Dynamics of the SU(1,1) Bloch vector

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In this paper we introduce the Bloch vector of the SU(1,1) space in full analogy with the well-known vector of the two-level systems.

The dynamics of generalized coherent states has been extensively studied in a number of recent papers. The theory of these states associated with SU(2) and SU(1,1) Lie algebras deserves particular interest in connection with the possibility of reducing the fluctuations.[1]

A large body of literature in this field has also been devoted to clarify the mathematical problems underlying the operatorial ordering of SU(2) and SU(1,1) algebraic structures.

In particular, Truax[2] has generalized the Weyl disentangling formula for exponential operators of the type

$$\hat{S}=e^{\alpha\hat{H}_1+\beta\hat{H}_2+\beta^*\hat{H}_3}\ , \tag{1}$$

where $\hat{H}_{1,2,3}$ are the generators of the SU(2) or SU(1,1) algebra, and α and β are complex numbers. Gerry[3] has discussed the dynamics of the SU(1,1) coherent states and has also developed a formalism to treat the evolution of quantum states driven by a Hamiltonian written in terms of the SU(1,1) generators. In Ref. 4 ordering techniques, relevant to Hamiltonian time-dependent linear combinations of SU(2) and SU(1,1) generators, have been discussed within the framework of the Wei-Norman algebraic method.[5]

The SU(2) coherent states have been extensively studied in quantum optics.[6] In particular, it is well known that their relevant dynamical properties can be studied following the evolution of the associated Bloch vector, which also played a crucial role in elucidating the dynamics of the two-level systems.[7]

In Ref. 1 it has been suggested that the SU(1,1) algebraic properties can be recovered from those of SU(2) (and vice versa) by means of a suitable complex rotation specified below. In this paper, according to the above suggestion, we construct explicitly the SU(1,1) Bloch vector and show that its dynamical properties can be directly "translated" from those well known, relevant to the SU(2) case.

Let us consider a Hamiltonian written as a linear combination of the generators of the SU(1,1) group, namely,[8]

$$\hat{H}=\alpha\hat{k}_0+\beta\hat{k}_+ +\beta^*\hat{k}_-\ , \tag{2}$$

where α and $\beta(=|\beta|e^{i\phi})$ are assumed to be constant, real, and complex numbers, respectively. The operators $\hat{k}_0,\hat{k}_\pm$ satisfy the commutation relations[8]

$$[\hat{k}_0,\hat{k}_\pm]=\pm\hat{k}_\pm,\quad [\hat{k}_-,\hat{k}_+]=2\hat{k}_0\ . \tag{3}$$

We introduce the operator vector $\hat{\mathbf{k}}$ with components

$$\hat{k}_1=\frac{\hat{k}_+ +\hat{k}_-}{2},\quad \hat{k}_2=\frac{\hat{k}_+ -\hat{k}_-}{2i},\quad \hat{k}_3=\hat{k}_0\ . \tag{4}$$

It is straightforwardly checked that the $\hat{\mathbf{k}}$ components obey the following commutation relations:

$$[\hat{k}_l,\hat{k}_m]=i\tilde{\epsilon}_{l,m,n}\hat{k}_n,\quad l,m,n=1,2,3\ , \tag{5}$$

where $\tilde{\epsilon}_{l,m,n}$, the structure constants of the SU(1,1) algebra, may be understood as the Ricci tensor of a non-Euclidean space. The components of the $\tilde{\epsilon}$ tensor read

$$\tilde{\epsilon}_{l,m,n}=(-1)^{\delta_{n,3}}\epsilon_{l,m,n}\ , \tag{6}$$

where δ is the Krönecker symbol and ϵ the Ricci tensor. The Heisenberg equations of motion for the components of the $\hat{\mathbf{k}}$ vector can be written in the following single equation:

$$\frac{d}{dt}\hat{k}_n=\tilde{\epsilon}_{l,m,n}\hat{k}_l\Omega_m\ , \tag{7}$$

where Ω_m are the components of the vector $\mathbf{\Omega}$

$$\mathbf{\Omega}=(2|\beta|\cos\phi,\ -2|\beta|\sin\phi,\ -\alpha)\ . \tag{8}$$

Equation (7) can be understood as a vector product in the SU(1,1) space. More precisely it turns out that the expectation value of the $\hat{\mathbf{k}}$ vector obeys the following equation

$$\left\langle\frac{d}{dt}\hat{\mathbf{k}}\right\rangle=\langle\hat{\mathbf{k}}\rangle\times\Omega\ , \tag{9}$$

where the vector product is defined in the non-Euclidean SU(1,1) space according to the new structure constants.

The metric of the SU(1,1) space is suggested by the relevant Casimir invariant, which can also be deduced as a dynamical invariant from Eq. (7), i.e.,[3]

$$\hat{C}=\hat{k}_3^2-(\hat{k}_1^2+\hat{k}_2^2)\ , \tag{10}$$

which also defines the "norm" of the $\hat{\mathbf{k}}$ vector. Going a step further it is immediately realized that the scalar product in SU(1,1) must be defined as

$$\mathbf{u}\cdot\mathbf{v}=-u_1v_1-u_2v_2+u_3v_3\ . \tag{11}$$

Following a well-known procedure, the Euclidean metric can be regained by means of the following transformation:

$$\mathbf{v}\rightarrow\tilde{\mathbf{v}}=(iv_1,iv_2,v_3) \tag{12}$$

(i.e., the rotation suggested in Ref. 1).

Thus we find

$$[\hat{\tilde{k}}_l,\hat{\tilde{k}}_m]=i\epsilon_{l,m,n}\hat{\tilde{k}}_n\ , \tag{13}$$

and therefore

$$\frac{d}{dt}\hat{\tilde{k}} = \hat{\tilde{\mathbf{k}}} \times \tilde{\mathbf{\Omega}} \quad , \tag{14}$$

according to the Euclidean metric involved.

In the hypothesis of constant "torque" vector $\tilde{\mathbf{\Omega}}$, a solution of the system (14) can be found by exploiting the Rabi method of the successive rotations.[7] In our case three rotations are required: the first around the 3-axis through the angle ϕ, the second around the 2-axis through an angle θ, defined by

$$\tan\theta = \frac{i\alpha}{2|\beta|} \quad , \tag{15}$$

and finally, the third around the 1-axis through the angle $-[\tilde{\mathbf{\Omega}}]\tau$, where [ ] denotes the norm in the pseudo-Euclidean space, namely,

$$[\tilde{\mathbf{\Omega}}] = \sqrt{\alpha^2 - 4|\beta|^2} \quad . \tag{16}$$

These rotations yield the solution of the Heisenberg equations of motion (14) as

$$\hat{\tilde{k}}_l(\tau) = \sum_{m=1}^{3} \tilde{R}_{l,m}(\tau,\phi)\tilde{k}_l \quad , \tag{17}$$

where $\tilde{R}_{l,m}(\tau,\phi)$ are the elements of the rotations matrix $\underline{\tilde{R}}(\tau,\phi)$ and explicitly read:

$$\begin{aligned}
&\tilde{R}_{1,1}(\tau,\phi) = \tilde{R}_{1,1}(\tau,0)\cos^2\phi + \cos([\tilde{\mathbf{\Omega}}]\tau)\sin^2\phi \quad , \\
&\tilde{R}_{1,2}(\tau,\phi) = \tilde{R}_{1,2}(\tau,0) + \tfrac{1}{2}[\cos([\tilde{\Omega}]\tau) - \tilde{R}_{1,1}(\tau,0)] \quad , \\
&\tilde{R}_{1,3}(\tau,\phi) = \tilde{R}_{1,3}(\tau,0)\cos\phi + \tilde{R}_{2,3}(\tau,0)\sin\phi \quad , \\
&\tilde{R}_{2,1}(\tau,\phi) = \tilde{R}_{2,1}(\tau,0) \\
&\qquad - \tfrac{1}{2}[\tilde{R}_{1,1}(\tau,0) - \cos([\tilde{\Omega}]\tau)]\sin^2\phi \quad , \\
&\tilde{R}_{2,2}(\tau,\phi) = \tilde{R}_{2,2}(\tau,0)\cos^2\phi + \tilde{R}_{1,1}(\tau,0)\sin^2\phi \quad , \\
&\tilde{R}_{2,3}(\tau,\phi) = \tilde{R}_{2,3}(\tau,0)\cos\phi - \tilde{R}_{1,1}(\tau,0)\sin\phi \quad , \\
&\tilde{R}_{3,1}(\tau,\phi) = \tilde{R}_{3,1}(\tau,0)\cos\phi - \tilde{R}_{2,3}(\tau,0)\sin\phi \quad , \\
&\tilde{R}_{3,2}(\tau,\phi) = \tilde{R}_{3,2}(\tau,0)\cos\phi - \tilde{R}_{3,1}(\tau,0)\sin\phi \quad , \\
&\tilde{R}_{3,3}(\tau,\phi) = \tilde{R}_{3,3}(\tau,0) \quad .
\end{aligned} \tag{18}$$

Furthermore, $\tilde{R}_{l,m}(\tau,0)$ are the matrix element for β real ($\phi=0$), i.e.,

$$\underline{\tilde{R}}(\tau,0) = \begin{pmatrix}
\dfrac{-4|\beta|^2 + \alpha^2\cos([\tilde{\mathbf{\Omega}}]\tau)}{[\tilde{\mathbf{\Omega}}]^2} & -\dfrac{\alpha}{[\tilde{\Omega}]}\sin([\tilde{\Omega}]\tau) & -\dfrac{4i\alpha|\beta|}{[\tilde{\Omega}]^2}\sin^2\left(\dfrac{[\tilde{\Omega}]\tau}{2}\right) \\
\dfrac{\alpha}{[\tilde{\mathbf{\Omega}}]}\sin([\tilde{\mathbf{\Omega}}]\tau) & \cos([\tilde{\mathbf{\Omega}}]\tau) & \dfrac{2i|\beta|}{[\tilde{\mathbf{\Omega}}]}\sin([\tilde{\mathbf{\Omega}}]\tau) \\
-\dfrac{4i\alpha|\beta|}{[\tilde{\mathbf{\Omega}}]^2}\sin^2\left(\dfrac{[\tilde{\Omega}]\tau}{2}\right) & -\dfrac{2i|\beta|}{[\tilde{\mathbf{\Omega}}]}\sin([\tilde{\mathbf{\Omega}}]\tau) & \dfrac{\alpha^2 - 4|\beta|^2\cos([\tilde{\mathbf{\Omega}}]\tau)}{[\tilde{\mathbf{\Omega}}]^2}
\end{pmatrix} \quad . \tag{19}$$

The above result is the complete solution to the problem of the time evolution of the SU(1,1) Bloch vector in the hypothesis of constant torque.

Identical results can be obtained, however, using the disentanglement theorem discussed in Refs. 1 and 3 or the more general Wei-Norman technique exploited in Ref. 4.

The technique we have discussed can be extended to more complicated cases, e.g. Hamiltonian operators written as a linear combination of the SU(1,1) group and Weyl-Heisenberg algebra $(a,a^+,\hat{\mathbf{I}})$,[8] i.e.,

$$\hat{H} = \alpha\hat{k}_0 + \beta\hat{k}_+ + \beta^*\hat{k}_- + \gamma\hat{a}^+ + \gamma^*\hat{a}, \quad (\beta = |\beta|e^{i\theta}, \quad \gamma = |\gamma|e^{i\chi}) \quad , \tag{20}$$

with the following representation of the SU(1,1) group:[1,3]

$$\hat{k}_+ = \tfrac{1}{2}\hat{a}^{+2}, \quad \hat{k}_- = \tfrac{1}{2}\hat{a}^2, \quad \hat{k}_3 = \tfrac{1}{4}(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) \quad . \tag{21}$$

Introducing the Hermitian squeeze operators

$$\hat{a}_1 = \frac{\hat{a}^+ + \hat{a}}{2}, \quad \hat{a}_2 = \frac{\hat{a}^+ - \hat{a}}{2i} \quad , \tag{22}$$

one finds that their time evolution is given by

$$\begin{aligned}
\hat{a}_1(t) &= [\tfrac{1}{2}[\mathbf{\Omega}]c([\mathbf{\Omega}]) + |\beta|\sin\phi\, s([\mathbf{\Omega}])]\hat{a}_1(t_0) - s([\mathbf{\Omega}])\left(\frac{\alpha}{2} - |\beta|\cos\phi\right)\hat{a}_2(t_0) + |\gamma|\sin\chi\, s([\mathbf{\Omega}]) \\
&\quad + \frac{|\gamma|}{4}\left(-\frac{\alpha}{2}\cos\chi + |\beta|\cos(\phi-\chi) + \sin\chi\right)[1 - c([\mathbf{\Omega}])] \quad , \\
\hat{a}_2(t) &= [\tfrac{1}{2}[\mathbf{\Omega}]c([\mathbf{\Omega}]) - |\beta|\sin\phi\, s([\mathbf{\Omega}])]\hat{a}_2(t_0) + s([\mathbf{\Omega}])\left(\frac{\alpha}{2} + |\beta|\cos\phi\right)\hat{a}_1(t_0) \\
&\quad + |\gamma|\cos\chi\, s([\mathbf{\Omega}]) + \frac{|\gamma|}{4}\left(\frac{\alpha}{2}\sin\chi - |\beta|\sin(\phi-\chi) + \cos\chi\right)[1 - c([\mathbf{\Omega}])] \quad .
\end{aligned} \tag{23}$$

It can be straightforwardly shown that starting from the vacuum state we have

$$\Delta a_1^2(t)=\frac{1}{4}\left\{\cos[\mathbf{\Omega}]t+\frac{\alpha^2}{2}s([\mathbf{\Omega}])^2+\frac{1}{2}[\mathbf{\Omega}]|\beta|\sin\phi c([\mathbf{\Omega}])s([\mathbf{\Omega}])-\alpha|\beta|\cos\phi s([\mathbf{\Omega}])^2\right\},$$

$$\Delta a_2^2(4)=\frac{1}{4}\left\{\cos[\mathbf{\Omega}]t+\frac{\alpha^2}{2}s([\mathbf{\Omega}])^2-\frac{1}{2}[\mathbf{\Omega}]|\beta|\sin\phi c([\mathbf{\Omega}])s([\mathbf{\Omega}])+\alpha|\beta|\cos\phi s([\mathbf{\Omega}])^2\right\}, \quad (24)$$

$$c(x)=\frac{\cos[(xt)/2]}{x/2},\quad s(x)=\frac{\sin[(xt)/2]}{x/2}.$$

The method of the Rabi rotations works in the hypothesis that α and β are not time-dependent; if not so, different techniques, as the algebraic ordering procedure exploited in Ref. 4, should be used.

It has been shown that in this case the solution of the problem depends on a single Riccati's equation with time-dependent coefficients. An analytical solution can therefore be found only in a restricted number of cases.

However, exploiting the analogy discussed so far, one could speculate about the possibility of finding exact solutions, e.g., generalizing, to the SU(1,1) case, the well-known hyperbolic-secant time dependence of the pulse or the more general solutions discussed in Ref. 9.

Let us finally point out that the analogy we have discussed in this paper can be generalized to higher groups. To give an example, the properties of the SU(3) "Bloch vector"[10] can be exploited to discuss the evolution of the corresponding SU(2,1) vector.

These last two problems, namely, the time-dependent case and the higher group analogy, will be discussed in a forthcoming paper.

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[1]K. Wod Kiewicz and J. J. Eberly, J. Opt. Soc. B **2**, 458 (1985).
[2]D. R. Truax, Phys. Rev. D **31**, 1988 (1985).
[3]C. C. Gerry, Phys. Rev. A **31**, 2721 (1985).
[4]G. Dattoli, J. Gallardo, and A. Torre, J. Math. Phys. (to be published).
[5]J. Wei and E. Norman, J. Math. Phys. A **4**, 575 (1963).
[6]F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).
[7]L. Allena and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).
[8]G. Dattoli, F. Orsitto, and A. Torre (unpublished).
[9]F. T. Hioe and J. H. Eberly, Phys. Rev. Lett. **47**, 838 (1981); J. N. Elgin, Phys. Lett. **80A**, 140 (1980); G. Dattoli and R. Mignani, J. Math. Phys. (to be published).
[10]A. Bambini and P. Berman, Phys. Rev. A **23**, 2496 (1981); A. Bambini and M. Lindberg, *ibid.* **30**, 794 (1984).