Relativistic generalization of the Wigner function and its interpretation in the causal stochastic formulation of quantum mechanics

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We show how the causal interpretation provides a physical basis for the Wigner phase-space formulation of quantum mechanics by analyzing the case of a relativistic spin-zero particle in an external electromagnetic field. The Wigner function can be interpreted as both a generalized transformation and a field which defines a quantum potential in phase space. We also propose a Lorentz-vector distribution function which always associates positive probability densities with the real motions of particles and antiparticles in phase space.

The Wigner-Moyal[1] phase-space approach casts quantum mechanics into a representation suitable for comparison with classical mechanics expressed in the same language. The existence of such a formalism involving the simultaneous use of position X and momentum P variables to represent quantum motions, and the introduction of a quasidistribution function $F(X,P)$ in phase space, has always been something of an enigma since it appears to conflict with the basic tenets of the Copenhagen interpretation. However, when one reflects that the latter view involves philosophical assumptions which cannot be derived from the quantum-mechanical formalism, it would seem that the Wigner-Moyal theory (and other similar approaches) should most appropriately be viewed as a representation of quantum phenomena complementary to that provided by the causal interpretation of quantum mechanics.[2] The latter theory, which reproduces all of the usual results of quantum mechanics, introduces the notion that a quantum particle always has simultaneously well-defined position and momentum coordinates (although the motion it executes is of course nonclassical due to the quantum potential derived from the wave function) and so it might be expected to provide a physical basis for the Wigner-Moyal method.

The relation between these two representations has been examined in the nonrelativistic one-body case by Takabayasi[3] who showed that while the description given by the quantum Liouville equation is mathematically equivalent to that provided by the Schrödinger equation (when the density matrix and distribution function satisfy the pure state condition), the physical picture implied by each is different. On the one hand, a particle motion is represented in a Hamilton-Jacobi phase-space formalism by a mean continuous trajectory which is subject to action by the quantum potential and the external potential; on the other hand, the coordinate X of the particle changes continuously with velocity $\dot{X}=P/m$ while its momentum P in general jumps with a certain "transition probability" dependent on the external potential. In fact, using the Wigner function as a weight, the variables X,P are subject to the uncertainty relations $\Delta X \Delta P \geq \hbar/2$ and we are led to a stochastic picture in phase space, although it is necessary to admit negative "probabilities." Mathematically, quantum mechanics becomes a nonlocal matrix algebra, in contrast to the point-to-point canonical transformations of classical mechanics.[4] The aim of the present paper is to extend this investigation to the relativistic case by a consideration of the motion of a spin-zero particle in an external electromagnetic field.

The relativistic generalization of the Wigner-Moyal formulation may be sought in several different ways. Firstly, one may proceed via Dirac's formulation of relativistic particle dynamics in which a proper treatment of the constraints allows a Hamiltonian description of different physical systems for which the so-called "no-go theorem" maybe circumvented. This is the case with the motion of a charged particle in an external electromagnetic field for which, as pointed out by Dirac,[5] the constrained Hamiltonian may be written as

$$H=\frac{\lambda}{2}[m^2-(P-eA)^2]\approx 0 \ ,$$

where $\lambda>0$ is a Lagrange multiplier and the weak equality $\approx$ means that the X and P have to be considered as independent parameters when one calculates the Poisson brackets. In general, for a spinless particle one constructs an eight-dimensional enlarged phase space $\Gamma=T^*M$ from the four-dimensional pseudoriemannian manifold M and its tangent bundle T^*M and imposes on Γ a symplectic structure with canonical 2 form $\omega=dX\wedge dP$. However, the proper choice of the state in the quantum version of the theory with the required classical limit which could serve to form the density operator and consequently the Wigner distribution function remains an open problem, in particular with respect to the coherence of the transformation applied to the whole operator algebra and to the set of state vectors. Secondly, one may try to elaborate the relativistic generalization of the Feynman path integral in phase space[6] by focusing attention on the properties of the evolution operator and its time dependence. Thirdly, for boson particles an obvious generalization can be obtained in the framework of the Feshbach-Villars formalism[7] by applying simultaneously the Weyl transformation to the operators and to the density matrix formed from the two-component wave function. Finally, one may define in a straightforward way a relativistic scalar generalization of the standard Wigner function.[8]

In our approach we introduce instead a Lorentz-vector distribution function and bring out the two distinct ways in which negative "probabilities" appear in such an approach. One, connected with the second-order nature of the Klein-Gordon equation, can be causally interpreted in terms of antiparticle motions. The other, associated with the very nature of the Wigner distribution (in both the relativistic and nonrelativistic cases) expressed as the mean value of the probabilistic characteristic function operator in a given pure quantum state,[9] arises from the constraints imposed by the uncertainty principle.

We start with the Klein-Gordon equation in an external electromagnetic field [in units $\hbar=c=1$ and with metric $(1,-1,-1,-1)$]:

$$(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu)\psi(x) = m^2\psi(x) . \qquad (1)$$

Substituting $\psi = Re^{iS}$ we obtain Hamilton-Jacobi and conservation equations:

$$(\partial_\mu S + eA_\mu)(\partial^\mu S + eA^\mu) = m^2 + \Box R/R \equiv M^2(x) , \qquad (2)$$

$$\partial_\mu j^\mu = 0,\quad j^\mu = \frac{1}{2m}\psi^*(i\overleftrightarrow{\partial}^\mu - 2eA^\mu)\psi = \frac{-R^2}{m}(\partial^\mu S + eA^\mu) , \qquad (3)$$

where M is de Broglie's "variable mass" involving the quantum potential. In the stochastic interpretation, $(p^\mu + eA^\mu)$ $(p_\mu = \partial_\mu S)$ is the drift momentum lying along the lines of flow defined by the probability density current j^μ. The particle thus acquires a drift 4-velocity

$$\dot{x}^\mu = dx^\mu/d\tau = (p^\mu + eA^\mu)/M ,$$

τ being the proper time. In this formulation the momentum is uniquely fixed as a function of x once $S(x)$ and the initial conditions are known and the phase-space distribution function is given by

$$(-1/m)R^2(x)[p^0 + eA^0(x)]\delta(p_\mu - \partial_\mu S) .$$

The scheme may be written in terms of Hamilton's equations in the form

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu},\quad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} , \qquad (4)$$

where the Lorentz scalar

$$H(x,p) = [(p_\mu + eA_\mu)(p^\mu + eA^\mu)]^{1/2} - M(x) \approx 0 .$$

If we restrict the solutions to (1) to those whose initial conditions satisfy $M^2 > 0$ at $t=0$ for all $\mathbf{x}$, then an initial timelike particle motion will remain so for all times and no anomalies can arise (such as the evolution of a particle into an antiparticle solution).[10]

We now pass to a phase-space version of the above theory based on new independent coordinates X^μ, P_μ. In a relativistic theory it is appropriate to define a distribution function by a Lorentz vector and we choose

$$F_\mu(X,P,e) = \left[\frac{1}{2\pi}\right]^4 \int_{-\infty}^{\infty} \frac{1}{2m}\psi^*\left[X + \frac{z}{2}\right] \times \left[i\frac{\overleftrightarrow{\partial}}{\partial X^\mu} - 2eA_\mu(X)\right] \times \psi\left[X - \frac{z}{2}\right] e^{-iP_\mu z^\mu} d^4z . \qquad (5)$$

This vector is evidently real and although F^0 may become negative (see below) we propose to treat F^μ as a phase-space probability density current, this being justified by the relation

$$\int_{-\infty}^{\infty} F^\mu(X,P,e)\,d^4P = j^\mu(X) , \qquad (6)$$

the Klein-Gordon current. Integrating by parts, the vector (5) may be expressed in terms of the relativistic scalar Wigner function

$$F(X,P) = \left[\frac{1}{2\pi}\right]^4 \int_{-\infty}^{\infty} \psi^*\left[X + \frac{z}{2}\right]\psi\left[X - \frac{z}{2}\right] e^{-iP_\mu z^\mu} d^4z \qquad (7)$$

as

$$F_\mu(X,P,e) = (1/m)[P_\mu - eA_\mu(X)]F(X,P) . \qquad (8)$$

The apparent divergence of the integral (5) may be dealt with by a "normalization in a box" argument as has been discussed elsewhere.[11]

Writing now Eq. (1) as two relations in the scalar density matrix $\rho(x,x') = \psi(x)\psi^*(x')$, we pass to c.m. and relative coordinates $X = \frac{1}{2}(x + x')$, $z = x' - x$ and apply the transformation (7). Assuming the external field varies only slowly with z, i.e.,

$$A_\mu\left[X \pm \frac{z}{2}\right] = A_\mu(X) \pm \frac{1}{2}z^\nu\frac{\partial A_\mu}{\partial X^\nu}(X) , \qquad (9)$$

we find the following two phase-space relations:

$$[P_\mu - eA_\mu(X)][P^\mu - eA^\mu(X)] = m^2 + \frac{1}{4}\frac{\Box_X F}{F} + \frac{e}{2F}\frac{\partial A^\mu(X)}{\partial X^\nu}\frac{\partial^2 F}{\partial X^\mu \partial P_\nu} \equiv N^2(X,P) , \qquad (10)$$

$$[P^\mu - eA^\mu(X)]\frac{\partial F}{\partial X^\mu} + e[P^\mu - eA^\mu(X)]\frac{\partial A_\mu}{\partial X^\nu}\frac{\partial F}{\partial P_\nu} = 0 . \qquad (11)$$

Clearly, if we had included higher-order terms in Eq. (9), or not assumed any approximation, we would have obtained higher-order momentum derivatives or nonlocal integrodifferential equations, respectively, for Eqs. (10) and (11).

It is important to realize that in these relations X^μ, P_μ, $\partial/\partial X^\mu$, and $\partial/\partial P_\mu$ act as a set of relativistic superoperators,[12] the original density matrix $\rho(x,x')$ now being represented by a vector $F(X,P)$ in a higher vector space.[13] The scheme is mathematically equivalent to that of Eqs. (2) and (3) if $F(X,P)$ satisfies the Wigner-Moyal equivalent of the pure state condition

$$\int \rho(x,x')dx'\rho(x',x'') = \int \rho(x',x')dx'\rho(x,x'') .$$

For then, starting from Eqs. (10) and (11), we may retrace our steps via an inverse Wigner-Moyal transformation to deduce Eq. (1).

The physical interpretation of the theory is as follows. The variables X^μ, $P_\mu - eA_\mu(X)$ represent the simultaneously existent position and momentum of a particle in an ensemble. According to Eq. (10) the particle motion lies on the quantum mass shell (N being the phase-space analogue of de Broglie's variable mass) and the term $\Box_x F/4F$ derived from the Wigner function acts on free particles as a quantum potential in phase space, with supplementary terms appearing when an external field is present. The particle 4-velocity is $\dot{X}^\mu = (P^\mu - eA^\mu)/N$. Equation (11) divided by N then becomes the Liouville equation

$$\frac{dF}{d\tau} = LF = 0 , \tag{12}$$

where the Liouville operator

$$L = \frac{\partial H'}{\partial P_\mu}\frac{\partial}{\partial X^\mu} - \frac{\partial H'}{\partial X^\mu}\frac{\partial}{\partial P_\mu}$$

has a classical form associated with the scalar Hamiltonian

$$H'(X,P) = \{[P_\mu - eA_\mu(X)][P^\mu - eA^\mu(X)]\}^{1/2} \approx N(X,P) .$$

Note that although the motion is governed by Hamilton's equations $\dot{X}^\mu = \partial H'/\partial P_\mu$, $\dot{P}_\mu = -\partial H'/\partial X^\mu$, the system does not reduce to that described by the classical Lorentz-force law due to the mass shell constraint (10) and the uncontrollability of the initial conditions.

We thus have a scheme based on Eqs. (10) and (11) analogous to that based on Eqs. (2) and (3), in which the Wigner function plays three distinct roles. Firstly, $F(X,P)$ does not define a point-to-point canonical transformation but rather a nonlocal unitary (through the term $e^{-iP_\mu z^\mu}$) transformation which goes outside of the usual local unitary operations of the quantum theory in that it represents the passage from an operator to a superoperator theory. Secondly, $F(X,P)$ is a physically real field in phase space which guides particles through its determination of the quantum potential. Thirdly, $F(X,P)$ enables one to define a probability density function.

To complete the physical interpretation we have to show that the current (5) satisfies a conservation law in phase space, and we have to explain the fact that our chosen density $F^0(X,P,e)$ may take on negative values in certain regions. The first problem is easy to solve. Employing Eq. (8) in Eq. (11) shows that we may write the latter as

$$\frac{\partial f^A}{\partial y^A} = 0 \tag{13}$$

where $y^A = (X^\mu, P_\mu)$ and $f^A = (F^\mu, e(\partial A^\nu/\partial X^\mu)F_\nu)$. The conserved charge Q associated with this 8-vector is then obtained by integrating f^0 over the seven-dimensional hypersurface orthogonal [with respect to the phase-space metric $(1,-1,-1,-1,1,-1,-1,-1)$] to X^0. We find using Eq. (6)

$$Q(X^0) = \int_{-\infty}^{\infty} f^0 d^4P\, d^3\mathbf{X} = \int_{-\infty}^{\infty} F^0 d^4P\, d^3\mathbf{X} = \int_{-\infty}^{\infty} j^0(X) d^3\mathbf{X} ,$$

that is, the usual Klein-Gordon probability. From the vector f^A we may derive an 8-velocity vector $v^A = dy^A/d\lambda$ (i.e., the unit tangent vector to a trajectory in phase space) so that Eq. (13) may be written

$$\frac{\partial}{\partial y^A}(\rho v^A) = 0 , \tag{14}$$

where $\rho = (f_A f^A)^{1/2}$ is an invariant density and $d\lambda^2 = d\tau^2 + dP_\mu dP^\mu$. Following the analysis of Halbwachs[14] for space-time, we may express Eq. (14) in the form

$$\frac{1}{\omega}\frac{d}{d\lambda}(\rho\omega) = 0 , \tag{15}$$

where ω is an 8-volume element such that $(1/\omega)d\omega/d\lambda = \partial v^A/\partial y^A$. For simplicity, let us consider the case of a free particle [$A_\mu(X) = 0$] so that $\rho = NF$ from Eqs. (8) and (10) and $\lambda = \tau$. Equation (15) then says that

$$NF\omega = K, \quad K \text{ constant} \tag{16}$$

along a phase-space trajectory. By arguments similar to those presented elsewhere[10] for the case of Eqs. (2) and (3), we can now show from Eq. (16) that initial timelike future pointing motion is preserved all along a trajectory, on which F^0 is a constant of the motion.

To deal with "negative densities," consider the charge conjugate solution to Eq. (1), $\psi^c(x) = \psi^*(x)$. Constructing a vector distribution function

$$F_\mu^c(X,P,-e) = (1/m)[P_\mu + eA_\mu(X)]F^c(X,P)$$

from this antiparticle wave function, where $F^c(X,P)$ satisfies (10) and (11) with $e \to -e$ it is easy to see from Eq. (5) that

$$F_\mu^c(X,P,e) = -F_\mu(X,-P,-e) . \tag{17}$$

Thus, a negative value of the probability density $F^0(X,P,e)$ for particles may be interpreted as a positive probability density for antiparticle motions with opposite 4-momenta. In general both kinds of motion must be integrated over at a point X^μ in order to obtain, say, a given positive particle density $j^0(X) > 0$ in the Hamilton-Jacobi representation. In this way we always have positive density real motions in phase space and $F^0(X,P,e)$ may be viewed as a genuine probability function.

It is of interest to see why our interpretation breaks down in the nonrelativistic limit. The Wigner function for the Schrödinger wave function

$$F_s(\mathbf{X},\mathbf{P},t) = \left[\frac{1}{2\pi}\right]^3 \int_{-\infty}^{\infty} \psi^*(\mathbf{X}+\tfrac{1}{2}\mathbf{z},t)\psi(\mathbf{X}-\tfrac{1}{2}\mathbf{z},t)e^{i\mathbf{P}\cdot\mathbf{z}}d^3\mathbf{z}$$

may be derived from Eq. (7),

$$F_s(\mathbf{X},\mathbf{P},t) = \int_{-\infty}^{\infty} F(X,P)dP^0 , \tag{18}$$

where $X^0 = t$ and has the well-known property

$$\int_{-\infty}^{\infty} F_s(\mathbf{X},\mathbf{P},t)d^3\mathbf{P} = |\psi(\mathbf{X},t)|^2 .$$

From Eq. (18) we see that the sign of F_s is correlated with that of F. However, our interpretation based on antiparticles concerns the sign of F^μ rather than F itself, there not being apparently any physically meaningful operation which reverses the sign of F. The existence of a negative "probability" density in phase space in Eq. (19) therefore cannot be explained by our method in nonrelativistic theory. (Royer[15] has shown that F_s is the expectation value of the parity operator about the point $\mathbf{X},\mathbf{P}$ and may be expressed as the difference between two positive densities corresponding to symmetric and antisymmetric parts of the wave function.)

To summarize, we argue that the only consistent physical interpretation of phase-space formulations of quantum mechanics (of the type introduced by Wigner) is to assume that quantum particles actually possess real trajectories in space-time so that the notion of a simultaneous probability distribution for position and momentum may be given a meaning. It is in no way a requirement of a trajectory interpretation that the distribution function be a delta function (we have replied to such an objection[12,16] elsewhere[17]). In the causal interpretation the uncertainty relations represent dispersion conditions and do not imply any inherent restriction on the simultaneous physical reality of variables associated with noncommuting operators. In our view the Wigner function has three aspects: it determines a generalized transformation between two causal representations of quantum mechanics, it is a conserved field which defines a quantum potential in phase space, and it is involved in the definition of probability. We do not imply, of course, that one could set up an experiment to "measure" the distribution function, in either the usual approach of the causal interpretation or in the Wigner-Moyal method. Rather, these functions relate to an objective physical process into which they enter both as physical field guiding the motion and as density functions, and the process so described has an existence independent of any observer.

Our approach puts on a firm theoretical basis the phase-space plots of the Wigner function[18] which were made precisely in order to give a classical-type trajectory view of quantum phenomena, and can be extended to underpin those phase-space treatments which employ quadrilinear distribution functions in order to define a positive definite probability function,[19] as well as the superoperator theory of Prigogine *et al.*[12] In all these approaches it seems that the same basic identification of the (X,P) variables as simultaneously existent parameters associated with real motions subject to action by the quantum potential will remain valid.

The present work complements that of a recent paper[20] where we showed, following a suggestion of Guerra and Marra,[21] how to give a canonical phase-space representation of relativistic quantum mechanics by taking as variables the density and phase fields.

The derivation of a superoperator version of the spin-$\frac{1}{2}$ Feynman-Gell-Mann equation in phase space has been given elsewhere.[22] The generalization of the present techniques to the relativistic n-body problem treated according to the methods of predictive mechanics[23] will be discussed in a further paper.

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