Origin of solitons in the "real" world

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In experiments, solitons emerge from arbitrary initial data even when the Hamiltonian perturbations are quite large. In this paper, it is shown explicitly through lowest order that the perturbations which appear in ion acoustic wave and shallow-channel water-wave experiments do not destroy, but merely renormalize, their velocity and shape. Comparisons with other perturbation approaches are made.

I. INTRODUCTION

Solitons, nonlinear wave packets which propagate without dispersing, are a beautiful phenomenon, and have been observed experimentally in many different physical settings including plasmas, glass fibers, and solids.[1] Since they were first observed by Russel[2] in 1834, they have been a continual source of fascination for physicists. Some years ago, Gardner *et al.*[3] showed that solitons emerge from arbitrary initial data in a completely calculable way when the underlying equation is the Korteweg–de Vries equation and the initial data tend to zero at $\pm\infty$. Shortly thereafter, Zakharov and Fadeev[4] showed that the Korteweg–de Vries equation is a completely integrable, Hamiltonian system and that the spectral transform and its inverse, by which means the Korteweg–de Vries equation is solved, are canonical transformations. Similar results have since been obtained for other special field equations, such as the nonlinear Schrödinger equation.[5]

In the physical world, none of these integrable field equations is ever realized exactly (except perhaps at the microscopic level of elementary particles). Instead, they must be obtained by making a small parameter expansion. For example, in the case of ion acoustic plasma waves, one begins with the two-fluid equations, ignoring electron inertia, and arrives at the Korteweg–de Vries equation by expanding, in the small parameters $\delta n/n$, the pulse density divided by the undisturbed plasma density, and T_i/T_e, the ion temperature divided by the electron temperature.[6] Similarly, in the case of shallow-channel water waves, one begins with Euler's equation, along with free-surface boundary conditions on the upper channel surface. One arrives at the Korteweg–de Vries equation by expanding, in the small parameters L/d, the pulse length divided by the channel height, and d/h, the channel height divided by the pulse height.[7]

The systems of equations from which the Korteweg–de Vries equation is derived are Hamiltonian. Continuing the small-parameter expansion beyond the Korteweg–de Vries equation, one finds corrected equations which are no longer completely integrable, but which are still Hamiltonian. Moreover, the corrected equations have the same Poisson bracket as the Korteweg–de Vries equation at every order.[8]

In experiments both in plasmas[9] and in water channels,[10] it is found that solitons emerge from arbitrary initial data even when the small parameters are quite large—as large as 0.3 in the plasma experiments and 0.6 in the water-channel experiments. The large deviations from integrability appear to have no adverse effect on the solitons, merely renormalizing their shapes and velocities. By contrast, non-Hamiltonian, dissipative perturbations lead to radiation tails and soliton destruction. Hence, the dissipation must be kept quite small in experiments.

Why do solitons emerge from arbitrary initial data in the "real" world of experiments, where Hamiltonian deviations from the Korteweg–de Vries equation are often quite large? In order to address this question, we[11] recently showed that solitons emerge from arbitrary initial data *to all orders* in the small parameter in a completely calculable way. We also discussed evidence which indicates that with suitable restrictions on the initial data, the perturbation series may actually be convergent when ε is sufficiently small. Kodama[12] has also recently addressed this question by showing that an infinite series of constants of the motion can be constructed through the two lowest orders in ε.

Our theory, because it was developed to deal with a large class of perturbations and arbitrarily high orders, is complicated and does not permit easy comparison with other perturbation theories, like those developed by Karpman and Maslov,[13] Kaup and Newell,[14] Kodama and Taniuti,[15] Ko and Kuehl,[16] and Kodama and Ablowitz.[17] The primary purpose of this paper is to demonstrate explicitly, for perturbations of practical interest, that soliton solutions exist through lowest order and will emerge from initial data in a completely calculable way. By explicitly exhibiting the apparatus of the theory, the underlying physics behind the demonstration that these results hold to all orders should become more transparent. A secondary purpose of this paper is to compare our theory with other perturbation theories.

The remainder of the paper is organized as follows: In Sec. II, the apparatus of the perturbation theory is constructed. In Sec. III, we derive single soliton solutions and compare our results to those of Kodama and Taniuti.[15] In Sec. IV, we discuss solutions whose initial shapes are proportional to $\text{sech}^2(\kappa x)$. Comparisons are made with the theory of Karpman and Maslov.[13] Section V contains the conclusions.

II. PERTURBATION THEORY

Both ion acoustic plasma waves and shallow-channel water waves are described through first order in the small parameter, which we will designate ε, by the Hamiltonian structure[8]

$$H = H_0[u] + H_1[u] , \tag{1a}$$

$$[F,G] = \int_{-\infty}^{\infty} dx \left[\frac{\delta F}{\delta u} \frac{\partial}{\partial x} \frac{\delta G}{\delta u} \right] , \tag{1b}$$

where $\delta/\delta u$ indicates the usual Fréchet, or functional derivative, and

$$H_0[u] = \int_{-\infty}^{\infty} dx (u^3 + u_x^2/2) , \tag{2a}$$

$$H_1[u] = \int_{-\infty}^{\infty} dx (au^4 + bu^2 u_{xx} + cu_{xx}^2) . \tag{2b}$$

The coefficients a, b, and c differ in the two cases mentioned previously. Using the relation

$$u_t = [u,H] , \tag{3}$$

we find that Eqs. (1) and (2) yield the equations of motion

$$u_t = 6uu_x - u_{xxx} + \varepsilon(12au^2u_x + 8bu_xu_{xx} + 4buu_{xxx} + 2cu_{xxxxx}) . \tag{4}$$

The theory of this paper and our preceding work[11] can be extended to deal with more general Poisson brackets than that of Eq. (1b). We do not do so, since these more general brackets do not appear to be necessary in practice.[8]

Our first task is to express the Hamiltonian, Eq. (2), in terms of the spectral data. Since we will be studying solutions close to a single soliton, we must first establish the transformation $u(x) \leftrightarrow [r(k), \kappa_\alpha, c_\alpha]$, where $r(k)$ is the continuous spectrum and κ_α and c_α are the spectral parameters corresponding to the soliton.[3–5] The appropriate canonical variables corresponding to the spectral data are

$$\begin{aligned} p(k) &= -\frac{2k}{\pi} \ln[1 - r(k)r(-k)] , \\ q(k) &= -(1/2i)\ln[r(k)/r(-k)] , \\ p_\alpha &= 2\kappa_\alpha^2, \quad q_\alpha = \ln c_\alpha . \end{aligned} \tag{5}$$

Using the relationship $r(k) = r(-k)^*$ on the real k axis, we find that $p(k)$ and $q(k)$ are real on the real k axis and that $p(k) > 0$ when $k > 0$. One may show that the Hamiltonian structure, when $\varepsilon = 0$, is given by[4,11]

$$H_0 = \int_0^{\infty} dk (8k^3) p(k) - (4\sqrt{2}/5) p_\alpha^{5/2} , \tag{6a}$$

$$[F,G] = \int_0^{\infty} dk \left[\frac{\partial F}{\partial q(k)} \frac{\partial G}{\partial p(k)} - \frac{\partial F}{\partial p(k)} \frac{\partial G}{\partial q(k)} \right] + \frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} . \tag{6b}$$

We evaluate the integrals in Eq. (6) over positive k rather than over all k, as do Zakharov and Fadeev,[4] in order to ensure that the perturbation theory yields real values for $p(k)$ and $q(k)$ at high order. The partial derivatives in Eq. (6b), with respect to $p(k)$ and $q(k)$, are to be interpreted as Fréchet derivatives. The zeroth-order system is clearly integrable since the Hamiltonian depends only on the canonical momenta. The equations of motion are

$$\begin{aligned} \dot{p}(k) &= -\frac{\partial H_0}{\partial q(k)} = 0 , \\ \dot{q}(k) &= \frac{\partial H_0}{\partial p(k)} = 8k^3 , \\ \dot{p}_\alpha &= -\frac{\partial H_0}{\partial q_\alpha} = 0 , \\ \dot{q}_\alpha &= \frac{\partial H_0}{\partial p_\alpha} = -2\sqrt{2} p_\alpha^{3/2} = -8\kappa_\alpha^3 . \end{aligned} \tag{7}$$

In order to write u and ultimately $H_1[u]$ as a function of the canonical variables, we first write[4,11]

$$u(x) = u_\alpha(x) + 2\frac{d}{dx} K(x,x) , \tag{8}$$

where

$$u_\alpha(x) = -2\kappa_\alpha^2 \operatorname{sech}^2(\kappa_\alpha x + q_\alpha/2) \tag{9}$$

is the single-soliton potential and $K(x,y)$ is given by the solution to the Marchenko equation

$$K(x,y) + F(x,y) + \int_{-\infty}^{x} dz\, K(x,z) F(z,y) = 0 . \tag{10}$$

The kernel $F(x,y)$ is given by the relation

$$F(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} r(k) g_\alpha(x,k) g_\alpha(y,k) , \tag{11}$$

where

$$g_\alpha(x,k) = \exp(-ikx) \left[\frac{k - i\kappa_\alpha \tanh(\kappa_\alpha x + q_\alpha/2)}{k + i\kappa_\alpha} \right] \tag{12}$$

is the left Jost function corresponding to the single-soliton potential. Equation (10) can be solved for $K(x,y)$ by making the Neumann expansion

$$K(x,y) = -F(x,y) + \int_{-\infty}^{x} dz\, F(x,z) F(z,y) - \cdots , \tag{13}$$

which can be shown to always converge.[11] For the examples considered in this paper, it will be sufficient to use the approximation

$$K(x,y) = -F(x,y) , \tag{14}$$

so that Eq. (8) becomes

$$u(x) = u_\alpha(x) - 2\frac{d}{dx} \int_{-\infty}^{\infty} \frac{dk}{2\pi} r(k) g_\alpha^2(x,k) . \tag{15}$$

Defining

$$\xi = x + q_\alpha/2\kappa_\alpha , \tag{16}$$

Eq. (15) becomes, explicitly,

$$u(\xi)=-2\kappa_\alpha^2\mathrm{sech}^2(\kappa_\alpha\xi)-2\frac{d}{d\xi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,|r(k)|\exp[-iq(k)+ikq_\alpha/\kappa_\alpha]\exp(-2ik\xi)\frac{[k-i\kappa_\alpha\tanh(\kappa_\alpha\xi)]^2}{(k+i\kappa_\alpha)^2}\,, \tag{17}$$

where $|r(k)|=r(k)\exp[-iq(k)]$ depends only on $p(k)$. Substituting Eq. (17) into Eq. (2b), and using the Fourier-transform relations listed in the Appendix, we obtain

$$H_1=(\tfrac{512}{35}a+\tfrac{1024}{105}b+\tfrac{256}{21}c)\kappa_\alpha^7-\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,|r(k)|\exp(-i\phi)k(k-i\kappa_\alpha)^2$$
$$\times i[\tfrac{256}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{512}{15}b(2k^2+3\kappa_\alpha^2)-\tfrac{256}{3}c(3k^2+2\kappa_\alpha^2)]\pi k\,\mathrm{csch}(\pi k/\kappa_\alpha)\,, \tag{18}$$

where $\phi=q(k)=kq_\alpha/\kappa_\alpha$.

In this work, we will carry out Hamiltonian perturbation theory using the Lie approach that was first invented by Hori[18] and Deprit.[19] The notation which we will use is due to Dragt.[20] Given any pair of functionals F and G, we let

$$:F:G=[F,G]\,. \tag{19}$$

The operator $:F:$ is referred to as a Lie operator. The Lie approach is based on the following two theorems.

(1) The transformation $\overline{p(k)}=\exp(:F:)p(k)$, $\overline{q_\alpha}=\exp(:F:)q_\alpha$ is a symplectic transformation for any functional F. Hence, the Poisson bracket of Eq. (6b) is preserved.

(2) Given an arbitrary pair of functionals F and G, it follows that

$$\exp(-:F:)G[\exp(:F:)p(k),\exp(:F:)q(k),\exp(:F:)p_\alpha,\exp(:F:)q_\alpha]=G[p(k),q(k),p_\alpha,q_\alpha]\,. \tag{20}$$

The goal of Hamiltonian perturbation theory is to make a variable transformation which eliminates all coordinate dependences in the Hamiltonian through the order to which one is working. The system is then formally integrable through that order. Explicitly, to lowest order in $\varepsilon r(k)$, we first divide the perturbation H_1 into two pieces,

$$H_1=\hat{H}_1+\tilde{H}_1\,, \tag{21}$$

where

$$\hat{H}_1[p(k),p_\alpha]=(\tfrac{512}{35}a+\tfrac{1024}{105}b+\tfrac{256}{21}c)\kappa_\alpha^7 \tag{22}$$

depends only on the momenta and is independent of the coordinates, and

$$\tilde{H}_1[p(k),q(k),p_\alpha,q_\alpha]=-\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,|r(k)|\exp(-i\phi)k(k-i\kappa_\alpha)^2$$
$$\times i[\tfrac{256}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{512}{15}b(2k^2+3\kappa_\alpha^2)+\tfrac{256}{3}c(3k^2+2\kappa_\alpha^2)]\pi k\,\mathrm{csch}(\pi k/\kappa_\alpha) \tag{23}$$

is coordinate dependent. We now search for a generating functional F_1 which has the property

$$H^{(1)}\equiv\exp(-\varepsilon:F_1:)H=H_0+\varepsilon\hat{H}_1+O(\varepsilon^2)\,, \tag{24}$$

or, through lowest order in ε,

$$[F_1,H_0]=\tilde{H}_1\,. \tag{25}$$

Integrating over the unperturbed orbits, we obtain

$$F_1=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,|r(k)|\exp(-i\phi)\left[\frac{k-i\kappa_\alpha}{k+i\kappa_\alpha}\right]$$
$$\times[\tfrac{32}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{64}{15}b(2k^2+3\kappa_\alpha^2)$$
$$+\tfrac{32}{3}c(3k^2+2\kappa_\alpha^2)]\pi k\,\mathrm{csch}(\pi k/\kappa_\alpha)\,. \tag{26}$$

It follows from Eq. (20) that the desired variable transformation is given by

$$\begin{aligned}p^{(1)}(k)&=\exp(\varepsilon:F_1:)p(k)\,,\\ q^{(1)}(k)&=\exp(\varepsilon:F_1:)q(k)\,,\\ p_\alpha^{(1)}&=\exp(\varepsilon:F_1:)p_\alpha\,,\\ q_\alpha^{(1)}&=\exp(\varepsilon:F_1:)q_\alpha\,.\end{aligned} \tag{27}$$

Through the order to which we are working, one finds

$$\kappa_\alpha^{(1)}=\kappa_\alpha,\quad q_\alpha^{(1)}=q_\alpha\,, \tag{28}$$

since the lowest-order corrections are proportional to $\varepsilon|r(k)|$. Henceforth, except in certain instances where it is retained for emphasis, the superscript (1) will be dropped from κ_α and q_α. One further finds

$$r(k)=\exp(-\varepsilon:F_1:)r^{(1)}(k)=r^{(1)}(k)+i\varepsilon\frac{1}{k}\left[\frac{k+i\kappa_\alpha}{k-i\kappa_\alpha}\right][\tfrac{8}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{16}{15}b(2k^2+3\kappa_\alpha^2)$$
$$+\tfrac{8}{3}c(3k^2+2\kappa_\alpha^2)]\pi k\,\mathrm{csch}(\pi k/\kappa_\alpha)\exp(-ikq_\alpha/\kappa_\alpha)\,. \tag{29}$$

Substitution of Eq. (29) into Eq. (17) yields $u^{(1)}(\xi^{(1)})=u(\xi)$. [Equivalently, we can obtain $u^{(1)}$ directly by using the relationship $u^{(1)}=\exp(-:F:)u$, as described in Ref. 11].

We now obtain the time variation of the canonical variables and from that

$$r^{(1)}(k)=|r^{(1)}(k)|\exp[-iq^{(1)}(k)]$$

and

$$\xi^{(1)}=\xi=x+q_\alpha/2\kappa_\alpha .$$

Using these results in Eq. (17) yields the time variation of $u^{(1)}(\xi^{(1)})=u(\xi)$. From Eq. (24), it follows that

$$H^{(1)}=H_0+\varepsilon\hat{H}_1 , \tag{30}$$

so that

$$\begin{aligned}\dot{p}^{(1)}(k)&=-\frac{\partial H^{(1)}}{\partial q^{(1)}(k)}=0 ,\\ \dot{q}^{(1)}(k)&=\frac{\partial H^{(1)}}{\partial p^{(1)}(k)}=8k^3 ,\\ \dot{p}_\alpha^{(1)}&=-\frac{\partial H^{(1)}}{\partial q_\alpha^{(1)}}=0 ,\\ \dot{q}_\alpha^{(1)}&=\frac{\partial H^{(1)}}{\partial p_\alpha^{(1)}}=-8\kappa_\alpha^3[1-\varepsilon\kappa_\alpha^2(\tfrac{16}{5}a+\tfrac{32}{15}b+\tfrac{8}{3}c)] .\end{aligned} \tag{31}$$

It immediately follows that

$$\begin{aligned}r^{(1)}(k,t)&=r^{(1)}(k,0)\exp(-8ik^3t) ,\\ \xi(t)&=\xi(0)-4\kappa_\alpha^2[1-\varepsilon\kappa_\alpha^2(\tfrac{16}{15}a+\tfrac{32}{15}b+\tfrac{8}{3}c)]t .\end{aligned} \tag{32}$$

In the remainder of this paper, we will be primarily concerned with two special solutions. In the first, we set $r^{(1)}(k)=0$. This choice leads to solitary waves and allows us to compare our theory to that of Kodama and Taniuti.[15] In the second, we set $r(k)=0$ initially. This choice corresponds to setting $u(\xi)=-2\kappa_\alpha^2\mathrm{sech}^2(\kappa_\alpha\xi)$ initially and allows us to compare our theory to that of Karpman and Maslov.[13]

III. SOLITARY WAVE SOLUTION

From Eq. (32), it immediately follows that if $r^{(1)}(k,0)=0$, then $r^{(1)}(k,t)=0$ for all time. It is interesting to note that $r(k)$ obtained from Eq. (29) is no longer legitimate spectral data in the sense of Fadeev[21] and Deift and Trubowitz.[22] Substituting $r(k)$ into Eq. (17), we obtain

$$\begin{aligned}u(\xi)=&-2\kappa_\alpha^2\mathrm{sech}^2(\kappa_\alpha\xi)+i\varepsilon\frac{d}{d\xi}\int_{-\infty}^{\infty}\frac{dk}{2\pi k}\frac{[k-i\kappa_\alpha\tanh(\kappa_\alpha\xi)]^2}{k^2+\kappa_\alpha^2}\pi k\,\mathrm{csch}(\pi k/\kappa_\alpha)\exp(-2ik\xi)\\ &\times[\tfrac{16}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{32}{15}b(2k^2+3\kappa_\alpha^2)+\tfrac{16}{3}c(3k^2+2\kappa_\alpha^2)] .\end{aligned} \tag{33}$$

The quantity $q(k)$ does not appear in Eq. (33); the only time dependence in Eq. (33) is through the dependence of ξ on q_α. Hence, Eq. (33) represents a solitary wave solution.

Using the integrals in the appendix to explicitly evaluate Eq. (33), we obtain

$$\begin{aligned}u(\xi)=&-2\kappa_\alpha^2[1+\varepsilon\kappa_\alpha^2(\tfrac{4}{5}a+\tfrac{88}{15}b+\tfrac{76}{3}c)]\mathrm{sech}^2(\kappa_\alpha\xi)-\varepsilon\kappa_\alpha^4(\tfrac{32}{5}a+\tfrac{64}{15}b-\tfrac{32}{3}c)\kappa_\alpha\xi\,\mathrm{sech}^2(\kappa_\alpha\xi)\tanh(\kappa_\alpha\xi)\\ &-2\varepsilon\kappa_\alpha^3(\tfrac{32}{15}a+\tfrac{64}{15}b+\tfrac{32}{3}c)\mathrm{sech}^2(\kappa_\alpha\xi)\tanh(\kappa_\alpha\xi)+\varepsilon\kappa_\alpha^4(4a+16b+60c)\mathrm{sech}^4(\kappa_\alpha\xi) ,\end{aligned} \tag{34}$$

where we have integrated *over* the pole at $k=0$. While $u(\xi)\to 0$ as $\xi\to\infty$ in Eq. (34), the explicit appearance of the factor $\kappa_\alpha\xi$ makes this result nonuniform. We may render Eq. (34) uniform by replacing it with the equivalent expression through $O(\varepsilon)$,

$$\begin{aligned}u(\xi)=&-2\kappa_\alpha^2[1+\kappa_\alpha^2(\tfrac{4}{5}a+\tfrac{88}{15}b+\tfrac{76}{3}c)]\mathrm{sech}^2\{\kappa_\alpha[1-\varepsilon\kappa_\alpha^2(\tfrac{8}{5}a+\tfrac{16}{15}b-\tfrac{8}{3}c)]\xi+\theta\}\\ &+\varepsilon\kappa_\alpha^4(4a+16b+60c)\mathrm{sech}^4\{\kappa_\alpha[1-\varepsilon\kappa_\alpha^2(\tfrac{8}{5}a+\tfrac{16}{15}-\tfrac{8}{3}c)]\xi+\theta\} ,\end{aligned} \tag{35}$$

where

$$\theta=-\varepsilon\kappa_\alpha^2(\tfrac{16}{15}a+\tfrac{8}{5}b+\tfrac{8}{3}c) .$$

If we had integrated *under* the pole at $k=0$, instead of over, we would still have obtained Eq. (35). The quantity θ would simply have been reversed in sign. Since it merely represents a shift in the origin of the x axis, it may be safely dropped, which we will do henceforth.

In order to compare our result to the predictions of the theory of Kodama and Taniuti,[15] we make the transformation

$$\kappa=\kappa_\alpha[1-\varepsilon\kappa_\alpha^2(\tfrac{8}{5}a+\tfrac{16}{15}b-\tfrac{8}{3}c)] . \tag{36}$$

Through lowest order in ε, Eq. (35) becomes

$$u(\xi)=-2\kappa^2[1+\varepsilon\kappa^2(4a+8b+20c)]\mathrm{sech}^2(\kappa\xi)+\varepsilon\kappa^4(4a+16b+60c)\mathrm{sech}^4(\kappa\xi)\ . \quad (37)$$

From Eq. (32), we further find

$$\xi(t)=x-4\kappa^2(1-8\varepsilon\kappa^2 c)t\ , \quad (38)$$

where we have set $q_\alpha=0$ at $t=0$.

Reduced to its bare essentials, the approach of Kodama and Taniuti consists of trying to solve the perturbed equations with the following ansatz: We may define the degree of any term as $[(d-1)/2]+p$, where d is the total number of derivatives and p is the number of factors. All the perturbations in Eq. (4) are of degree 3. We next assume that any term of degree $J+2$ is multiplied by ε^J. This last assumption is not really necessary, but appears to always be fulfilled in real, physical systems.[6–8] We now substitute into the perturbed equation the assumed form

$$u=\sum_{j=0}^{N}\left[\sum_{l=j}^{N}\varepsilon^l B_{jl}\right]\mathrm{sech}^{2j}\left[\kappa x+\left[\sum_{m=0}^{N}\varepsilon^m\gamma_m\right]t\right]\ , \quad (39)$$

where N is the maximum order to which we wish to carry out the perturbation theory. Kodama and Taniuti showed that it is always possible to satisfy the perturbed equation through order N by an appropriate choice of the β_{jl} and γ_m, as long as all terms in the perturbed equation are of integral degree. For the case we are considering here, $N=1$, and the ansatz becomes

$$u=(-2\kappa_\alpha^2+\varepsilon\beta_{01})\mathrm{sech}^2(\kappa x-4\kappa^3 t+\varepsilon\gamma_1 t)+\varepsilon\beta_{11}\mathrm{sech}^4(\kappa x-4\kappa^3 t+\varepsilon\gamma_1 t)\ . \quad (40)$$

After straightforward but tedious algebra, one can verify that this ansatz yields Eq. (37). The connection of this ansatz with Hamiltonian perturbations is not altogether clear. Dissipative perturbations like εu and εu_{xx} are clearly excluded since they are not of integral degree. It appears that the systems treatable by this ansatz may all be Hamiltonian systems in the generalized sense of Kodama,[23] but this result remains to be demonstrated.

The Lie approach appears to require somewhat more algebra than the approach of Kodama and Taniuti, but it has the advantage that it allows us to explicitly relate the soliton shape and velocity to the initial data. To lowest order in $\varepsilon|r(k)|$, any initial choice of $r(k)$ will lead to a soliton shape and velocity given by Eqs. (32) and (35). This point will become more clear in the next section.

In the original spectral space $[r(k),\kappa_\alpha,q_\alpha]$, the effect of the perturbation is to generate a radiation cloud which travels along with the soliton, renormalizing its velocity and shape. Analogies can be made with the radiation cloud in quantum electrodynamics or the Debye cloud in plasma physics. In the former case, the analogy is good in the sense that the radiation cloud is an integral part of the electron or soliton to which it is attached. It is a bad analogy in the sense that the renormalization is finite in the case of solitons, but infinite in the case of the electrons. In the case of Debye clouds, the situation is reversed since Debye clouds lead to a finite renormalization of the electron potential through screening by positive ions, but there is a clear physical distinction between the electron and the Debye cloud which screens it.

IV. SOLUTION WITH ARBITRARY INITIAL RADIATION

From Eqs. (29) and (32), it immediately follows that

$$r(k,t)=r(k,0)\exp(-8ik^3t)+i\varepsilon\frac{1}{k}\left[\frac{k+i\kappa_\alpha}{k-i\kappa_\alpha}\right][\tfrac{8}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{16}{15}b(2k^2+3\kappa_\alpha^2)+\tfrac{8}{3}c(3k^2+2\kappa_\alpha^2)]\times\pi k\,\mathrm{csch}(\pi k/\kappa_\alpha)\{\exp[-ikq_\alpha(t)/\kappa_\alpha]-\exp(-8ik^3t)\}\ , \quad (41)$$

where for convenience we have set $q_\alpha(0)=0$. We then find, by substitution into Eq. (17),

$$\begin{aligned}u(\xi)=&-2\kappa^2[1+\varepsilon\kappa^2(4a+8b+20c)]\mathrm{sech}^2(\kappa\xi)+\varepsilon\kappa^4(4a+16b+20a)\mathrm{sech}^4(\kappa\xi)\\&-2\frac{d}{d\xi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\{-8ik^3t-8ik\kappa_\alpha^2[1-\varepsilon\kappa_\alpha^2(\tfrac{16}{15}a+\tfrac{32}{15}b+\tfrac{8}{3}c)]t\}\frac{[k-i\kappa_\alpha\tanh(\kappa_\alpha\xi)]^2}{(k+i\kappa_\alpha)^2}\exp(-2ik\xi)\\&\times\left[r(k,0)-i\varepsilon\left[\frac{k+i\kappa_\alpha}{k-i\kappa_\alpha}\right]\pi\,\mathrm{csch}(\pi k/\kappa_\alpha)[\tfrac{8}{15}a(k^2+4\kappa_\alpha^2)+\tfrac{16}{15}b(2k^2+3\kappa_\alpha^2)+\tfrac{8}{3}c(3k^2+2\kappa_\alpha^2)]\right]\ ,\end{aligned} \quad (42)$$

where we recall from Eq. (36) that

$$\kappa=\kappa_\alpha[1-\varepsilon_\alpha^2(\tfrac{8}{5}a+\tfrac{16}{15}b-\tfrac{8}{3}c)] .$$

Equation (42) consists of the solitary wave which we found in the preceding section, along with a radiative contribution which phase mixes and ultimately disappears as $t\to\infty$, leaving only the solitary waves. Equation (42) is sufficient to demonstrate through lowest order in $|r(k)|$ and ε that a solitary wave will emerge from any initial data whose spectral transform contains a single κ_α and q_α. In Ref. 11, this result is extended to arbitrarily high order in ε and $|r(k)|$ and to arbitrary initial data which can contain any number of solitons.

If we set $r(k,0)=0$, then at $t=0$

$$u=-2\kappa_\alpha^2\operatorname{sech}^2(\kappa_\alpha\xi) . \tag{43}$$

At succeeding times, Eq. (42) implies that u breaks up into the solitary wave solution and a radiative correction which ultimately vanishes. Karpman and Maslov[13] have extensively considered the evolution of initial data given by Eq. (43) under the influence of a variety of perturbations. The bulk of their attention has been devoted to dissipative perturbations, and they have shown that these perturbations lead to radiation tails and soliton destruction. However, their approach applies equally well to the Hamiltonian perturbations considered in this paper. While in this paper we compare our results to those of Karpman and Maslov, we note that Kaup and Newell[14] independently obtained results equivalent to those of Karpman and Maslov, and our results could be just as well compared to those of Kaup and Newell.

Writing quite generally

$$u_t=6uu_x-u_{xxx}+\varepsilon R[u] , \tag{44}$$

Karpman and Maslov derive the relations for the case $u(x,0)=u_\alpha=-2\kappa_\alpha^2\operatorname{sech}^2(\kappa_\alpha x)$,

$$\dot{\kappa}_\alpha=-\frac{\varepsilon}{4}\int_{-\infty}^{\infty}d\xi\,R[u_\alpha]\operatorname{sech}^2(\kappa_\alpha\xi) , \tag{45a}$$

$$\dot{q}_\alpha=-8\kappa_\alpha^3+\frac{\varepsilon}{2\kappa_\alpha}\int_{-\infty}^{\infty}d\xi\,R[u_\alpha]\operatorname{sech}^2(\kappa_\alpha\xi)[\kappa_\alpha\xi+\tfrac{1}{2}\sinh(2\kappa_\alpha\xi)] , \tag{45b}$$

$$\dot{r}(k)=-8ik^3r(k)-\frac{i\varepsilon}{2k}\int_{-\infty}^{\infty}d\xi\,R[u_\alpha]\frac{[k+i\kappa_\alpha\tanh(\kappa_\alpha\xi)]}{(k-i\kappa_\alpha)^2}\exp(2ik\xi)\exp(-ikq_\alpha/\kappa_\alpha) . \tag{45c}$$

When comparing Eq. (45c) to Karpman and Maslov's work, it should be recalled that they use right Jost functions, while we use left Jost functions, so that their definition of $r(k)$ differs from ours.

In order to show the equivalence of Eq. (45) with our earlier results, we first note $R[u]=(\partial/\partial\xi)\delta H_1[u]/\delta u$. It follows, then, that

$$\dot{\kappa}_\alpha=\frac{\varepsilon}{8\kappa_\alpha^2}\int_{-\infty}^{\infty}d\xi\,u_\alpha\frac{d}{d\xi}\frac{\delta H_1[u_\alpha]}{\delta u}=-\frac{\varepsilon}{8\kappa_\alpha^2}\int_{-\infty}^{\infty}d\xi\frac{dH_1[u_\alpha]}{d\xi}=0 . \tag{46}$$

In similar fashion,

$$\dot{q}_\alpha+8\kappa_\alpha^3=-\frac{\varepsilon}{2\kappa_\alpha}\int_{-\infty}^{\infty}d\xi\left[\frac{d^2}{d\xi^2}\xi\tanh(\kappa_\alpha\xi)\right]\frac{\delta H_1[u_\alpha]}{\delta u}=\varepsilon\int_{-\infty}^{\infty}d\xi\frac{\delta u_\alpha}{\delta p_\alpha}\frac{\delta H_1[u_\alpha]}{\delta u}=\varepsilon\frac{\partial H_1}{\partial p_\alpha} . \tag{47}$$

Finally, to obtain Eq. (45c), we work backwards, starting from the expressions

$$\delta u=-2\frac{d}{d\xi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}r(k)\frac{[k-ik_\alpha\tanh(\kappa_\alpha\xi)]^2}{(k+i\kappa_\alpha)^2}\exp(-2ik\xi)\exp(ikq_\alpha/\kappa_\alpha) \tag{48}$$

and

$$\begin{aligned}\dot{r}(k)+8ik^3r(k)&=\varepsilon[r(k),H_1]\\&=\varepsilon\int_{-\infty}^{\infty}d\xi\frac{\delta H_1[u_\alpha]}{\delta u}[r(k),\delta u]\\&=\frac{i\varepsilon}{2k}\int_{-\infty}^{\infty}d\xi\frac{\delta H_1[u_\alpha]}{\delta u}\frac{d}{d\xi}\left[\frac{[k+i\kappa_\alpha\tanh(\kappa_\alpha\xi)]^2}{(k-i\kappa_\alpha)^2}\exp(2ik\xi)\exp(-ikq_\alpha/\kappa_\alpha)\right] .\end{aligned} \tag{49}$$

Integrating by parts, we obtain Eq. (45c). Additionally, Karpman and Maslov have shown that the amplitude of the radiation tails produced by the perturbations is proportional to

$$q=\frac{1}{4\kappa_\alpha^4}\int_{-\infty}^{\infty}d\xi\,R[u_\alpha]\tanh^2(\kappa_\alpha\xi) . \tag{50}$$

By analogy with Eq. (46), one readily obtains $q=0$ for Hamiltonian perturbations, indicating that no radiation tails are produced, consistent with this paper's result.

The perturbation approach of Karpman and Maslov has the advantage over ours that it allows one to treat both dissipative and Hamiltonian perturbations, rather than just the latter. Unfortunately, their approach tends to obscure the very important qualitative distinction between the two sorts of perturbations. Additionally, it is very difficult to extend their approach to higher order in $\varepsilon |r(k)|$, or even to deal with more general pulses than those which initially have the form u_α. In particular, it is very difficult to extract the solitary wave solutions which we found in the preceding section. Using the Hamiltonian approach, it is possible to treat the solitary wave solutions and the solutions for which $u=u_\alpha$ at $t=0$ in a unified fashion. For Hamiltonian perturbations, the Hamiltonian approach is thus simpler to use, particularly at high order, and more revealing.

V. CONCLUSIONS

In experimental observations of ion acoustic plasma waves or shallow-channel water waves, it is observed that solitons emerge from initial data even when the Hamiltonian deviations from the Korteweg–de Vries equation are quite large, although their velocities and shapes can be substantially changed. By contrast, only a relatively small amount of dissipation is needed to prevent the emergence of solitons.

To explain this result in a qualitative fashion, we[11] have shown how to construct a Hamiltonian perturbation theory for a large class of possible perturbations. We then demonstrated that solitons emerge to all orders in the small parameters. In this paper, we explicitly apply this approach to the lowest order perturbations which appear in ion acoustic plasma-wave and shallow-channel water-wave systems. The solitary wave solution is obtained and it is shown that for initial data close to this solitary wave, the solitary wave will always emerge from the initial data.

Comparisons are made to the perturbation approaches of Kodama and Taniuti,[15] which can be used to obtain the solitary wave solution, and the approach of Karpman and Maslov,[13] which is used to study pulses for which $u=u_\alpha$ at $t=0$. The approach of Kodama and Taniuti[15] is less algebraically complex but does not allow one to relate the solitary wave solution to the initial data. The approach of Karpman and Maslov[13] is more algebraically complex than the Hamiltonian approach and less revealing. It is, however, not restricted to systems which are Hamiltonian.

Much work remains to be done—experimentally, computationally, and theoretically—before a full quantitative understanding of how solitons emerge from arbitrary initial data in the "real" world of experiments is achieved. This work represents a step along that path.

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APPENDIX

In obtaining Eq. (18) from Eq. (17), we used the relations

$$\int_{-\infty}^{\infty} d\xi\,\operatorname{sech}^{2j}(\kappa\xi)\tanh(\kappa\xi)\exp(-2ik\xi) = -\frac{i\pi 2^{2j}}{\kappa[(2j)!]}\prod_{i=1}^{j}\left[\frac{k^2}{\kappa^2}+(i-1)^2\right]\operatorname{csch}(\pi k/\kappa)\,,$$
$$\int_{-\infty}^{\infty} d\xi\,\operatorname{sech}^{2j}(\kappa\xi)\exp(-2ik\xi) = \frac{\pi 2^{2j}}{k(2j-1)!}\prod_{i=1}^{j}\left[\frac{k^2}{\kappa^2}+(i-1)^2\right]\operatorname{csch}(\pi k/\kappa)\,, \tag{A1}$$

which may be obtained from the basic integral

$$\int_{-\infty}^{\infty} d\xi\,\tanh(\kappa\xi)\exp(-2ik\xi) = -\frac{\pi i}{\kappa}\operatorname{csch}(\pi k/\kappa) \tag{A2}$$

by repeated integration by parts. Equation (A2) can be found by using contour integration.

In obtaining Eq. (34) from Eq. (33), we used the basis integrals

$$\int_{-\infty}^{\infty} dk\,\frac{\operatorname{csch}(\pi k/\kappa)}{(k^2+\kappa^2)}[k-i\kappa\tanh(\kappa\xi)]^2\exp(-2ik\xi) = i\kappa[1+\tanh(\kappa\xi)-(1+2\kappa\xi)\operatorname{sech}^2(\kappa\xi)]\,, \tag{A3a}$$

$$\int_{-\infty}^{\infty} dk\,\operatorname{csch}(\pi k/\kappa)[k-i\kappa\tanh(\kappa\xi)]^2\exp(-2ik\xi) = i\kappa^3[1+\tanh(\kappa\xi)-\operatorname{sech}^2(\kappa\xi)-\tfrac{5}{2}\operatorname{sech}^2(\kappa\xi)\tanh(\kappa\xi)]\,. \tag{A3b}$$

In evaluating the integrals in Eq. (A3), we integrated over the poles at $k=0$.

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