapped onto the coin standing on its edge. This boundary is a set of measure zero and thus with probability one the coin ends up either heads or tails (we assume an infinitely thin coin).

In order to illustrate the situation, in Sec. III we produce a definite model of a true coin toss. We consider it important to have a fairly realistic model, as it greatly aids in thinking clearly about the problem. The details of the model are not important as long as it has the generic properties of physical coin tosses, with the approximations just discussed. (Other models with different details give, numerically, the same generic results.)

It is very well known that deterministic nonlinear systems (DNS) are quite capable of displaying intrinsic randomness, so it is perhaps natural to think that systems describing coin flips have this property. We shall now argue that this is not the case.

Technical definitions of randomness can be reduced to statements about arbitrarily long sequences of zeros and ones. Such a sequence can be interpreted as a real number in the binary representation. [We exclude definitions of randomness (if any exist) which are able to distinguish between random and nonrandom finite sequences. Otherwise our discussion is independent of the precise technical definition of randomness.] It can also be regarded as the output of a dynamic process. For example, it could represent the sequence of heads and tails in the indefinitely repeated flip of a coin. Or it could represent an indefinitely long sequence of measurements on a dynamical process which does not stop. An essential aspect is that rigorous concepts of randomness do not apply to a finite sequence of results.

A DNS can be regarded as a map of an initial (real) value or values to an output consisting of a finite or infinite sequence of zeros and ones. This description assumes that the output consists of a sequence of coarse-grained results with each element of the sequence on an equal footing with all other elements. The input parameters, on the other hand, are as a rule approximate. Two cases can be distinguished. In one the output sequence is bounded in length, while in the other it is unbounded. Coin flips and all the other common mechanical randomizers are in the first category. Periodically repeated measurements on conservative systems or on driven systems provide examples of the second category.

Mechanical randomizers are in the first category because they conveniently give a single, definite and practically unpredictable outcome in a finite time. Physically this behavior is based on the fact that the system's energy is dissipated rapidly enough that it comes to rest (or at any rate has its final configuration obviously determined) in a finite time.

Those DNS possessing arbitrarily long output sequences have the capability of having a random output. If the output is random, even if the input initial condition is a finite set of real numbers, we may say that the DNS is *intrinsically random.* This case is characterized by the fact that the output sequence is extremely sensitive to the input value.

The Bernoulli shift is a trivial example of an intrinsically random DNS. It consists of a rule which rewrites the input real in binary notation and interprets the result as an unbounded sequence. An intrinsically random DNS based on Newton's equations can be constructed from Sinai's billiard, or a periodically kicked rotor. [Sinai's billiard moves without friction on a square elastic table with one fixed ball of finite radius at the center from which it can also scatter elastically. An elementary measurement gives the result one (zero) if the billiard is in the upper (lower) half of the table. Repeating this measurement periodically (with sufficiently long period) gives, for almost all initial conditions, a random sequence as output.]

Actually, even an intrinsically random DNS does not "create" randomness, but merely transforms one (assumed) random sequence (the initial real value) into a random output sequence. The terminology "intrinsic" is justified because in this case one cannot help but choose a real represented by a random sequence, if by "choose" one means to "determine physically the initial conditions." In fact, physical determination can at best set the initial conditions to some accuracy which we denote by ϵ. A given determination can then be regarded as picking a typical real from the region characterized by ϵ. Note that ϵ need not be, and generally is not, very precisely determined. We note that Wolfram[6] has suggested that appropriate definitions of randomness exist which allow DNS's to produce a random sequence from a finite (and therefore nonrandom) set of input integers. The correctness of his suggestion has no influence on our argument.

II. BASINS OF ATTRACTION

The appropriate concept to describe deterministic systems whose output is one of a finite set of possibilities is that of basin of attraction (BA). One should investigate BA's for a given system and classify randomness according to the degree of their interpenetration.[6] To make the discussion more concrete we give in Fig. 1 the basin of attraction for heads (tails) of the much oversimplified model coin first considered by Keller.[4] This model does not allow the coin to bounce and determines heads or tails by the angle of the coin at the instant it first touches the surface. The basin boundaries are the family of equidistant hyperbolas, $zE=(\pi n)^2/12$ where n is an integer corresponding to a vertical coin at the first touch. [Actually, Fig. 1 (and Fig. 2) fixes all initial variables but two, namely the initial potential energy and the initial rotational energy.] The dark regions are the set of initial conditions which lead to the result "heads," the light regions are "tails," and the mutual boundary of these regions is the unstable result of the coin standing on edge.

As illustrated in Fig. 1, the coin toss of Ref. 4 cannot be arbitrarily sensitive to initial conditions. Rather, any initial condition leading to heads will have a neighborhood of nearby initial conditions also leading to heads. If a coin is repeatedly tossed with initial parameters determined sufficiently accurately to lie in this neighborhood it will always give heads.

If a long sequence of coin tosses is to give a random result, therefore, it can only be because the initial conditions vary sufficiently from toss to toss. Clearly, they must also vary randomly. In other words, *it is crucial to specify how the coin is tossed.* Clearly a sequence of coin tosses will be random if the uncertainty ϵ of the initial conditions is large compared to the width W of the stripes characterizing the basins of attraction. (Of course, it must be *assumed* that an uncertainty in making the flip

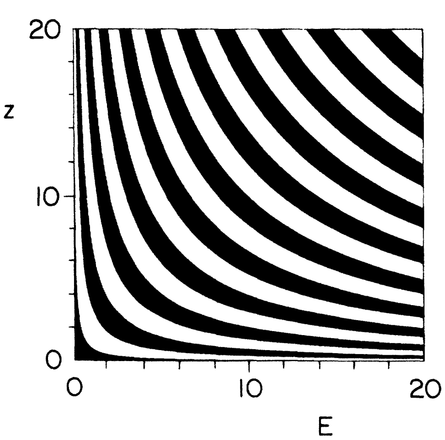

FIG. 1. Outcome of a simplified model coin toss of Ref. 4 as a function of the initial excess height z and the initial rotational energy E. (Dark regions correspond to heads.)

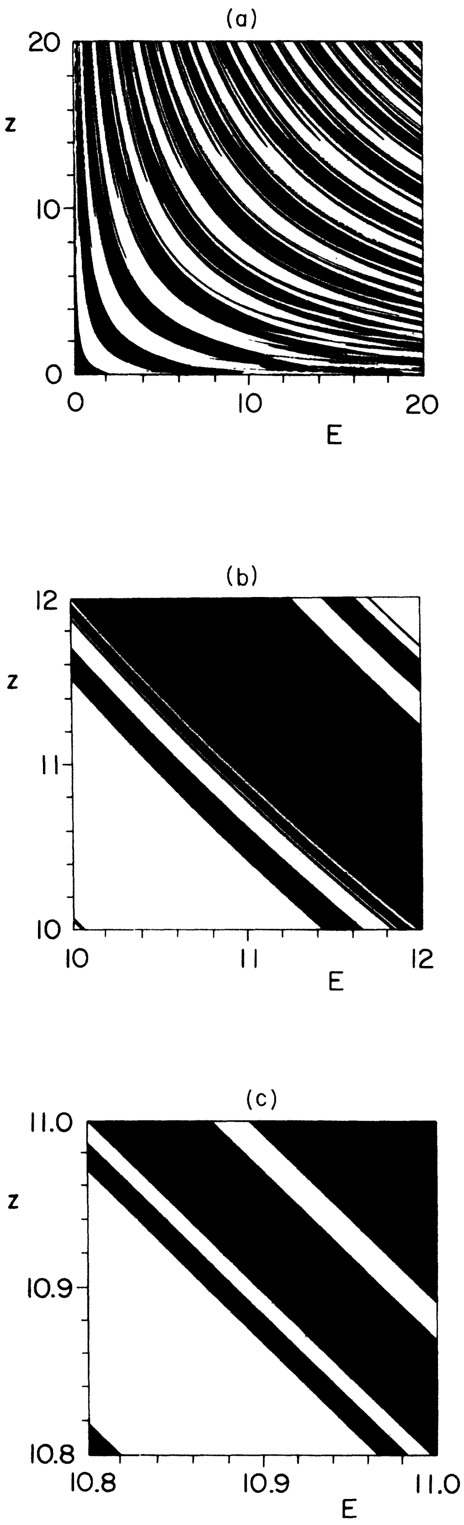

FIG. 2. (a) Same as Fig. 1 but for a more realistic model coin toss described in the text. Surface parameters are $\gamma=0.3$ and $\mu=1$. (b) Inset from (a), enlarged. Fuzzy boundaries and isolated points of (a) here reveal additional structure. (c) Inset from (b), enlarged. Apart from graininess due to the finite number of points shown, the boundaries are smooth. Under further magnification no new structure can be resolved.

picks out *at random* a set of initial values in a region whose size is characterized by ϵ). *This is the fundamental reason that a coin toss is perceived to be random.* Namely, the deliberate inability of the tosser to choose the initial conditions precisely enough so that the coin toss is statistically repeatable leads to the unpredictability and randomness of the result. That is, *the condition* $W \ll \epsilon$ *is essential* for a random output, where ϵ depends on the mechanism of flipping, while W is intrinsic to the coin.

This conclusion is disappointingly trivial and may indeed have been the conclusion reached by many, in times before modern ideas about random sequences became fashionable. However, it is worthwhile noting that by a continuous decrease in ϵ, which can be achieved without any change in the coin itself, the situation $W \gg \epsilon$ can be achieved. This is the condition for many devices which are meant to be deterministic, for example, keys on a calculator. The distance between keys is W, while ϵ is related to the diameter of the fingertip. A wristwatch calculator is an example of a device which is nearly on the borderline between mechanically deterministic and mechanically random because $W \gtrsim \epsilon$.

However, Fig. 1 depicts a much simpler BA than that of more realistic models. BA's for a more realistic model (Sec. III) which allows for bouncing of the coin from the surface are shown in Fig. 2. While the details of our model toss are postponed for Sec. III, we notice here that in our model, and in any other realistic coin toss model, the outcomes heads and tails are the stable point attractors. Since stable attractors have open basins of attraction[5] it then follows that for a general choice of initial conditions there is a sufficient precision ϵ ($\epsilon \neq 0$) to ensure that all successive tosses give the same outcome. We expect, though, that actual coins with more degrees of freedom would have more complicated BA's than for the model of Fig. 2.

The BA of a finite dynamical process is the most complete intrinsic measure of what it means for the process to be random, or more precisely, how that process converts uncertainty in the initial conditions into uncertainty of the final result. It does not seem to be possible to give a meaningful classification of BA's. Nevertheless, it is of some interest to consider ways in which it could be said that a degree of intrinsic randomness can attach to the basins of attraction themselves.

More generally, the parameter W (and ϵ) must be regarded as a vector of parameters which varies with position in initial value space. Suppose that W decreases exponentially as some conveniently controlled parameter is varied; for example, it could decrease exponentially with initial total energy. Then the BA could be considered to have a sort of intrinsic randomness in somewhat the same sense that we say the weather is unpredictable, namely, any improvement in ϵ could be counteracted by raising the height a bit or by trying to predict weather farther into the future.

Even if a DNS is nonrandom, there is a possibility that basin boundaries separating BA's have complicated, in particular, fractal structure. Grebogi *et al.*[7] have produced and studied such systems. They show that final-state sensitivity depends on the fractal characteristics of the basin boundaries (BB). Their analysis gives a generic result for fixed BA by considering the limit $\epsilon \to 0$. This limit is not really applicable to the case of a system which is perceptually random, but rather consists of the error analysis for a system which is nearly predictable. Nevertheless, it is worth investigating the possibility that BB's of a coin toss are fractal. [Near a given BB, if initial conditions are taken with error ϵ, then a fraction $f(\epsilon)$ of initial points give uncertain outcome. In the limit $\epsilon \to 0$, $f(\epsilon) \sim \epsilon^{\alpha}$ where $\alpha < 1$ for fractal and $\alpha = 1$ for smooth boundary.]

Fractal BB's can be classified further as discontinuous (e.g., an uncountable sequence of disjoint stripes) or continuous (e.g., a snowflake structure). For our model coin toss we have numerically determined that the BB is smooth, although at low levels of resolution it appears as a disjoint fractal (see Fig. 2). We also argue on general grounds that the BB for coin tosses and other common randomizers is smooth.

It would be of interest to determine the deviations of the system from randomness as $W/\epsilon \to \infty$. We have not found a way to do this which did not depend very importantly on the form assumed for the uncertainty in the initial conditions, suggesting that the characteristics of the tossing mechanism are much more important than those of the coin. However, we speculate that if the uncertainty in initial conditions has the form of a Gaussian distribution of scale ϵ about the nominal initial condition, then the corrections to "true" randomness are small in proportion to $\exp[-(W/\epsilon)^2]$. This may explain why coins behave in practice as perfect randomizers.

III. DESCRIPTION OF THE MODEL

To illustrate and expand on the points just discussed we need a fairly realistic model, in particular one which has the important nonlinearities involved in describing the bouncing of the coin from the surface. As mentioned earlier, we regard the coin as a rigid body, falling in vacuum onto a perfectly flat surface. In order to further simplify calculations, our model coin has only one axis of rotation (one of the coin diameters). In other words our model coin cannot wobble. It is equivalent to a rod which can move in one vertical plane only (two-dimensional coin toss). While this two-dimensional (2D) coin toss cannot be compared meaningfully with experiments done on three-dimensional (3D) coins, it should be a good model for a physically realizable system (although we do not advocate trying to build such a system). In other words, we have every reason to expect that this model describes a system in the same class as coins and dice. We give some details of the model below in case someone wants to check our results, and because there are a few amusing features which were not obvious before doing the calculation.

We choose units such that mass of the coin, its radius, and the acceleration of gravity are all equal to 1. Gravity acts along the negative-z axis, the surface against which the coin collides is a plane perpendicular to the z axis and the coin being two dimensional can move only in the xz plane. Motion between the collisions is trivial and the only effort required is in finding the times of collisions. We also need to specify laws which govern the collision. Assuming that the collision is instantaneous we have

$$u_z' = u_z + a p_z + b p_x ,$$

$$u_x' = u_x + b p_z + c p_x ,$$

where $\mathbf{u}$ and $\mathbf{u}'$ are endpoint velocities before and after the application of momentum transfer $\mathbf{p}$, $a = 1 + \sin(\theta)^2/I$, $b = \sin(\theta)\cos(\theta)/I$, $c = 1 + \cos(\theta)^2/I$, I is the moment of inertia, and the angle θ gives the orientation of a coin at the collision. θ is measured from the z axis so that $\theta = \pi/2$ corresponds to a coin parallel to the floor with heads facing up, and a counterclockwise rotation around the y axis corresponds to a positive change in θ.

We assume an inelastic surface with friction described by the inelasticity parameter γ $(0 \le \gamma < 1)$ and the coefficient of friction μ $(\mu > 0)$. Following Whittaker[8] and Roth[9] we choose $u_z' = -\gamma u_z$ and $p_x = -\mathrm{sgn}(u_x)\mu p_z$ if that leads to u_x' such that $u_x' u_x \ge 0$. In case this gives $u_x' u_x < 0$ we separate the collision into one or two parts. In the first part there is a transfer of momentum $\mathbf{p}^{(1)}$ such that $p_x^{(1)} = -\mathrm{sgn}(u_x)\mu p_z^{(1)}$ and the endpoint velocity at the end of the first part of collision, $\mathbf{u}^{(1)}$, satisfies $u_x^{(1)} = 0$. If $u_z^{(1)} \ge 0$ we assume the collision has ended. If $u_z^{(1)} < 0$ we assume that the second part of collision occurs, at the end of which endpoint velocity is specified by $u_x^{(2)} = 0$ and $u_z^{(2)} = -\gamma u_z^{(1)}$.

While the above definition of the collision is somewhat arbitrary, it satisfies the following physical requirements: energy after the collision is never greater than before the collision; p_z is positive; p_x acts as a friction (tends to decrease $|u_x|$); u_z' is positive.

IV. RESULTS OF NUMERICAL SIMULATIONS

Numerical simulations were done for $\gamma = 0.3$ and $\mu = 1$ and we have used $I = \frac{1}{3}$ characteristic for a uniform rod of mass 1 and length 2 (the 2D coin). In Fig. 2 we show BA in the (z,E) plane. $z+1$ is the initial height of the center of mass of the coin above the surface and the initial rotational energy is $E = 0.5 I \omega^2$. The initial velocity v and the initial angle θ are chosen to be zero and the initial angular velocity ω to be non-negative. The general pattern of Fig. 2(a) is clearly related to Fig. 1. Namely, at the energies under consideration, for most tosses the final outcome is determined by the first collision, i.e., in a typical toss the total energy of the coin decreases below 1 after the first collision.

From Fig. 2(a) we see that there are large regions in the phase space where even significant changes in the initial conditions (z,E) do not change the outcome. It cannot be seen from Fig. 2 but one expects and we have checked that the wide stripes do not suddenly disappear but continuously change positions as the initial angle θ is varied. This means that a two-dimensional section of the phase space shown in Fig. 2 is a good indication of what happens in the entire phase space.

To check for the possibility that the basin boundary separating heads from tails is fractal, in Fig. 2(b) we show an enlargement of a typical boundary region of Fig. 2(a).

It is seen that the boundary region of Fig. 2(a) consists of many little stripes. However, in a further enlargement of Fig. 2(b) shown in Fig. 2(c), no new satellite stripes are found. This shows that our model coin toss has the smooth BB. Below, we connect this result to the concept of significant collision and argue that there are only a finite number of stripes in a given energy region because there can be only a finite number significant collisions of the coin with the surface.

Even for DNS which have only stable point attractors and have smooth BB's there is a further meaningful description based on the (average) final-state sensitivity when some initial parameters of the system are being varied. For an arbitrary point in a BA there exists a maximal error in the initial data which still does not change the outcome. We may analyze how this maximal allowed uncertainty U (and its relevant moments) varies with initial data. The average uncertainty $\overline{U}$ is related to the characteristic size of the basins of attraction. While the model coin toss of Ref. 4 obviously has $\overline{U}=\text{const}$, in order to find $\overline{U}=\overline{U}(E)$ for our model toss we had to use a computer to determine positions of boundaries in the direction of increasing energy. We choose to count the number and positions of the boundaries along the main diagonal of Fig. 2(a).

The total number N of boundaries crossed, along the line segment from the origin $z=E=0$ to $z=E$, is shown in Fig. 3. An approximate exponential dependence of $N=N(E)$ is seen explicitly in the inset of Fig. 3 where $\ln(N)$ is shown versus E. For $15>E>3$ one has approximately $N=\exp(0.4+0.273E)$. However, the data for the highest energies probed $(18>E>15)$ suggest a steeper than exponential asymptotic $N=N(E)$ dependence.

The fact that the average width of the stripes decreases with the initial energy slightly faster than exponentially means that in some sense a coin toss is random. However, things are much more complicated. The point is that the distribution of stripe widths is very broad. There are wide stripes whose width decreases on average very slowly with energy. The precision tosser would have an easy task achieving a certain outcome if aimed for the center of one of these regions. On the other hand, there are fractal-like regions, located in the vicinity of tosses where a coin at the first collision lands perpendicularly to the surface, where the stripe width decreases with energy *faster* than exponentially.

Figure 4 is a semilogarithmic (base 10) plot of stripes' widths as a function of E at the stripes' right boundaries. The fact that for larger energies $(E\geq 10)$ the lower envelope of points on Fig. 4 decreases somewhat faster than linearly suggests that the width of the narrowest stripe decreases with energy faster than exponentially.

We were not able to fit the decrease of the width of the widest stripe to either a power law or an exponential dependence with energy. The absence of a simple fit is caused by the toungelike domains, protruding from high-z regions, which first cross the $z=E$ line around $z=14.1$ [see Fig. 2(a)].

The abundance of boundaries in a given (small) energy region is determined by the maximal number of significant collisions for a toss in that region. We define N_s, the number of significant collisions for a given toss, as the number of collisions after which the coins orientation changes quadrant. This is based on the expectation that flat and perpendicular collisions behave as branching points for the further evolution, and the observed fact that repeated consecutive collisions in the same quadrant cost little energy and do not affect future evolution so drastically.

In Fig. 5 we plot N_s corresponding to different boundaries as a function of E. We notice that for larger energies $(E\geq 10)$ the upper envelope of points in Fig. 5 increases slightly faster than linearly with energy. These points correspond to tosses at the very narrowest stripes—and for such tosses, except for the first collision when the coin loses a significant portion of its energy, in

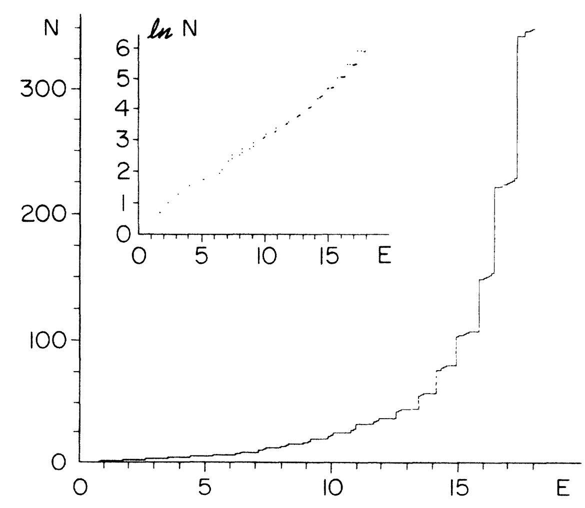

FIG. 3. N, the number of boundaries crossed along the main diagonal of Fig. 2(a), from the origin to the point with rotational energy E. An approximate exponential dependence of N on E in the energy interval $3<E<15$ is seen clearly in the inset.

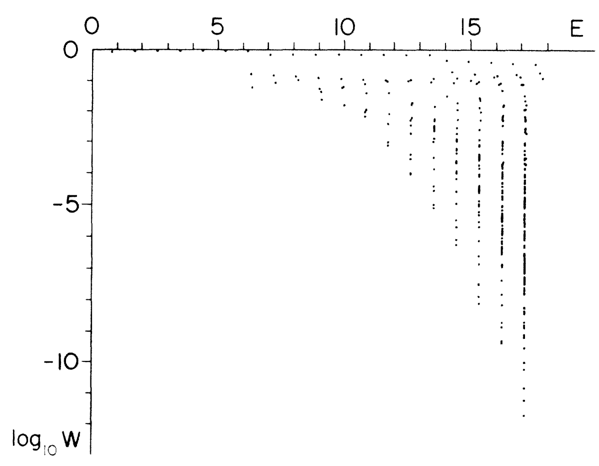

FIG. 4. A semilogarithmic plot of stripe widths W along the main diagonal of Fig. 2(a), as a function of the rotational energy E at the stripes' right boundaries.

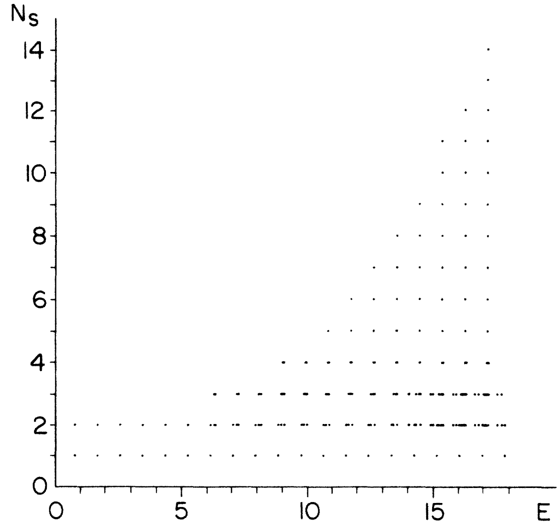

FIG. 5. N_s, the number of significant collisions for tosses at different boundaries, along the main diagonal of Fig. 2(a), as a function of rotational energy E at a given boundary.

each of the subsequent collisions energy decreases by a small, though weakly increasing amount.

We have also investigated how N_a, the number of significant collisions per toss, averaged over 100 equally spaced initial angles between 0° and 180°, changes as a function of energy. From Fig. 6 we see that N_a increases logarithmically with E, meaning that in a typical toss, at each significant collision, the coin on average loses a fixed fraction of its energy.

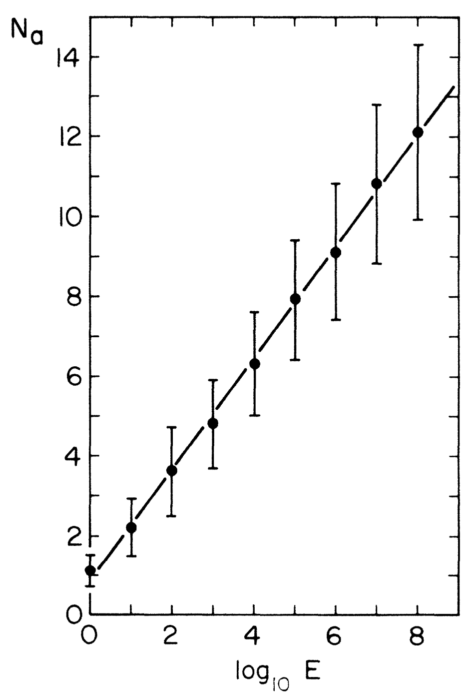

FIG. 6. N_a, the number of significant collisions per toss, averaged over 100 equally spaced initial angles between 0° and 180°, as a function of $z=E$. Error bars are equal to two standard deviations of the N_s distribution.

V. ANALYSES OF THE RESULTS

In this section we describe the critical point which characterizes the stripes' boundaries and discuss some additional approximations used to handle consecutive soft collisions.

From numerical solutions one finds that between successive collisions, in tosses with the greatest number of significant bounces, the angle changes by almost exactly 180°. This requires ωv_z to be nearly constant, $\mp\pi/2$, which for large total energy means that $|v_z|$ is small and $|\omega|$ large, or vice versa. We always find that $|\omega|$ is large and nearly equal to $|v_x|$ so that u_x is very small and $u_x'=0$. Collisions are such that the coin hits the surface nearly upright and $|\omega|$ decreases a bit.

For each boundary found in numerical simulations, tosses very near the boundary involve very many soft collisions, all in the same quadrant, with the effect of orienting a coin closer to being orthogonal to the surface. A collision is called soft if the momentum transfer during the collision is very small. Such a collision is followed by many other soft collisions, occurring in exponentially decreasing time intervals ("chattering collisions").

After very many soft collisions, a stage is reached, however, where either the total energy of a coin is decreased below 1 and one can stop the execution, or the upper bound on the error for u_z' exceeds u_z'. In the second case we usually already have $u_x'=0$ so we set u_z' to zero and proceed through a rotation with $u_z=u_x=0$ fixed. In very few cases would one have nonzero u_x'. In these cases we assume that the surface first instantly supplies the necessary momentum to set u_x' to zero and then again proceed with a rotation with $u_z=u_x=0$ fixed. Once a coin is rotating, for tosses near the boundary it does so so as to increase its height and then eventually flips over and falls flat on the other side. We have made sure that in all such cases the lower bound on energy (error estimation) is larger than 1 so that the flip is really possible.

Our simulations reveal that the critical point which determines whether the final outcome will be heads or tails consists of many chattering collisions all of which have the effect of orienting the coin closer to being perpendicular to the surface. Tosses where such collisions never make the coin stand up on its edge form one side of a given boundary, while in tosses on the other side of a boundary, after many chattering collisions the coin starts rotating and finally flips over.

The *ad hoc* approximation which we used to treat the very few cases of nonzero u_x' is expected at most to slightly displace those few boundaries. One may also object to the rule which keeps $u_z=u_x=0$ fixed during the rotation since in general this may require a force **F** supplied by the surface such that $|F_x|>|F_z|$ or $F_z<0$. However, this does not happen for rotations entailed in tosses close to the basin boundaries. These rotations are characterized by $\omega^2 \ll 1$, $|\sin(\theta)| \ll 1$ so that $F_z \cong 1$ and $|F_x| \ll 1$.

The error estimation is an essential part of numerical simulation for coin tosses used to generate Figs. 2–5, and while we have made sure that the outcomes of all tosses used to generate these figures are certain, Fig. 6 is an exception, however. Almost all tosses used to generate this figure occur at enormous initial energies so that when finally the total energy of a coin decreases below 1 the upper bound on the error in the orientation of a coin is so large that one cannot tell if a coin is heads or tails. Fortunately, to find N_a it is not crucial to know whether a particular toss ends up heads or tails. Rather, we assume that numerical uncertainties in the orientations of a coin are uncorrelated and equivalent to averaging over different nearby initial conditions of a coin. Since N_a is by definition the average of N_s this additional numerical "self-averaging" should not change its value.

We have also analyzed a coin toss of frictionless surface ($p_x = 0$) with a collision law $u_z' = -\gamma u_z$ and demonstrated that it is a predictable process.[10] There again a long sequence of chattering collisions represents a critical point, and on one side of each boundary is followed by sliding (instead of rotation) which finally turns the coin over. The quantitative numerical analysis has been done only for the surface with friction, however.

It would be interesting to consider other classical dissipative systems, and in particular determine the distribution of the sizes of the BA's and its dependence on total energy.

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