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Quantum optical tunneling: A representation-free theory valid near the state-equation turning points

P. D. Drummond
University of Auckland, Private Bag, Auckland, New Zealand
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The problem of representation dependence in the treatment of quantum tunneling in absorptive optical bistability is considered. Disregarding this problem can lead to order-of-magnitude errors in calculated tunneling rates. A new description is presented, valid near the state-equation turning points, in which regime the discrepancies between different representations are shown to disappear. This description is also useful for treating critical fluctuations and dispersive cases.

Quantum tunneling in open, multistable, driven systems has become a topic of great interest recently. This effect is straightforward to observe, in principle, and gives a stringent test of theories of large-scale cooperation in open quantum systems. The case of optical bistability[1-10] quantum tunneling[11-13] is of interest because of the extreme simplicity of the Hamiltonian, which has no adjustable parameters. Thus an experiment would involve less theoretical parameters than Josephson junction tunneling.[14]

In this Rapid Communication it is shown that there is a hidden subtlety in using Fokker-Planck equations for the calculation of quantum tunneling. This occurs when approximations are used which are not representation invariant. For this reason, the tunneling rates calculated in previous work[8,11-13] depend on the representation[15] of operator orderings. This completely unphysical dependence can change the predicted tunneling rates through orders of magnitude.

Obviously this subtlety is extremely significant in quantitative terms, and leads to a useful criterion for determining the acceptability of a calculation, which is that the results must not depend on the operator representation. This type of invariance principle is applicable in more general problems involving coherent states or operator representations. I examine here the problem of quantum optical tunneling from this invariance principle viewpoint, and develop a description valid near the state-equation turning points; in this limit, the description turns out to be the same for different representations. The new theory has the advantage that it can be used to treat more general types of quantum tunneling than just absorptive optical bistability. This is shown directly by calculating tunneling rates in the dispersive case, which have not been treated previously.

In order to calculate tunneling rates in open quantum systems it is usual to represent the relevant operator equations by a Fokker-Planck equation for a distribution $P(\boldsymbol{\alpha})$. Different representations are transformable from one to the other via transformations[15,16] of the general type,

$$Q(\beta,\beta')=\int P(\boldsymbol{\alpha})\exp\left[-\frac{1}{\epsilon}(\alpha-\beta)(\alpha'-\beta')\right]\times d\mu(\boldsymbol{\alpha})/(\pi\epsilon)\ . \tag{1}$$

Suppose that a particular distribution $P(\boldsymbol{\alpha})$ is known to have a discrete eigenvalue spectrum given by $\{\lambda_j\}$ under the action of a time-development functional $L\{\ \}$. That is,

$$P(\boldsymbol{\alpha},t)=\sum_{j=0}^{\infty}\exp(-\lambda_j t)P_j(\boldsymbol{\alpha})\ . \tag{2}$$

A change of representation alters the coefficients $P_j(\boldsymbol{\alpha})$ to some new coefficients $Q_j(\boldsymbol{\alpha})$, while leaving λ_j unchanged. Thus the tunneling rate (λ_1) does not depend on the representation. This can be understood physically by noting that the observed tunneling rate is defined by a one-time correlation, which is simply proportional to the moment of a distribution (with a constant that is representation dependent). The moments therefore are directly observable, with representation-invariant time dependence.

In the case of absorptive optical bistability at zero temperature in a high-Q, plane-wave ring interferometer, equivalent Fokker-Planck equations were derived in the generalized P representation,[9] and in the Wigner representation,[8] via truncation of higher derivatives:

$$\frac{1}{k}\frac{\partial}{\partial t}P(\boldsymbol{\alpha})=\left[\frac{\partial}{\partial x_\mu}[A_\mu(\mathbf{x})-y_\mu]+\frac{1}{2n_0}\frac{\partial^2}{\partial x_\mu\partial x_\nu}D_{\mu\nu}(\mathbf{x})\right]P(\boldsymbol{\alpha})\ , \tag{3}$$

where

$$\boldsymbol{\alpha}=(\alpha,\alpha')=(x,x^*)\sqrt{n_0}=(x_1+ix_2,x_1-ix_2)\sqrt{n_0}\ .$$

Here k is the decay rate of the one-mode interferometer, $n_0\gg 1$ is the threshold photon number in the mode, while $\boldsymbol{\alpha},\mathbf{x}$ represent the unscaled and scaled field amplitudes, respectively, and $\mathbf{y}$ represents the input amplitude.

In both representations, the drift (A_μ) is identical, apart from terms of $O(1/n_0)$ which are neglected. A general relationship[16] is known that gives the diffusion term D^Q for a representation transformed according to Eq. (1):

$$D^Q_{\mu\nu}(\mathbf{x})=D_{\mu\nu}(\mathbf{x})+\frac{\epsilon}{2}\left[\frac{\partial A_\mu}{\partial x_\nu}+\frac{\partial A_\nu}{\partial x_\mu}\right]+O(1/n_0)\ . \tag{4}$$

Here $D_{\mu\nu}$ is the diffusion term in the original representation $P(\boldsymbol{\alpha})$, while $D^Q_{\mu\nu}$ is the diffusion term in the transformed representation $Q(\boldsymbol{\beta})$. Defining $\mathbf{x}=r(\cos\theta,\sin\theta)$, one can write down a unified Fokker-Planck equation for high-Q optical bistability in any representation as

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$$\frac{1}{k}\frac{\partial}{\partial t}P(\boldsymbol{\alpha})=\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\left(1+\frac{2C}{1+r^2}\right)r-y\cos\theta\right]+\frac{\partial}{\partial\theta}\left(\frac{y\sin\theta}{r}\right)+\frac{1}{2n_0}\frac{\partial^2}{\partial r^2}\left[\frac{Cr^2(1-2f+2r^2)}{(1+r^2)^3}+\epsilon\left(1+\frac{2C(1-r^2)}{(1+r^2)^2}\right)\right]\right.$$
$$\left.+\frac{1}{2n_0}\frac{\partial^2}{\partial\theta^2}\left[\frac{C}{1+r^2}+\epsilon\left(\frac{1+2C+r^2}{r^2(1+r^2)}\right)\right]\right\}rP(\boldsymbol{\alpha}) \ . \tag{5}$$

In the above (truncated) equations, $\epsilon=0$ corresponds to the P representation, and $\epsilon=\frac{1}{2}$ to the Wigner representation. The other symbols have a standard usage[8,9] in laser and optical bistability theory; terms of higher order in $(1/n_0)$ are omitted. Some early calculations of tunneling used a different Fokker-Planck equation,[11-13] which omitted atomic correlations. Equation (5) takes these correlations[8,9] fully into account.

A common approximation used to solve for the tunneling eigenvalue of this type of Fokker-Planck equation is to neglect the coefficient of phase diffusion.[7,8,11-13] An approximate calculation of the eigenvalue λ_1 from Eq. (5) using this technique leads to results which can be changed through orders of magnitude by changing ϵ. This can be demonstrated quantitatively in the radiative case with $n_0=10^4$, where the value of λ_1 apparently varies by a factor of up to 10^{5000} or more, depending on the representation.

The combination of the truncation of higher derivatives and the phase-pinning approximation is therefore inconsistent, as would be a gauge-dependent calculation in quantum electrodynamics. Since the same difficulty occurs even in truly one-dimensional Fokker-Planck equations, the problem seems to be chiefly due to the truncation of higher-order derivatives, which is not a good approximation during tunneling far from a turning point. Instead, an adiabatic expansion will be used around the turning points of the state equation, as higher derivative terms are then negligible. Since this also allows dispersive problems[9,10] to be handled, the general case of an arbitrary Fokker-Planck equation with the structure of Eq. (3) will be treated here.

Let $(\mathbf{x}_c,\mathbf{y}_c)$ be a turning point of the state equation $\mathbf{y}=\mathbf{A}(\mathbf{x})$, defined so that $[\partial/\partial x_\nu]A_\mu(\mathbf{x}_c)$ has one vanishing eigenvalue, whose eigenvector can always be taken in the x_1 direction. Let the input intensity $|\mathbf{y}_c|^2$ be varied slightly to $(1\pm\Delta^2)|\mathbf{y}_c|^2$, then

$$A_\mu(\mathbf{x})-y_\mu=-\delta y_\mu+\delta x_\nu a_{\mu\nu}+\tfrac{1}{2}\delta x_\nu\delta x_\rho a_{\mu\nu\rho}+O(\delta x^3) \ ,$$

where

$$\begin{aligned}\delta\mathbf{x}&=\mathbf{x}-\mathbf{x}_c \ ,\\ \delta\mathbf{y}&=\mathbf{y}-\mathbf{y}_c\approx\pm\tfrac{1}{2}\Delta^2\mathbf{y}_c \ ,\\ a_{\mu\nu}&=[\partial/\partial x_\nu]A_\mu(\mathbf{x}_c) \ ,\\ a_{\mu\nu\rho}&=[\partial^2/\partial x_\nu\partial x_\rho]A_\mu(\mathbf{x}_c) \ .\end{aligned}\tag{6}$$

Equation (6) is now substituted into Eq. (3) with new scaled variables $\tilde{x}=\delta x_1/\Delta$, $\tilde{x}_2=\delta x_2/\Delta^2$, $\tau=\Delta kt$. The scaling is chosen to give the simplest expansion uniform in Δ, noting that $a_{11}=a_{21}=0$, to give the vanishing eigenvalue. For optical bistability, the trace of $[a_{\mu\nu}]$ is real and positive,[10] and hence $[a_{22}]$ equals $\mathrm{Tr}[a_{\mu\nu}]$, and is nonvanishing. The variable $\tilde{x}_2$ therefore relaxes rapidly relative to $\tilde{x}$, and can be adiabatically eliminated.[17] The new Fokker-Planck equation valid for $\tilde{x}=0(1)$ and $\Delta\to 0$ is

$$\frac{\partial}{\partial\tau}\tilde{P}(\tilde{x})\approx\frac{1}{2}\frac{\partial}{\partial\tilde{x}}\left[-\tilde{y}+\tilde{a}\tilde{x}^2+\left(\frac{1}{\Delta^3 n_0}\right)\frac{\partial}{\partial\tilde{x}}\tilde{d}\right]\tilde{P}(\tilde{x}) \ , \tag{7}$$

where

$$\begin{aligned}\tilde{d}&=D_{11}(\mathbf{x}_c)-2[a_{12}/a_{22}]D_{12}(\mathbf{x}_c)+[a_{12}/a_{22}]^2D_{22}(\mathbf{x}_c) \ ,\\ \tilde{a}&=a_{111}-a_{211}[a_{12}/a_{22}] \ ,\\ \tilde{y}&=\pm(y_{c1}-y_{c2}[a_{12}/a_{22}]) \ .\end{aligned}$$

For $\tilde{x}$ initially on the stable branch and with $\mathbf{y}$ chosen so that $\tilde{y}\tilde{a}>0$, this equation describes a quantum penetration through a potential barrier. Higher derivative terms in Eq. (7) can only change higher orders of Δ in the potential, and are negligible as $\tilde{P}(\tilde{x})$ is slowly varying for $\tilde{x}=O(1)$. The rate can be calculated using standard methods,[18] with a result valid for $[1/n_0]<<\Delta^3<<1$, to leading order in Δ:

$$\ln(\lambda_1/k)=\left(\frac{-4\Delta^3 n_0|\tilde{y}^3/\tilde{a}|^{1/2}}{3\tilde{d}}\right) \ . \tag{8}$$

The change of representation equation [Eq. (4)] implies that $\tilde{d}$ and hence the tunneling eigenvalue λ_1 of the Fokker-Planck equation are representation invariant, as required. Note that the requirement of $\Delta^3>>[1/n_0]$ is equivalent to the requirement of a relatively deep potential, as usual[18] in this type of approximation.

Applying Eq. (8) to absorptive optical bistability gives, for the first time, the scaling of the tunneling rate with input intensity near the upper and lower turning points, for $C>>4$:

$$\begin{aligned}\ln(\lambda_1/k)&=-32\Delta^3 n_0/[9-6f] \quad (x_c,y_c=1,C)\\ &=-16C\Delta^3 n_0/3 \quad (x_c,y_c=\sqrt{2C},\sqrt{8C}) \ .\end{aligned}\tag{9}$$

The relative size of the critical fluctuations near $C=4$ can also be calculated using an adiabatic theory, just as before, except with $A_\mu(\mathbf{x})$ expanded to cubic order in $\tilde{x}$. The result is a large increase in the in-phase fluctuations, while the out-of-phase fluctuations remain of order $(1/n_0)$. This effect is similar to that occurring in a squeezed electromagnetic state.[19] The in-phase critical fluctuations $\langle\delta x_1^2\rangle$ can also be calculated in the rate-equation limit of a slowly relaxing population,[20] where the critical fluctuations scale with $(1/N)$ (N is the total number of interacting atoms, which is proportional to n_0):

$$\begin{aligned}\delta x_1^2&=\left(\frac{21-6f}{2n_0}\right)^{1/2}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \quad \text{(high } Q\text{)}\\ &=8\left(\frac{7}{N}\right)^{1/2}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \quad \text{(rate equation)} \ .\end{aligned}\tag{10}$$

Similarly, dispersive problems are tractable using this theory: Previous theories were not applicable to the dispersive case. A limiting Fokker-Planck equation can be derived at sufficiently large detunings, with the general form[9]

$$\frac{1}{\bar{k}}\frac{\partial}{\partial t}P(\boldsymbol{\alpha}) = \left[\frac{\partial}{\partial \bar{x}}\left((1+i\bar{\phi})\bar{x} - \bar{y} - i\bar{x}|\bar{x}|^2 + \frac{i}{2\bar{n}}\frac{\partial}{\partial \bar{x}}\bar{x}^2\right) + \text{c.c.}\right]P(\boldsymbol{\alpha}) \ . \quad (11)$$

Here $\bar{k},\bar{\phi}$ are the (renormalized) cavity-decay rate and detuning, respectively, while $\alpha = \bar{x}(\bar{n})^{1/2}$, where $\bar{n} = n_0[\delta + \delta^3/(2C)]$ in the notation of Ref. 9. Near the dispersive bistability critical point, which occurs at $\bar{\phi} = \sqrt{3}$, the tunneling rates are symmetric, with an eigenvalue at either turning point of

$$\ln(\lambda_1/\bar{k}) = -\tfrac{16}{9}\Delta^3\bar{n}\{[(\tfrac{1}{3}\bar{\phi}^2) - 1]/4\}^{-1/4} \ . \quad (12)$$

While this expression shows characteristic critical slowing-down effects near $\bar{\phi}^2 = 3$, it should be noted that the higher-order terms in Eq. (6) start to modify the tunneling in the immediate vicinity of the critical point, just as in the absorptive case. Equations (11) and (12) have more general applications as well: They describe a driven quantum anharmonic oscillator in the rotating-wave approximation.

In summary, the tunneling and critical fluctuations are a type of probe of the statistical ensemble in a quantum system that is far from thermal equilibrium; a calculation of these effects requires care, since the relevant ensemble is not known *a priori.* This Rapid Communication gives a new technique for calculations in one-mode problems that is valid close to a turning point, and can be easily extended to more general situations. The useful criterion of representation invariance would never occur in a thermal or classical process: It is a unique characteristic of quantum fluctuations.

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