Functional-calculus approach to stochastic differential equations

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The connection between stochastic differential equations and associated Fokker-Planck equations is elucidated by the full functional calculus. One-variable equations with either additive or multiplicative noise are considered. The central focus is on approximate Fokker-Planck equations which describe the consequences of using "colored" noise, which has an exponential correlation function and a correlation time τ. To leading order in τ, the functional-calculus approach generalizes the τ-expansion result and produces an approximate Fokker-Planck equation free from certain difficulties which have plagued the less general approximations. Mean first-passage-time behavior for bistable potentials, an additive case, is discussed in detail. The new result presented here leads to a mean first-passage-time formula in quantitative agreement with the results of numerical simulation and in contrast with earlier theoretical conclusions. The theory provides new results for the multiplicative case as well.

INTRODUCTION

The functional derivative and the path integral are the basic tools of functional calculus.[1,2] With them, the connection between a stochastic differential equation and its associated Fokker-Planck equation can be made transparent. When a stochastic process incorporates "colored" noise, i.e., noise with an exponential correlation function instead of the Dirac-δ-function correlation of white noise, the process is no longer Markovian. Consequently, it no longer is characterized by a Fokker-Planck equation. However, to leading order in the correlation time τ of the colored noise, an approximate Fokker-Planck equation can be found.[3-5] This τ-expansion result has been achieved using the Furutsu-Novikov method and the cumulant method. The Furutsu-Novikov[6,7] method utilizes the functional derivative but does not involve the path integral.[8] In this paper, the full functional calculus is used to derive a generalization of the τ-expansion result.[5] This generalization removes certain technical difficulties found in the τ-expansion method.[9] Moreover, it solves a recent conundrum which arose in the study of mean first-passage times for bistable potentials.[10]

In this paper attention is restricted to stochastic differential equations in only one variable, x:

$$\frac{d}{dt}x = W(x) + g(x)f(t) , \qquad (1)$$

in which $W(x)$ and $g(x)$ may be nonlinear functions of x. When $g(x) = 1$, the process is "additive"; otherwise it is "multiplicative." The noise function $f(t)$ is assumed to be Gaussian, and it may be either "colored" or "white." In the case of white noise, Eq. (1) is to be interpreted in the sense of Stratonovich.[11,12] This interpretation is the one which generalizes unchanged to the colored-noise case. The Gaussian character of $f(t)$ is expressed in the functional calculus[7] by a probability distribution functional

$$P[f] = N \exp\left[-\tfrac{1}{2} \int ds \int ds' f(s) f'(s') K(s-s') \right] , \qquad (2)$$

in which K is the inverse of the f correlation function, and N is the normalization which is expressed by a path integral over f:

$$N^{-1} = \int\!\!\int \mathscr{D}f \exp\left[-\tfrac{1}{2} \int ds \int ds' f(s) f(s') K(s-s') \right]. \qquad (3)$$

The path integral is also used to define the probability distribution functional for $x(t)$, the solution to Eq. (1). This quantity is

$$P(y,t) = \int\!\!\int \mathscr{D}f\, P[f]\delta(y - x(t)) . \qquad (4)$$

It is shown below that for white noise, $P(y,t)$ satisfies the Fokker-Planck equation,

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P] + \lambda \frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}g(y)P , \qquad (5)$$

for which $K(s-s') \equiv (1/2\lambda)\delta(s-s')$ in (2). In the colored-noise case, an approximate Fokker-Planck equation is derived,

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P]$$
$$+ D\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}\left[\frac{g(y)}{1-\tau W'(y)} - \frac{\tau W(y)g'(y)}{[1-\tau W'(y)]^2} \right]P , \qquad (6)$$

for which the correlation function for $f(t)$ is

$$\langle f(t)f(s)\rangle = \frac{D}{\tau}\exp\left[-\frac{|t-s|}{\tau} \right]$$

such that τ is the correlation time. The primed functions, W' and g', are the first derivatives of W and g, respec-

tively, with respect to y. When the τ-dependent denominators in (6) are expanded to leading order in τ, then the τ-expansion result is

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P]$$

$$+D\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}$$

$$\times[g(y)+\tau g(y)W'(y)-\tau W(y)g'(y)]P \ . \qquad (7)$$

The problem of a bistable potential models many different physical problems.[9,10] Its standard form is given by the potential $U(y)$ which satisfies $W(y)=-U'(y)$ and is explicitly given by

$$U(y)=-\frac{a}{2}y^2+\frac{b}{4}y^4 \ . \qquad (8)$$

It has symmetric minima at $y=\pm\sqrt{a/b}$, with an intervening local maximum at $y=0$ of relative height $a^2/4b$. The mean first-passage time for a transition from one minimum over the local maximum into the other minimum may be relatively easily measured in both numerical simulations and with analog computers. Analytically, its calculation requires the solution to an ordinary differential equation, which in one variable is always formally tractable.[12] For these reasons, the mean first-passage time for the quartic potential in (8) has been studied with each of these approaches. For the white-noise case, there is complete agreement among all approaches. For the colored-noise case, however, a tantalizing conundrum has arisen.[10]

In the case of bistability, the simplest model has $W(y)=ay-by^3$ and $g(y)=1$. The τ expansion yields the approximate Fokker-Planck equation, which follows from[5] (7):

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[(ay-by^3)P]$$

$$+D\frac{\partial^2}{\partial y^2}\{[1+\tau(a-3by^2)]P\} \ . \qquad (9)$$

Hanggi et al.[9] have computed the mean first-passage time T for this equation to leading order in τ and obtained

$$T = \frac{\pi}{\sqrt{2}a}\left[\frac{1+2a\tau}{1-a\tau}\right]^{1/2}\exp\left[\frac{a^2}{4bD}\right] \ , \qquad (10)$$

which exhibits a weak τ dependence and no τ dependence in the exponential factor. Their numerical simulations, however, show instead a very clear τ dependence in the exponential which is quantitatively given by $\exp[(a^2/4bD)(1+2a\tau)]$. Hanggi et al.[10] have made a proposal to explain this discrepancy. They used a procedure developed by Hanggi[13] using functional derivatives to arrive at the approximate Fokker-Planck equation given by

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[(ay-by^3)P]$$

$$+D\frac{1}{1+\tau(3b\langle y^2\rangle-a)}\frac{\partial^2}{\partial y^2}P \ , \qquad (11)$$

in which $\langle y^2\rangle$ in the denominator is the mean square of y with respect to $P(y,t)$. It is assumed that the system is in

a steady state, on the average, and $\langle y^2\rangle$ is replaced by its steady-state value, which is a/b. With these plausible assumptions, Eq. (11) yields a mean first-passage time T involving the exponential factor $\exp[(a^2/4bD)(1+2a\tau)]$. This leaves the two puzzles: (1) Why does the τ expansion fail? (2) What is the justification for Hanggi's ansatz which produces Eq. (11)?

The functional-calculus approach of this paper answers both of these questions. First of all, it shows that the τ-expansion Fokker-Planck equation (9) is a limiting expression for a stronger result, the analogue of Eq. (6) for the bistable potential

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[(ay-by^3)P]$$

$$+D\frac{\partial^2}{\partial y^2}\left[\left[\frac{1}{1-\tau(a-3by^2)}\right]P\right] \ . \qquad (12)$$

The similarity of this equation to Eq. (11) is obvious. The calculation of the mean first-passage time requires an integration of the steady-state solution of the Fokker-Planck equation.[12] Generally, such integrals are not analytically tractable, and either numerical integration or an approximate, steepest-descent method is employed. When the steepest-descent method is applied[9] to (12), the result is again (10). However, in this case it is possible to execute the required integral exactly, using parabolic cylinder functions. Doing so yields the expression

$$T = \frac{\pi}{a\sqrt{2}}\left[\frac{1+2a\tau}{1-a\tau}\right]\exp\left[\frac{a^2}{4bD}(1+2a\tau)\right] \ , \qquad (13)$$

which quantitatively matches the numerical solutions.

These results provide confidence in the functional-calculus approach to stochastic differential equations involving colored noise. The bistable-potential analysis corroborates the use of Eq. (6) for the additive-noise case. The multiplicative-noise implications of Eq. (6) will be tested by measurements of laser noise.

I. FUNCTIONAL CALCULUS FOR NOISE

In Eq. (1), $f(t)$ can be either white noise or colored noise. Consider the white-noise case first. The conditions on $f(t)$ are that it is Gaussian and has first and second moments given by

$$\langle f(t)\rangle=0 \ , \qquad (14)$$

$$\langle f(t)f(s)\rangle=2\lambda\delta(t-s) \ . \qquad (15)$$

The probability distribution functional is given by

$$P[f]=N\exp\left[-K\int ds\,f^2(s)\right] \ , \qquad (16)$$

where

$$N^{-1}=\int\!\!\int \mathscr{D}f\exp\left[-K\int ds\,f^2(s)\right] \ . \qquad (17)$$

Using functional differentiation, one obtains

$$\frac{\delta P[f]}{\delta f(t)}=\frac{\delta N}{\delta f(t)}\exp\left[-K\int ds\,f^2(s)\right]$$

$$-NKf(t)\exp\left[-K\int ds\,f^2(s)\right] \qquad (18)$$

and

$$\frac{\delta N}{\delta f(t)} = -N^2 \iint \mathscr{D}f[-Kf(t)]\exp\left[-K\int ds\, f^2(s)\right]$$

$$= NK\langle f(t)\rangle = 0 .$$ (19)

Therefore, one also obtains

$$\frac{\delta^2 P[f]}{\delta f(s)\delta f(t)} = -K\delta(t-s)P[f]+K^2 f(t)f(s)P[f] .$$ (20)

By using (15), this yields

$$0 = \iint \mathscr{D}f \frac{\delta^2 P[f]}{\delta f(s)\delta f(t)}$$

$$= -K\delta(t-s)+K^2\langle f(t)f(s)\rangle$$

$$= -K\delta(t-s)+K^2 2\lambda\delta(t-s) .$$ (21)

This implies

$$K = \frac{1}{2\lambda}$$ (22)

and that the $K(s-s')$ of (2) is indeed $(1/2\lambda)\delta(s-s')$. Combining (22), (19), and (18) produces

$$\frac{\delta P[f]}{\delta f(t)} = -\frac{1}{2\lambda}f(t)P[f]$$ (23)

which will prove useful later.

Now, consider the colored-noise case. The conditions on $f(t)$ are the same as above except for (15) which becomes

$$\langle f(t)f(s)\rangle = C(t-s) .$$ (24)

The probability distribution functional is given in (2). From (3) one obtains

$$\frac{\delta N}{\delta f(t)} = -N^2 \int\int \mathscr{D}f\left[-\tfrac{1}{2}\int ds'f(s')K(t-s')+\int ds\, f(s)K(s-t)\right]\exp\left[-\tfrac{1}{2}\int ds\int ds'f(s)f(s')K(s-s')\right]$$

$$= N\int ds\, K(t-s)\langle f(s)\rangle = 0 .$$ (25)

Therefore, it follows that

$$\frac{\delta P[f]}{\delta f(t)} = -\left[\int ds\, K(t-s)f(s)\right]P[f] .$$ (26)

Consequently,

$$\frac{\delta^2 P[f]}{\delta f(t')\delta f(t)} = \left[\int ds\int ds'K(t-s)K(t'-s')f(s)f(s')\right]P[f]-K(t-t')P[f]$$ (27)

and

$$0 = \iint \mathscr{D}f \frac{\delta^2 P[f]}{\delta f(t')\delta f(t)} = \int ds\int ds'K(t-s)K(t'-s')\langle f(s)f(s')\rangle - K(t-t')$$

$$= \int ds\int ds'K(t-s)K(t'-s')C(s-s')-K(t-t') .$$ (28)

This implies

$$\int ds\, K(t-s)C(s-s') = \delta(t-s') ,$$ (29)

i.e., the kernel $K(s-s')$ is the inverse of the correlation function $C(s-s')$. This fact makes (26) useful later.

II. ADDITIVE WHITE NOISE

Start with (1) and choose $g(x)=1$. Let $f(t)$ be determined by (16), (17), and (22). This is the additive—white-noise case. From (4), it follows that

$$\frac{\partial}{\partial t}P = \iint \mathscr{D}f P[f]\left[-\frac{\partial}{\partial y}\delta(y-x(t))\dot{x}\right] ,$$ (30)

in which $\dot{x}=(d/dt)x$, which is replaced by the right-hand side of (1). Therefore,

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}\iint \mathscr{D}f P[f]\delta(y-x(t))[W(x)+f(t)]$$

$$= -\frac{\partial}{\partial y}[W(y)P]-\frac{\partial}{\partial y}\iint \mathscr{D}f P[f]f(t)\delta(y-x(t)) .$$ (31)

Now, use (23) and employ functional integration by parts to obtain

$$\iint \mathscr{D}f P[f]f(t)\delta(y-x(t))$$

$$= -2\lambda \iint \mathscr{D}f \frac{\delta P[f]}{\delta f(t)}\delta(y-x(t))$$

$$= 2\lambda \iint \mathscr{D}f P[f]\frac{\delta}{\delta f(t)}\delta(y-x(t))$$

$$= -2\lambda \iint \mathscr{D}f P[f]\frac{\partial}{\partial y}\delta(y-x(t))\frac{\delta x(t)}{\delta f(t)} .$$ (32)

From (1), one observes that

$$\frac{d}{dt}\frac{\delta x(t)}{\delta f(t')}=W'(x)\frac{\delta x(t)}{\delta f(t')}+\delta(t-t') .$$ (33)

This equation possesses the unique solution

$$\frac{\delta x(t)}{\delta f(t')}=\int_0^t ds \exp\left[\int_s^t ds' W'(x(s'))\right]\delta(s-t')$$

$$=\Theta(t-t')\exp\left[\int_{t'}^t ds\, W'(x(s))\right] ,$$ (34)

in which $\Theta(t-t')$ is defined by

$$\Theta(t-t')=\begin{cases}1, & t>t'\\ \frac{1}{2}, & t=t'\\ 0, & t<t' .\end{cases}$$ (35)

The value for $t=t'$ is a consequence of the δ function in (34) and the coincidence of its t' argument with the s-integration limit, t. In (32) the $t=t'$ value is needed and the replacement

$$\frac{\delta x(t)}{\delta f(t)}=\frac{1}{2}$$ (36)

has been justified. This completes the analysis started in (32). Therefore,

$$\iint \mathscr{D}f\, P[f]f(t)\delta(y-x(t))=-\lambda\frac{\partial}{\partial y}P .$$ (37)

Inserting this into (31) yields the Fokker-Planck equation

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]+\lambda\frac{\partial^2}{\partial y^2}P .$$ (38)

III. MULTIPLICATIVE WHITE NOISE

This case is the preceding case with $g(x)\neq 1$ in (1). Instead of (31) one now obtains

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]$$

$$-\frac{\partial}{\partial y}g(y)\iint \mathscr{D}f\, P[f]f(t)\delta(y-x(t)) .$$ (39)

Once again (32) follows. However, (33) is no longer correct and in turn becomes

$$\frac{d}{dt}\frac{\delta x(t)}{\delta f(t')}=W'(x)\frac{\delta x(t)}{\delta f(t')}+g(x)\delta(t-t')$$

$$+g'(x)\frac{\delta x(t)}{\delta f(t')}f(t) .$$ (40)

The solution to this equation generalizes (34), and is

$$\frac{\delta x(t)}{\delta f(t')}=\int_0^t ds \exp\left[\int_s^t ds'[W'(x(s'))+g'(x(s'))f(s')]\right]g(x(s))\delta(s-t')$$

$$=\Theta(t-t')g(x(t'))\exp\left[\int_{t'}^t ds[W'(x(s))+g'(x(s))f(s)]\right]$$ (41)

in which $\Theta(t-t')$ is again given by (35). Now, (36) becomes

$$\frac{\delta x(t)}{\delta f(t)}=\frac{1}{2}g(x(t)) .$$ (42)

When this is put into (32), and then into (39), the result is

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]+\lambda\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}g(y)P ,$$ (43)

the desired Fokker-Planck equation for multiplicative white noise.

IV. ADDITIVE COLORED NOISE

Consider (1) with $g(x)=1$ again, but use (24), (29), (2), and (3) instead of (16), (17), and (22). This is the additive—colored-noise case. Once again Eq. (31) is valid, but Eq. (32) is no longer tenable. By using (26) and (29), it is observed that

$$P[f]f(t)=P[f]\int ds\,\delta(t-s)f(s)=P[f]\int ds\int ds' C(t-s')K(s'-s)f(s)$$

$$=-\int ds' C(t-s')\frac{\delta P[f]}{\delta f(s')} .$$ (44)

Functional integration by parts, as in (31), yields

$$\iint \mathscr{D}f\, P[f]f(t)\delta(y-x(t))=-\int ds' C(t-s')\iint \mathscr{D}f\, P[f]\frac{\partial}{\partial y}\delta(y-x(t))\frac{\delta x(t)}{\delta f(s')} .$$ (45)

For $\delta x(t)/\delta f(s')$, the complete solution (34) is needed. Putting (34) into (45) and then into (31) produces

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]+\frac{\partial^2}{\partial y^2}\int_0^t ds' C(t-s')\iint \mathscr{D}f\, P[f]\exp\left[\int_{s'}^t ds\, W'(x(s))\right]\delta(y-x(t)) .$$ (46)

This is *not* a Fokker-Planck equation.[8,13] As it stands the last term cannot be reduced to a term containing $P(y,t)$ be-

cause of the non-Markovian dependence on $x(s)$ for $s < t$.

Consider the special case of colored noise with an exponentially decaying correlation function, and assume that the correlation decays rapidly. These conditions are met with

$$C(t-s) = \frac{D}{\tau} \exp\left[-\frac{|t-s|}{\tau} \right] \tag{47}$$

and τ small. In fact, the white-noise case is given by $D \equiv \lambda$ and the limit $\tau \to 0$. Change variables to $t' = t - s'$ and observe that

$$\int_0^t ds' C(t-s') \exp\left[\int_{s'}^t ds\, W'(x(s)) \right] = \int_0^t dt' C(t') \exp\left[\int_{t-t'}^t ds\, W'(x(s)) \right]$$

$$\cong \frac{D}{\tau} \int_0^t dt' \exp\left[-\frac{t'}{\tau} \right] \exp[t' W'(x(t)) - \tfrac{1}{2}(t')^2 W''(x(t))\dot{x}(t)] , \tag{48}$$

in which the integral over W' has been expanded in terms of t'. Neglecting the $(t')^2$ term in (48), which can be shown to be self-consistently valid for small τ, the Markov approximation is obtained from

$$\int_0^t dt' C(t') \exp\left[\int_{t-t'}^t ds\, W'(s) \right] \cong \frac{D}{\tau} \int_0^t dt' \exp\left[-\frac{t'}{\tau} + t' W'(x(t)) \right]$$

$$\cong \frac{D}{1 - \tau W'(x(t))} \tag{49}$$

for sufficiently large t. Putting this back into (46) does yield a bona fide Fokker-Plank equation

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial y}[W(y)P] + D\frac{\partial^2}{\partial y^2}\left[\frac{1}{1 - \tau W'(y)} P \right] . \tag{50}$$

The special case of bistability quoted in (12) is just this equation for $W(y) = ay - by^3$. If the τ dependence is formally expanded to first order in τ, then one gets

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial y}[W(y)P] + D\frac{\partial^2}{\partial y^2}\{[1 + \tau W'(y)]P\} , \tag{51}$$

which is identical with the τ-expansion result for additive colored noise. This equation becomes (9) for the special case of bistability. Notice that for sufficiently large y^2, the diffusive term in (9) becomes quite negative, which leads to technical difficulties in the mathematics and the interpretation of Eq. (9). However, the generalization given by (12), which is a special case of (50), is free from this behavior, the diffusive term remaining positive for all y^2, as long as τ is not too large. A return to the analysis of bistability using Eq. (5) comes later.

V. MULTIPLICATIVE COLORED NOISE

Once again start with (1) with $g(x) \neq 1$. The preceding analyses are valid insofar as yielding (39) and (45). Together this gives

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial y}[W(y)P] + \frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y} \int ds' C(t-s') \iint \mathscr{D}f\, P[f]\delta(y - x(t))\frac{\delta x(t)}{\delta f(s')} . \tag{52}$$

Equation (41) is also still valid. Therefore,

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial y}[W(y)P] + \frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y} \int_0^t ds' C(t-s') \iint \mathscr{D}f\, P[f]\delta(y - x(t))$$

$$\times \exp\left[\int_{s'}^t ds\, [W'(x(s)) + g'(x(s))f(s)] \right] g(x(s')) . \tag{53}$$

The restriction to an exponential decay for $C(t-s')$ is again made by assuming (47). By using the change of variables used in (48) yields, to dominant order in t', the result

$$\int_0^t ds' C(t-s') \exp\left[\int_{s'}^t ds\,[W'(x(s))+g'(x(s))f(s)]\right]g(x(s'))$$

$$\cong \frac{D}{\tau} \int_0^t dt' \exp\left[-\frac{t'}{\tau}\right] \exp\{t'[W'(x(t))+g'(x(t))f(t)]\}[g(x(t))-t'g'(x(t))W(x(t))]$$

$$\cong \frac{D}{\tau} \int_0^t dt' \exp\left[-\frac{t'}{\tau}+t'W'(x(t))\right][g(x(t))-t'g'(x(t))W(x(t))]$$

$$\cong \frac{D}{1-\tau W'(x(t))}g(x(t)) - \frac{D\tau}{[1-\tau W'(x(t))]^2}g'(x(t))W(x(t)),\tag{54}$$

wherein a $f(t)$ term has been neglected which can be shown to be self-consistently valid. It has also been assumed that t is sufficiently large compared to τ. Inserting the approximation expressed by (54) into (53) yields the Markovian approximation

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P]+D\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}\left[\left[\frac{g(y)}{1-\tau W'(y)}-\frac{\tau g'(y)W(y)}{[1-\tau W'(y)]^2}\right]P\right].\tag{55}$$

When the τ dependence is formally expanded to leading order, the τ-expansion result[5] is obtained

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P]+D\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}\{[g(y)+\tau g(y)W'(y)-\tau g'(y)W(y)]P\}.\tag{56}$$

Technical difficulties of the type referred to above[9] for (51) also plague (56), but are eliminated by the generalization given by (55).

It should not go unemphasized that the derivation of (55), which contains (5) when $g(y)=1$, is much easier using functional calculus than the derivation using either cumulants[5] or the Furnstsu-Novikov[3,4,8] identity. Moreover, the technical difficulties of the τ-expansion results are avoided by this generalization.

VI. FIRST-PASSAGE TIME DISTRIBUTION

Consider the general one-variable Fokker-Planck equation

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P]+\frac{\partial^2}{\partial y^2}[D(y)P].\tag{57}$$

Even Eq. (55) can be put into this form by making an Ito shift[11,12] on the streaming term in (56). The steady-state solution is

$$P_s(y)=N\frac{1}{D(y)}\exp\left[\int_{-\infty}^y dx\,\frac{W(x)}{D(x)}\right]\tag{58}$$

in which

$$N^{-1}=\int_{-\infty}^0 dy\,\frac{1}{D(y)}\exp\left[\int_{-\infty}^y dx\,\frac{W(x)}{D(x)}\right].\tag{59}$$

The lower limits in the integrals should be thought of as a limit

$$\int_{-\infty}^0 dx\cdots=\lim_{R\to\infty}\int_{-R}^0 dx\tag{60}$$

especially in the case where $U(y)$ ($U'=-W$) satisfies

$$\lim_{y\to-\infty}U(y)\to\infty.\tag{61}$$

Kolomogorov's backward equation[12,14] corresponding to (57) is its adjoint:

$$\frac{\partial}{\partial t}Q = W(y)\frac{\partial}{\partial y}Q+D(y)\frac{\partial^2}{\partial y^2}Q.\tag{62}$$

While the Fokker-Planck equation (57) describes the probability distribution $P(y,t,y',0)$ for the variable values between y and $y+dy$ at time t, starting from y' at $t=0$, the Kolmogorov equation (62) describes the probability distribution $Q(y',t,y,0)$ for the variable values between y' and $y'+dy'$ at t, starting from y at $t=0$. Therefore, the probability that the variable is still in the interval $(-\infty,0)$ at time t is given by

$$G(y,t)\equiv\int_{-\infty}^0 dy'Q(y',t,y,0).\tag{63}$$

From (62), it follows that

$$\frac{\partial}{\partial t}G = W(y)\frac{\partial}{\partial y}G+D(y)\frac{\partial^2}{\partial y^2}G\tag{64}$$

with initial condition

$$G(y,0)=\begin{cases}1 & \text{for } y\in(-\infty,0)\\ 0 & \text{otherwise}.\end{cases}\tag{65}$$

The boundary condition $G(-\infty,t)=0$ is also used. Let the time that the variable leaves $(-\infty,0)$ be called T. For small dt, the probability that the variable leaves $(-\infty,0)$ between t and $t+dt$ is given by

$$G(t)-G(t+dt)\sim-\frac{\partial}{\partial t}G\,dt.$$

Therefore, the average of any function of T, $f(T)$, is expressible as

$$\langle f(T)\rangle=-\int_0^\infty dt\,f(t)\frac{\partial}{\partial t}G(t)$$

$$=f(0)+\int_0^\infty dt\left[\frac{d}{dt}f(t)\right]G(t)\tag{66}$$

because $G(y,\infty)=0$. Consequently, the "mean first-passage time" is

$$T(y) \equiv \langle T \rangle = \int_0^\infty dt \, G(y,t) \tag{67}$$

starting from y.

The mean first-passage time $T(y)$ satisfies an equation which follows from (64) and (67):

$$W(y)\frac{\partial}{\partial y}T + D(y)\frac{\partial^2}{\partial y^2}T = -1 , \tag{68}$$

wherein the initial condition (65) was used. Let $\phi \equiv T'$. ϕ satisfies

$$W\phi + D\phi' = -1 \tag{69}$$

or

$$\phi' = -\frac{W}{D}\phi - \frac{1}{D} .$$

The solution is

$$\phi(y) = \exp\left[-\int_{-\infty}^y dx \frac{W(x)}{D(x)}\right]\phi(-\infty)$$
$$- \int_{-\infty}^y dx \exp\left[-\int_x^y dz \frac{W(z)}{D(z)}\right]\frac{1}{D(x)} . \tag{70}$$

These integrals should again be interpreted in the sense of (60). As a notational convenience, introduce

$$\psi(y) \equiv \exp\left[\int_\infty^y dx \frac{W(x)}{D(x)}\right] . \tag{71}$$

The solution to (68) can be written as

$$T(y) = \int_{-\infty}^y dx \frac{1}{\psi(x)}\phi(-\infty)$$
$$- \int_{-\infty}^y dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)}\frac{1}{D(z)} . \tag{72}$$

The boundary condition $T(-\infty) = 0$ is manifestly satisfied, but the boundary condition $T(0) = 0$ requires

$$\phi(-\infty) = \frac{\int_{-\infty}^0 dx \int_{-\infty}^x dz \, [\psi(z)/\psi(x)]1/D(z)}{\int_{-\infty}^0 dx \, [1/\psi(x)]} . \tag{73}$$

Thus the final result is

$$T(y) = \frac{1}{\int_{-\infty}^0 dx \frac{1}{\psi(x)}}\left[\left[\int_{-\infty}^y dx \frac{1}{\psi(x)}\right]\int_{-\infty}^0 dx \int_{-\infty}^x dx \frac{\psi(z)}{\psi(x)}\frac{1}{D(z)}\right] - \int_{-\infty}^y dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)}\frac{1}{D(z)} . \tag{74}$$

The boundary conditions, $T(-\infty) = T(0) = 0$, correspond to "absorbing" boundaries at $y = 0$ and $y = -\infty$. For a boundary such as $y = -\infty$ where $U(-\infty) = \infty$, a "reflecting" boundary is often used and is expressed by $T'(-\infty) = 0$ instead of $T(-\infty) = 0$. This means that instead of (74), one gets

$$T(y) = T(-\infty) - \int_{-\infty}^y dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)}\frac{1}{D(z)} . \tag{75}$$

Now, $T(0) = 0$ implies

$$T(-\infty) = \int_{-\infty}^0 dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)}\frac{1}{D(z)} . \tag{76}$$

Together, this yields

$$T(y) = \int_y^0 dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)}\frac{1}{D(z)} . \tag{77}$$

Notice also that

$$P_s(y) = \frac{\psi(y)/D(y)}{\int_{-\infty}^0 dx \, [\psi(x)/D(x)]} , \tag{78}$$

which converts (77) into its equivalent[9]

$$T(y) = \int_y^0 dx \frac{1}{D(x)P_s(x)}\int_{-\infty}^x dz \, P_s(z) . \tag{79}$$

Formula (74) reproduces (77) when $U(-\infty) = \infty$, provided one shows

$$\lim_{R \to \infty}\left[\int_{-R}^y dx \frac{1}{\psi(x)}\bigg/\int_{-R}^0 dx \frac{1}{\psi(x)}\right] = 1 . \tag{80}$$

This is the situation for the bistable potential discussed below.

Not only the mean first-passage time, $T(y)$, but the entire first-passage-time distribution is computable. Define $T_n(y)$ by

$$T_n(y) = \langle T^n \rangle = n \int_0^\infty dt \, t^{n-1}G(y,t) \quad \text{for } n > 0 . \tag{81}$$

From (66), $T_0(y) = \langle T^0 \rangle = 1$. From (64), it follows that

$$W(y)\frac{\partial}{\partial y}T_n + D(y)\frac{\partial^2}{\partial y^2}T_n = -nT_{n-1} \tag{82}$$

with boundary conditions $T_n(0) = 0$ and $T_n(-\infty) = 0$, or $T_n(0) = 0$ and $T_n'(-\infty) = 0$. The Fourier transform of the first-passage-time distribution is the characteristic function $\Phi(k,y)$ defined by

$$\Phi(k,y) \equiv \langle \exp(ikT) \rangle = \sum_{n=0}^\infty \frac{(ik)^n}{n!}T_n(y) . \tag{83}$$

It satisfies the equation which follows from (82)

$$W(y)\frac{\partial}{\partial y}\Phi + D(y)\frac{\partial^2}{\partial y^2}\Phi = -ik\Phi \tag{84}$$

with $\Phi(k,-\infty) = \Phi(k,0) = 1$, or $\Phi(k,0) = 1$ and $\Phi'(k,-\infty) = 0$. The second choice for boundary condi-

tion is of interest below. It yields an iterated solution to (84) generated by

$$\Phi(k,y) = 1 + ik \int_y^0 dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)} \frac{1}{D(z)} \Phi(k,z) .$$ (85)

Thus we define the integral kernel $K(y,z')$ by

$$K(y,z') \equiv \int_y^0 dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)} \frac{1}{D(z)} \delta(z - z') .$$ (86)

Therefore,

$$\Phi(k,y) = 1 + ik \int_{-\infty}^0 dz' K(y,z') + (ik)^2 \int_{-\infty}^0 dz_1 \int_{-\infty}^0 dz_2 K(y,z_1) K(z_1,z_2) + \cdots$$

$$+ (ik)^n \int_{-\infty}^0 dz_1 \cdots \int_{-\infty}^0 dz_n K(y,z_1) \cdots K(z_{n-1},z_n) + \cdots .$$ (87)

$\Phi(k,y)$ can either be Fourier transformed into the distribution function of first-passage times or it can be used as a generator for first-passage-time moments:

$$T_n(y) = (-i)^n \frac{\partial^n}{\partial k^n} \Phi(k,y) \bigg|_{k=0} .$$ (88)

VII. BISTABILITY

The case of the bistable potential is governed by (8). Recent work[10] has been concerned with the additive—colored-noise case, i.e., $g(x)=1$ in (1). The τ expansion[3-5] yields the Fokker-Planck approximation given in (51). This means that in (68), $D(y)$ is given by

$$D(y) = D[1 + \tau W'(y)] .$$ (89)

The functional-calculus result (50), which is stronger, implies

$$D(y) = \frac{D}{1 - \tau W'(y)} .$$ (90)

Hanggi's ansatz[10,13] implies

$$D(y) = \frac{D}{1 + \tau(3b\langle y^2 \rangle - a)} ,$$ (91)

in which the explicit expression $W(y) = ay - by^3$ has been used.

This situation is the case for which (77) is valid for the mean first-passage time. Moreover, $P_s(y)$ may be used to determine the most probable y for the initial value. $P_s(y)$ is maximum when

$$\frac{\psi'(y)}{D(y)} - \frac{\psi(y)D'(y)}{D^2(y)} = 0 .$$ (92)

This translates into

$$W(y) - D'(y) = 0 ,$$ (93)

which implies

$$y \cong -\sqrt{a/b} + \alpha\tau ,$$ (94)

where α is a constant. Therefore, it suffices to compute

$$T(-\sqrt{a/b}) = \int_{-\sqrt{a/b}}^0 dx \int_{-\infty}^x dz \frac{\psi(z)}{\psi(x)} \frac{1}{D(z)} .$$ (95)

Now, $1/\psi(x)$ for $x \in (-\sqrt{a/b}, 0)$ achieves its maximum at $x=0$ and has a Gaussian profile close to $x=0$. Using (71), we note that

$$\frac{\psi(z)}{\psi(x)} = \exp\left[\int_x^z dy \frac{W(y)}{D(y)}\right]$$

$$= \exp\left[-\int_0^x dy \frac{W(y)}{D(y)}\right] \exp\left[\int_0^z dy \frac{W(y)}{D(y)}\right] .$$ (96)

The approximation

$$\exp\left[-\int_0^x dy \frac{W(y)}{D(y)}\right] \cong \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2}\right]$$ (97)

is justified because $W(0)=0$, and σ^2 is given by

$$\frac{1}{\sigma^2} = \frac{W'(0)}{D(0)} - \frac{W(0)D'(0)}{D^2(0)} = \frac{W'(0)}{D(0)} .$$ (98)

The expression for $T(-\sqrt{a/b})$ in (95) is dominated by

$$T(-\sqrt{a/b}) \cong \int_{-\infty}^0 dz \frac{1}{D(z)} \exp\left[\int_0^z dy \frac{W(y)}{D(y)}\right]$$

$$\times \frac{1}{2}(2\pi\sigma^2)^{1/2} ,$$ (99)

in which (96) and (97) have been used. The z dependence of $D(z)$ is weak in each of the three cases (89)—(91). The exponential factor in (99) has its maximum at $z = -\sqrt{a/b}$ if τ-dependent corrections are ignored. This implies the further approximation, valid to leading order in τ for each of the three cases:

$$T(-\sqrt{a/b}) \cong \left[\frac{\pi D(0)}{2a}\right]^{1/2} \frac{1 + 2at}{D}$$

$$\times \int_{-\infty}^0 dz \exp\left[\int_0^z dy \frac{W(y)}{D(y)}\right] .$$ (100)

The remaining integrals have produced the conundrum discussed in the Introduction. Hanggi et al.[9,10] have treated (100) by the method of steepest descent. The maximum for the exponential integrand of (100) occurs when the exponential's argument is maximal, which occurs when its derivative vanishes and its second derivative is negative. This happens when $W(z)=0$ and

$$\frac{W'(z)}{D(z)} - \frac{W(z)D'(z)}{D^2(z)} = \frac{W'(z)}{D(z)} < 0 .$$ (101)

The solution is $z = -\sqrt{a/b}$ which is independent of the choice for $D(z)$. The steepest-descent approximation to (100) becomes

$$T(-\sqrt{a/b}) \cong \left[\frac{\pi D(0)}{2a}\right]^{1/2} \frac{1+2a\tau}{D} \exp\left[\int_0^{-\sqrt{a/b}} dy \frac{W(y)}{D(y)}\right] \int_{-\infty}^0 dz \exp\left[\frac{1}{2}\frac{W'(-\sqrt{a/b})}{D(-\sqrt{a/b})}(z+\sqrt{a/b})^2\right]$$

$$\cong \left[\frac{\pi D(0)}{2a}\right]^{1/2}\left[\frac{2\pi D(-\sqrt{a/b})}{2a}\right]^{1/2}\frac{1+2a\tau}{D}\exp\left[\int_0^{-\sqrt{a/b}}dy\frac{W(y)}{D(y)}\right]. \tag{102}$$

Now, even with the stronger version of (89) and (90), this yields

$$T(-\sqrt{a/b}) \cong \left[\frac{\pi D}{2a(1-\tau a)}\right]^{1/2}\left[\frac{2\pi D}{2a(1+2a\tau)}\right]^{1/2}\frac{1+2a\tau}{D}\exp\left[\frac{1}{D}\int_0^{-\sqrt{a/b}}dy[W(y)-\tau W(y)W'(y)]\right]$$

$$= \frac{\pi}{\sqrt{2}a}\left[\frac{1+2a\tau}{1-a\tau}\right]^{1/2}\exp\left[\frac{a^2}{4bD}\right] \tag{103}$$

because $W(-\sqrt{a/b})=W(0)=0$. This last pair of identities kills any τ dependence in the exponential factor! Hanggi's ansatz, (91), however, yields

$$T(-\sqrt{a/b}) \cong \frac{\pi}{\sqrt{2}a}\exp\left[\frac{a^2}{4bD}(1+2a\tau)\right] \tag{104}$$

because $D(0)=D(-\sqrt{a/b})=D/(1+2a\tau)$ when $\langle y^2\rangle=a/b$ is used. This result, of course, is preferred because it agrees with the numerical simulations.[9]

The conclusion[9,10] that the functional-calculus approach which produces (100) has somehow failed is unwarranted. Only the applicability of the method of steepest descent needs to be questioned. Instead, it can be shown that (100) may be integrated exactly, and yields a result commensurate with (104) when this is done. The method of steepest descent is insensitive to the quartic dependence in U which determines the behavior of the integrals in (100).

These facts are seen as follows. Using (90), we begin with

$$T(-\sqrt{a/b}) \cong \left[\frac{\pi D}{2a(1-a\tau)}\right]^{1/2}\frac{1+2a\tau}{D}\int_{-\infty}^0 dz\exp\left[\int_0^z dy\frac{W(y)}{D}[1-\tau W'(y)]\right]$$

$$= \left[\frac{\pi D}{2a(1-a\tau)}\right]^{1/2}\frac{1+2a\tau}{D}\int_{-\infty}^0 dz\exp\left[-\frac{1}{D}[U(z)+\tfrac{1}{2}\tau W^2(z)]\right] \tag{105}$$

and then look at the integral

$$I = \int_{-\infty}^0 dx\exp\left[-\frac{1}{D}[U(x)+\tfrac{1}{2}\tau W^2(x)]\right]$$

$$= \int_{-\infty}^0 dx\exp\left[-\frac{1}{D}\left[-\frac{a}{2}x^2+\frac{b}{4}x^4+\tfrac{1}{2}\tau(a^2x^2+b^2x^6-2abx^4)\right]\right]$$

$$= \int_{-\infty}^0 dx\exp\left[\left[\frac{a}{2D}-\frac{a^2\tau}{2D}\right]x^2-\left[\frac{b}{4D}-\frac{ab\tau}{D}\right]x^4\sum_{n=0}^\infty\left[\frac{-b^2\tau}{2D}\right]^n\frac{1}{n!}x^{6n}\right]. \tag{106}$$

We then change variables to $y=x^2$, which implies $dy=2x\,dx$. Therefore, I becomes

$$I = \tfrac{1}{2}\int_0^\infty dy\,y^{-1/2}\exp\left[\frac{a-a^2\tau}{2D}y-\frac{b-4ab\tau}{4D}y^2\right]$$

$$\times\sum_{n=0}^\infty\left[\frac{-b^2\tau}{2D}\right]^n\frac{1}{n!}y^{3n}. \tag{107}$$

The parabolic cylinder functions satisfy the identities[15]

$$U(n,-x)=\frac{\pi}{\Gamma(n+\tfrac{1}{2})}V(n,x)$$

$$\text{for } x>0 \text{ and } n \text{ an integer} \tag{108}$$

and

$$U(n,-x)=\frac{1}{\Gamma(n+\tfrac{1}{2})}\exp(-\tfrac{1}{4}x^2)$$

$$\times\int_0^\infty ds\,s^{n-(1/2)}\exp(xs-\tfrac{1}{2}s^2).$$

This means that

$$\int_0^\infty dx\,s^{n-(1/2)}\exp(xs-\tfrac{1}{2}s^2)=\pi\exp(\tfrac{1}{4}x^2)V(n,x). \tag{109}$$

If $s=[(b-4ab\tau)/2D]^{1/2}y$ is substituted into (107), then

$$I = \frac{1}{2} \left[\frac{2D}{b - 4ab\tau} \right]^{1/4}$$

$$\times \sum_{n=0}^{\infty} \left[\frac{-b^2\tau}{2D} \right]^n \frac{1}{n!} \left[\frac{2D}{b - 4ab\tau} \right]^{3n/2}$$

$$\times \pi \exp(\tfrac{1}{4}x^2) V(3n,x) \,, \tag{110}$$

where

$$x = \frac{a - a^2\tau}{2D} \left[\frac{2D}{b - 4ab\tau} \right]^{1/2} . \tag{111}$$

Since only the dominant τ dependence is desired, this yields the approximation

$$I = \frac{1}{2} \left[\frac{2D}{b - 4ab\tau} \right]^{1/4} \pi \exp(\tfrac{1}{4}x^2) V(0,x) . \tag{112}$$

The numerical simulations were performed for $a = b = 1$, $\tau \le 0.05$, and $D = 0.05$ and 0.1. The values for x in (111) are, respectively, $\sqrt{10}$ and $\sqrt{5}$. These values justify the asymptotic expansion for $V(0,x)$ given by

$$V(0,x) \cong \sqrt{2/\pi} \exp(\tfrac{1}{4}x^2) x^{-1/2}$$

$$\times \left[1 + \frac{3}{8} \frac{1}{x^2} + \frac{105}{128} \frac{1}{x^4} + \cdots \right] . \tag{113}$$

Inserting this into (112) gives

$$I \cong \left[\frac{\pi D}{a - a^2\tau} \right]^{1/2} \exp \left[\frac{a^2}{4bD} \frac{(1 - a\tau)^2}{1 - 4a\tau} \right] . \tag{114}$$

The τ dependence in the argument of the exponential has the τ expansion $1 + 2a\tau$. Returning to (105), the mean first-passage time becomes

$$T(-\sqrt{a/b}) \cong \frac{\pi}{a\sqrt{2}} \left[\frac{1 + 2a\tau}{1 - a\tau} \right] \exp \left[\frac{a^2}{4bD}(1 + 2a\tau) \right] . \tag{115}$$

This proves that the functional-calculus approach agrees with the numerical simulations. Hanggi's ansatz has been justified, but it need not be invoked since (12) has been shown to work without approximation. The general result embodied by Eq. (6) will be tested in the near future.

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