SU(2) and SU(1,1) interferometers

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A Lie-group-theoretical approach to the analysis of interferometers is presented. Conventional interferometers such as the Mach-Zehnder and Fabry-Perot can be characterized by SU(2). We introduce a class of interferometers characterized by SU(1,1). These interferometers employ active elements such as four-wave mixers or degenerate-parametric amplifiers in their construction. Both the SU(2) and SU(1,1) interferometers can in principle achieve a phase sensitivity $\Delta\phi$ approaching $1/N$, where N is the total number of quanta entering the interferometer, provided that the light entering the input ports is prepared in a suitable quantum state. SU(1,1) interferometers can achieve this sensitivity with fewer optical elements.

I. INTRODUCTION

In a conventional interferometer such as the Mach-Zehnder[1] depicted in Fig. 1 light is fed into one of the input ports. The light beam is split into two beams which propagate along different paths and suffer a phase shift relative to each other of ϕ. The light beams are combined and interfere with each other at a second beam splitter. The relative phase shift ϕ can be determined by measuring the position of the interference fringes in the output beams. Such an interferometer can achieve a phase sensitivity

$$\Delta\phi = \frac{1}{\sqrt{N}} , \tag{1.1}$$

where N is the total number of photons that have passed through the interferometer during the measurement time. Caves[2] has pointed out that by feeding suitably constructed squeezed states into both input ports of the interferometer the phase sensitivity can approach

$$\Delta\phi = \frac{1}{N} . \tag{1.2}$$

Bondurant and Shapiro[3,4] and Ni[5] have also investigated the use of squeezed states in increasing interferometer sensitivity.

The interferometers considered by Caves[2] and Bondurant and Shapiro[3,4] were primarily passive lossless devices with two input ports and two output ports. We will show that the group SU(2) naturally characterizes such interferometers and present group-theoretical arguments indicating the ultimate sensitivity that can be achieved by such devices. We will then introduce a class of active lossless interferometers characterized by the group SU(1,1). In these devices the interference arises not from recombining light beams via beam splitters, but from the phase-sensitive response of active elements such as degenerate-parametric amplifiers and four-wave mixers. In contrast to SU(2) interferometers, SU(1,1) interferometers can achieve a phase sensitivity of $1/N$ with only vacuum fluctuations entering the input ports and coherent light pumping the active devices. SU(1,1) interferometers can achieve a phase sensitivity of $1/N$ with fewer optical elements than the SU(2) interferometers and hence present a more practical way of doing sensitive interferometry, once sufficiently low-noise parametric amplifiers or four-wave mixers become available.

II. SU(2) CHARACTERIZATION OF PASSIVE LOSSLESS DEVICES WITH TWO INPUT AND TWO OUTPUT PORTS

In this section the connection between a linear lossless passive device having two input ports and two output ports and the group SU(2) is presented. Since SU(2) is equivalent to the rotation group in three dimensions, this will allow one to visualize the operations of beam splitters and phase shifters as rotations in 3-space. This insight will be exploited in the next section to discuss the performance of the Mach-Zehnder interferometer.

Let a_1 and a_2 denote the annihilation operators for two light beams which may be, for example, the two light beams entering a beam splitter or the two light beams leaving a beam splitter. These operators and their Hermitian conjugates satisfy the boson commutation relations:

$$\begin{aligned} &[a_i,a_j]=[a_i^\dagger,a_j^\dagger]=0 , \\ &[a_i,a_j^\dagger]=\delta_{ij} , \end{aligned} \tag{2.1}$$

where i and j take on the values 1 and 2. One can introduce the Hermitian operators

$$\begin{aligned} J_x &= \tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) , \\ J_y &= -\frac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1) , \\ J_z &= \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) , \end{aligned} \tag{2.2}$$

and

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 . \tag{2.3}$$

The operators (2.2) satisfy the commutation relations for the Lie algebra of SU(2):

$$[J_x, J_y] = iJ_z \ ,$$
$$[J_y, J_z] = iJ_x \ , \tag{2.4}$$
$$[J_z, J_x] = iJ_y \ .$$

The Casimir invariant for this group, using (2.2) and (2.3), can be put into the form

$$J^2 = \frac{N}{2}\left[\frac{N}{2}+1\right] , \tag{2.5}$$

in fact, N itself commutes with all the operators of (2.2).

Why one should want to characterize a lossless passive device with two input ports and two output ports with the operators (2.2) and (2.3) will now be explained. Let $a_{1\,\mathrm{in}}$ and $a_{2\,\mathrm{in}}$ denote the annihilation operators for the light entering the two input ports and similarly let $a_{1\,\mathrm{out}}$ and $a_{2\,\mathrm{out}}$ denote the annihilation operators for two light beams leaving the two output ports. The scattering matrix for the device will have the form

$$\begin{bmatrix} a_{1\,\mathrm{out}} \\ a_{2\,\mathrm{out}} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} a_{1\,\mathrm{in}} \\ a_{2\,\mathrm{in}} \end{bmatrix} . \tag{2.6}$$

Since the creation and annihilation operators for the two input beams and the two output beams must satisfy (2.1) the matrix

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \tag{2.7}$$

must be unitary. Such a transformation will in general transform J_x, J_y, and J_z among themselves.

How $\mathbf{J}=(J_x, J_y, J_z)$ transforms under U will now be determined for some common optical elements.

Consider a beam splitter with the scattering matrix

$$U = \begin{bmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix} . \tag{2.8}$$

This transformation will transform $\mathbf{J}$ according to

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{out}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{in}} . \tag{2.9}$$

That is, the abstract angular momentum vectors are rotated about the x axis by an angle α. This transformation can be expressed in the form

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{out}} = e^{i\alpha J_x} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} e^{-i\alpha J_x} , \tag{2.10}$$

where the angular momentum operators on the right-hand side are evaluated for the input beams $a_{1\,\mathrm{in}}$ and $a_{2\,\mathrm{in}}$. The equivalence of (2.9) and (2.10) can be checked using the operator identity

$$e^{\xi A} B e^{-\xi A} = B + \xi[A,B] + \frac{\xi^2}{2!}[A,[A,B]] + \cdots \ . \tag{2.11}$$

One can alternatively work in a Schrödinger picture where the operators J_x, J_y, and J_z remain unchanged but the state vector, after interacting with the beam splitter, becomes

$$|\,\mathrm{out}\rangle = e^{-i\alpha J_x}\,|\,\mathrm{in}\rangle \ , \tag{2.12}$$

where $|\,\mathrm{in}\rangle$ is the state vector for the light before it has interacted with the beam splitter. Throughout this paper we will hop back and forth between the Heisenberg picture where $\mathbf{J}$ is rotated while the state vector remains fixed and the Schrödinger picture where $\mathbf{J}$ remains fixed depending on which picture is most convenient for the discussion at hand.

Another realizable scattering matrix for a beam splitter is

$$U = \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} . \tag{2.13}$$

At radio frequencies devices with the scattering matrix Eq. (2.8) and Eq. (2.13) would be distinguished, respectively, as 90° and 180° couplers. The scattering matrix Eq. (2.13) transforms $\mathbf{J}$ according to

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{out}} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{in}} . \tag{2.14}$$

This transformation represents a rotation of $\mathbf{J}$ about the y axis by an angle β. This transformation can be written as

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{out}} = e^{i\beta J_y} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} e^{-i\beta J_y} . \tag{2.15}$$

Hence in the Schrödinger picture where $\mathbf{J}$ remains fixed the state vector for the light after interacting with the beam splitter is

$$|\,\mathrm{out}\rangle = e^{-i\beta J_y}\,|\,\mathrm{in}\rangle \ . \tag{2.16}$$

How $\mathbf{J}$ transforms under a phase shift or change in optical path length is now determined. Let light beams 1 and 2 incur a phase shift γ_1 and γ_2, respectively. The unitary matrix associated with this process is

$$U = \begin{bmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{bmatrix} . \tag{2.17}$$

Under this transformation $\mathbf{J}$ transforms as

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\mathrm{out}} = \begin{bmatrix} \cos(\gamma_2-\gamma_1) & -\sin(\gamma_2-\gamma_1) & 0 \\ \sin(\gamma_2-\gamma_1) & \cos(\gamma_2-\gamma_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} . \tag{2.18}$$

This represents a rotation about the z axis by the angle

$\gamma_2-\gamma_1$ corresponding to the relative phase shift between the two light beams. This transformation can be expressed as

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{\text{out}} = e^{i(\gamma_2-\gamma_1)J_z} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} e^{-i(\gamma_2-\gamma_1)J_z} . \quad (2.19)$$

Hence in the Schrödinger picture this represents a transformation of the incoming state vector $|\text{in}\rangle$ according to

$$|\text{out}\rangle = e^{-i(\gamma_2-\gamma_1)J_z}|\text{in}\rangle . \quad (2.20)$$

It is worth noting that under the full transformation Eq. (2.17) the incoming state transforms as $|\text{out}\rangle = e^{i(\gamma_1+\gamma_2)N/2}e^{-i(\gamma_2-\gamma_1)J_z}|\text{in}\rangle$ but since N commutes with $\mathbf{J}$ the operator $e^{i(\gamma_1+\gamma_2)N/2}$ gives rise to phase factors which do not contribute to the expectation values or moments of number-conserving operators such as $\mathbf{J}$ and N. In fact, it is the insensitivity of photodetectors (photon counters) to the extra phase $e^{i(\gamma_1+\gamma_2)N/2}$ that allows one to fully characterize an interferometer by the SU(2) transformations described above. It has now been shown that the transformations the beam splitters and phase shifters perform on the two incoming light beams can be visualized as rotations of the vector $\mathbf{J}$. Further, since the operator ($a_1^\dagger a_1$ or $a_2^\dagger a_2$) characterizing the number of photons counted by a photodetector placed in one of the light beams can be expressed in terms of the operator N and J_z, interferometry can be visualized as the process of measuring rotations of $\mathbf{J}$. The operators giving rise to the mode transformations of Eqs. (2.12), (2.16), and (2.20) have also been recently discussed by Schumaker in Ref. 6 where they are referred to as two-mode mixing operators.

III. THE MACH-ZEHNDER INTERFEROMETER

The formalism of the last section will now be applied to the Mach-Zehnder[7] interferometer. This interferometer is depicted in Fig. 1. It consists of two 50-50 beam splitters S1 and S2. The relative phase shift $\phi=\phi_2-\phi_1$ is measured by observing the interference fringes in the light leaving S2. Here, as depicted in Fig. 1, the case will be considered where the photodetector is placed in each of the two output beams $a_{1\,\text{out}}$ and $a_{2\,\text{out}}$. By counting the number of photoelectrons generated by each detector, D_1 and D_2, separately, one measures the operators $N_1=a_{1\,\text{out}}^\dagger a_{1\,\text{out}}$ and $N_2=a_{2\,\text{out}}^\dagger a_{2\,\text{out}}$. From Eqs. (2.2) and (2.3) one sees that this is equivalent to measuring both N_{out} and $J_{z\,\text{out}}$.

A geometrical picture of the operation of the interferometer will now be developed. For definiteness the beam splitters S1 and S2 will be chosen to have scattering matrices of the form (2.8). For a 50-50 beam splitter α must take on the value $\pi/2$ or $-\pi/2$. For the beam splitter S1 we take $\alpha=+\pi/2$, for the beam splitter S2 we take $\alpha=-\pi/2$. Let $|\text{in}\rangle$ denote the state vector for the light in the two light beams entering the interferometer. From Eq. (2.12) the state $|\psi\rangle$ of the light

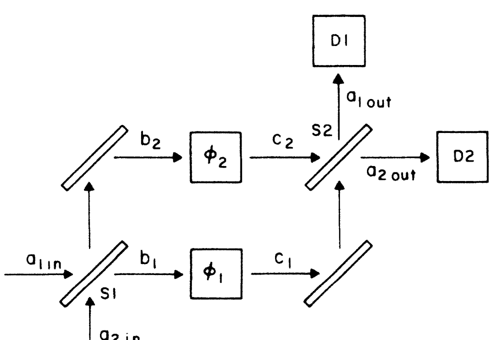

FIG. 1. A Mach-Zehnder interferometer. Light entering one of the two input ports $a_{1\,\text{in}}$ or $a_{2\,\text{in}}$ is split into two beams by beam splitter S1. The two light beams b_1 and b_2 accumulate a phase shift ϕ_1 and ϕ_2, respectively, before entering beam splitter S2. The photons leaving the interferometer are counted by detectors D1 and D2.

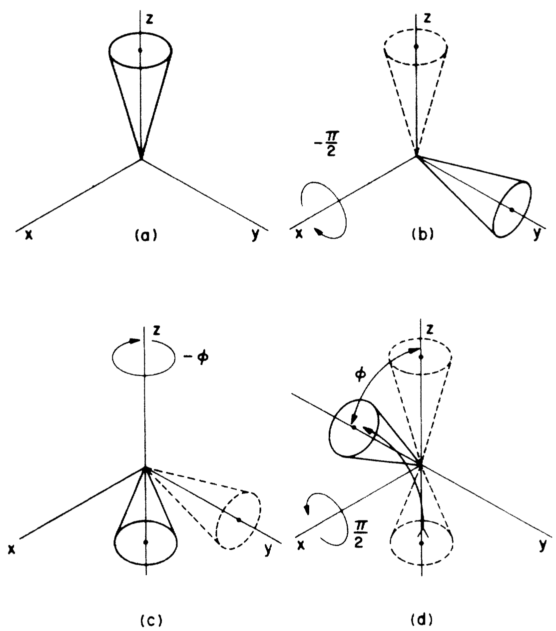

FIG. 2. A rotation-group picture of the performance of a Mach-Zehnder interferometer. When light enters only one input port of the interferometer the input state has the form $|j,m\rangle=|j,j\rangle$ in a fictitious (J_x,J_y,J_z) space and can be represented by a cone centered along the z axis with height j (a). The first beam splitter performs a $-\pi/2$ rotation about the x axis. The cone now lies along the y axis (b). The phase shifts accumulated by the two light beams in the interferometer correspond to a rotation $-\phi$ about the z axis (c). The second beam splitter performs a $\pi/2$ rotation about the x axis (d). Since J_z is proportional to the difference in the number of photons counted by the two photodetectors in the interferometer output beam the interferometer can resolve states whose overall rotation is sufficiently far from the z axis so that on average the J_z measured will differ from j by one. In order for this to be the case the cone must be rotated by approximately the width of its base which is $\sqrt{j}$. Hence the minimum detectable ϕ is of order $1/\sqrt{j}$.

upon leaving the S1 is

$$|\psi\rangle = e^{-i(\pi/2)J_x}|\text{in}\rangle \tag{3.1}$$

which amounts to a rotation of the state vector about the x axis by an amount $-\pi/2$. This is depicted in Figs. 2(a) and 2(b) where for definiteness $|\text{in}\rangle$ was chosen to be the state $|j,m=j\rangle$, that is, **J** lies on the circle surrounding the base of the cone in Fig. 2(a). With a $-\pi/2$ rotation about the x axis this cone now lies along the y axis.

Upon reaching the input ports of S2 one light beam c_1 has undergone a phase shift of ϕ_1 while the other c_2 has undergone a phase shift ϕ_2. Thus, from Eq. (2.19), upon arriving at S2 the light is in the state $|\psi'\rangle$:

$$\begin{aligned}|\psi'\rangle &= e^{-i(\phi_2-\phi_1)J_z}|\psi\rangle \\ &= e^{-i(\phi_2-\phi_1)J_z}e^{-i(\pi/2)J_x}|\text{in}\rangle \ .\end{aligned} \tag{3.2}$$

Hence, as depicted in Fig. 2(c), the phase shift rotates the state vector about the z axis by an amount $-\phi=-(\phi_2-\phi_1)$.

The second beam splitter rotates the state vector $|\psi'\rangle$ about the x axis by an amount $\pi/2$. The state vector $|\text{out}\rangle$ for the light leaving the interferometer is thus

$$\begin{aligned}|\text{out}\rangle &= e^{i(\pi/2)J_x}|\psi'\rangle \\ &= e^{i(\pi/2)J_x}e^{-i(\phi_2-\phi_1)J_z}e^{-i(\pi/2)J_x}|\text{in}\rangle \ .\end{aligned} \tag{3.3}$$

As shown in Fig. 2(d), the net result of this sequence of rotations is a rotation of the initial state vector about the y axis by an amount ϕ.

As pointed out earlier, by placing photodetectors in both of the output beams one can measure both N (the total number of photons passing through the interferometer) and J_z (the difference in the number of photons arriving at each detector divided by 2). Because N commutes with **J** it by itself gives one no useful information about ϕ. It does, however, give one useful information about $|\text{in}\rangle$, in particular the total number of photoelectrons n counted after the light has passed through the interferometer tells one that $|\text{in}\rangle$ was in an eigenstate of N:

$$N|\text{in}\rangle = n|\text{in}\rangle \ . \tag{3.4}$$

From this, using (2.5) one concludes

$$j(j+1)=\frac{n}{2}\left[\frac{n}{2}+1\right] \tag{3.5}$$

or

$$j=\frac{n}{2} \ , \tag{3.6}$$

that is, $|\text{in}\rangle$ was an eigenstate of J^2.

The other variable measured, J_z, allows one to infer what the value of ϕ was. In particular,

$$\langle\text{out}|J_z|\text{out}\rangle = \langle\text{in}|e^{i(\pi/2)J_x}e^{i\phi J_z}e^{-i(\pi/2)J_x}J_z e^{i(\pi/2)J_x}e^{-i\phi J_z}e^{-i(\pi/2)J_x}|\text{in}\rangle \ . \tag{3.7}$$

One can show

$$e^{i(\pi/2)J_x}e^{i\phi J_z}e^{-i(\pi/2)J_x}J_z e^{i(\pi/2)J_x}e^{-i\phi J_z}e^{-i(\pi/2)J_x} = -(\sin\phi)J_x+(\cos\phi)J_z \ . \tag{3.8}$$

Hence

$$\begin{aligned}\langle J_z\rangle &= \langle\text{out}|J_z|\text{out}\rangle \\ &= -\sin\phi\langle\text{in}|J_x|\text{in}\rangle+\cos\phi\langle\text{in}|J_z|\text{in}\rangle\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\langle J_z^2\rangle &= \langle\text{out}|J_z^2|\text{out}\rangle \\ &= \sin^2\phi\langle\text{in}|J_x^2|\text{in}\rangle \\ &\quad -\sin\phi\cos\phi\langle\text{in}|J_xJ_z+J_zJ_x|\text{in}\rangle \\ &\quad +\cos^2\phi\langle\text{in}|J_z^2|\text{in}\rangle \ .\end{aligned} \tag{3.10}$$

To proceed further one needs additional information on $|\text{in}\rangle$. Let us suppose the interferometer is operated in the usual manner where light enters the interferometer only along one of the input beam paths, say a_1. Then from the total number of photons n counted by D1 and D2 one knows that there were n photons in the incoming light beam. Hence $|\text{in}\rangle$ is an eigenstate of J_z:

$$J_z|\text{in}\rangle = \frac{n}{2}|\text{in}\rangle \ . \tag{3.11}$$

From (3.6) and (3.11) one concludes that the incoming light beam was in an eigenstate of $|j,m\rangle$

$$|\text{in}\rangle = |j=n/2, m=n/2\rangle \ . \tag{3.12}$$

Hence the incoming light is in the eigenstate that was depicted in Fig. 2.

Intuitively the smallest ϕ that can be measured is one where the cones of Fig. 2(d) do not appreciably overlap. The distance from the apex of one of the cones to a point on the circle of the cone's base is the square root of the eigenvalue of J^2 or $\sqrt{j(j+1)}$. The distance from the apex of one of the cones to the center of its base is the eigenvalue of J_z or j. Hence the radius of the base of one of the cones is $[j(j+1)-j^2]^{1/2}=\sqrt{j}$. The minimum detectable ϕ is thus of order $\phi_{\min}\simeq j^{-1/2}$, and since from Eq. (3.12) $j=n/2$,

$$\phi_{\min}\simeq n^{-1/2} \ . \tag{3.13}$$

Hence the sensitivity of an interferometer operated in the mode where light enters only one of the two input ports has a sensitivity that goes as the square root of the number of photons passing through the interferometer.

Equation (3.13) is now made more rigorous by a direct calculation from Eq. (3.9) and Eq. (3.10). For the state (3.12) the mean value of J_z is

$$\bar{J}_z = \frac{n}{2}\cos\phi \ . \tag{3.14}$$

The mean-square fluctuation $(\Delta J_z)^2$ about this value is

$$(\Delta J_z)^2 \equiv \overline{J_z^2} - \bar{J}_z^2 = \frac{n}{4}\sin^2\phi \; . \tag{3.15}$$

The mean-square noise in ϕ is thus

$$(\Delta\phi)^2 = \frac{(\Delta J_z)^2}{\left[\dfrac{\partial \bar{J}_z}{\partial\phi}\right]^2} = \frac{1}{n} \; . \tag{3.16}$$

Hence the rms fluctuation of ϕ due to photon noise goes as $n^{-1/2}$,

$$\Delta\phi = n^{-1/2} \tag{3.17}$$

in agreement with the intuitive argument based on Fig. 2. We will refer to (3.17) as the "standard noise limit" for an interferometer.

Note that no assumption was made about the quantum statistics of the source of light entering the interferometer. The total number of photons n entering the interferometer completely characterizes the ultimate sensitivity that can be achieved with an interferometer in which light is fed into only one input port. If instead of using photodetectors in both output ports and measuring N and J_z one chooses to use only one photodetector or to measure only J_z, then one is throwing away information. In this case knowledge about the photon statistics of the source becomes important. For this situation the performance of the interferometer will generally degrade although for some particular values of $\phi_1 - \phi_2$ the $n^{1/2}$ phase sensitivity can still be achieved.

As will be pointed out in Sec. V, the sensitivity of an interferometer can be dramatically improved if photons are allowed to enter both input ports provided the photons are prepared in the right quantum state.

IV. THE FABRY-PEROT INTERFEROMETER

In the last section a geometrical picture of the operation of the Mach-Zehnder interferometer was presented in terms of rotations of the operators (J_x, J_y, J_z) defined by Eq. (2.2). Photodetectors placed in the output beam of the interferometer measure the operator J_z. A relative phase shift between two optical beams produces a rotation of $\mathbf{J}$ about the z axis, see Eq. (2.19). A measurement of J_z, however, is only sensitive to rotations in a plane containing the z axis. The function of the two 50-50 beam splitters is thus to convert a rotation about the z axis into a rotation in a plane containing the z axis. For the particular set of beam splitters chosen in the last section this corresponds to a net rotation in the x-z plane as depicted in Fig. 2.

The Fabry-Perot interferometer,[7] by employing semitransparent mirrors, also converts a rotation about the x axis into a rotation lying in a plane containing the z axis. One can also take advantage of the multiple passes of the light between the two mirrors to enhance the sensitivity of the interferometer, but at the expense of the interferometer's bandwidth. Here expressions are obtained for the phase sensitivity of a Fabry-Perot.

A Fabry-Perot interferometer is depicted in Fig. 3(a). It consists of semitransparent mirrors M1 and M2. This interferometer measures the phase shift ϕ suffered by light as it propagates from one mirror to the other. This device has two input ports $a_{1\,\mathrm{in}}$ and $a_{2\,\mathrm{in}}$, and two output ports $a_{1\,\mathrm{out}}$ and $a_{2\,\mathrm{out}}$. Although $a_{1\,\mathrm{in}}$ and $a_{2\,\mathrm{out}}$ and $a_{2\,\mathrm{in}}$ and $a_{1\,\mathrm{out}}$ are collinear they can be separated with optical circulators as shown in Fig. 3(b). In this manner one can place photodetectors in both beams $a_{1\,\mathrm{out}}$ and $a_{2\,\mathrm{out}}$ without obstructing the light injected into $a_{1\,\mathrm{in}}$ or $a_{2\,\mathrm{in}}$. Hence one is allowed to measure N and J_z for the two output beams.

An analysis of the Fabry-Perot interferometer is now carried out. The mirrors M1 and M2 will be taken to have scattering matrices of the form Eq. (2.8). In particular for the mirror M1 we take

$$\begin{aligned} b_1 &= \cos(\tfrac{1}{2}\beta)a_{1\,\mathrm{in}} + i\sin(\tfrac{1}{2}\beta)b_2 \; , \\ a_{2\,\mathrm{out}} &= +i\sin(\tfrac{1}{2}\beta)a_{1\,\mathrm{in}} + \cos(\tfrac{1}{2}\beta)b_2 \; , \end{aligned} \tag{4.1}$$

and for the mirror M2

$$\begin{aligned} a_{1\,\mathrm{out}} &= \cos(\tfrac{1}{2}\beta)c_1 - i\sin(\tfrac{1}{2}\beta)a_{2\,\mathrm{in}} \; , \\ c_2 &= -i\sin(\tfrac{1}{2}\beta)c_1 + \cos(\tfrac{1}{2}\beta)a_{2\,\mathrm{in}} \; . \end{aligned} \tag{4.2}$$

In writing (4.1) and (4.2) it has been assumed that both mirrors have the same transmission coefficient

$$T = \cos^2(\tfrac{1}{2}\beta) \; . \tag{4.3}$$

The phase shift θ sustained by the light as it propagates between the two mirrors is given by

$$\begin{aligned} c_1 &= e^{i\theta}b_1 \; , \\ b_2 &= e^{i\theta}c_2 \; . \end{aligned} \tag{4.4}$$

Equations (4.1)–(4.4) can be solved to obtain $a_{1\,\mathrm{out}}, a_{2\,\mathrm{out}}$ in terms of $a_{1\,\mathrm{in}}$ and $a_{2\,\mathrm{in}}$. One finds

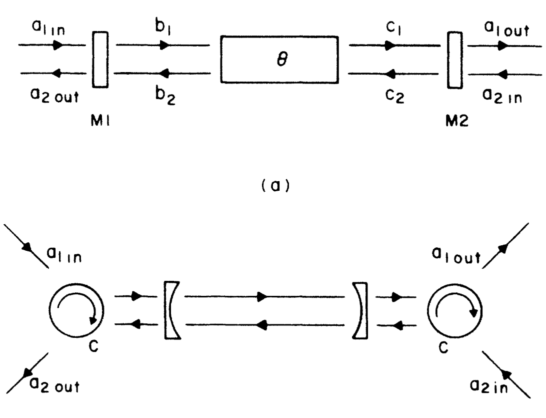

FIG. 3. A Fabry-Perot interferometer. The device (a) consists of two semitransparent mirrors M1 and M2. The light propagating between M1 and M2 suffers a one-way phase shift θ. In (b) circulators C have been placed behind the mirrors to physically separate the incoming and outgoing light beams. It is evident that the Fabry-Perot interferometer has two input ports and two output ports.

$$\begin{pmatrix} a_{1\,\text{out}} \\ a_{2\,\text{out}} \end{pmatrix} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \begin{pmatrix} a_{1\,\text{in}} \\ a_{2\,\text{in}} \end{pmatrix} , \tag{4.5}$$

where

$$\mu = \frac{\cos^2(\frac{1}{2}\beta)e^{i\theta}}{1-\sin^2(\frac{1}{2}\beta)e^{2i\theta}} ,$$
$$\nu = -\frac{i\sin(\frac{1}{2}\beta)(e^{2i\theta}-1)}{1-\sin^2(\frac{1}{2}\beta)e^{2i\theta}} . \tag{4.6}$$

The scattering matrix of Eq. (4.5) is unitary. Using Eq. (2.2) one can determine how **J** transforms under this unitary transformation. One finds

$$\begin{pmatrix} J_x \\ J_y \\ J_x \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}_{\text{in}} , \tag{4.7}$$

where

$$\cos\phi = |\mu|^2 - |\nu|^2 ,$$
$$\sin\phi = \mu^*\nu + \nu^*\mu . \tag{4.8}$$

So the Fabry-Perot interferometer, for the mirrors chosen, performs a rotation of **J** about the y axis. Hence, following the same line of reasoning as in the last section, if light enters the Fabry-Perot in only one input port the ultimate phase sensitivity $\Delta\phi$ is given by

$$\Delta\phi = n^{-1/2} . \tag{4.9}$$

In order to determine what this implies for the ultimate phase sensitivity $\Delta\theta$ one needs to evaluate $|d\phi/d\theta|$. With Eq. (4.6), Eq. (4.8) becomes

$$\cos\phi = \frac{\cos^4(\frac{1}{2}\beta) - 4\sin^2(\frac{1}{2}\beta)\sin^2\theta}{\cos^4(\frac{1}{2}\beta) + 4\sin^2(\frac{1}{2}\beta)\sin^2\theta} , \tag{4.10}$$

$$\sin\phi = +\frac{4\cos^2(\frac{1}{2}\beta)\sin(\frac{1}{2}\beta)\sin\theta}{\cos^4(\frac{1}{2}\beta) + 4\sin^2(\frac{1}{2}\beta)\sin^2\theta} , \tag{4.11}$$

and thus ϕ is given by

$$\phi = \arctan\left[\frac{4\cos^2(\frac{1}{2}\beta)\sin(\frac{1}{2}\beta)\sin\theta}{\cos^4(\frac{1}{2}\beta) - 4\sin^2(\frac{1}{2}\beta)\sin^2\theta}\right] . \tag{4.12}$$

Differentiating this equation with respect to θ one obtains

$$\frac{d\phi}{d\theta} = \frac{4\cos^2(\frac{1}{2}\beta)\sin(\frac{1}{2}\beta)\cos\theta}{\cos^4(\frac{1}{2}\beta) + 4\sin^2(\frac{1}{2}\beta)\sin^2\theta} . \tag{4.13}$$

The Fabry-Perot is most sensitive for those angles θ for which $|d\phi/d\theta|$ is maximized. From Eq. (4.13) one sees that the sensitivity is greatest when $|\cos\theta| = 1$ and $\sin\phi = 0$. So

$$\left|\frac{d\phi}{d\theta}\right|_{\max} = \frac{4|\sin(\frac{1}{2}\beta)|}{\cos^2(\frac{1}{2}\beta)} , \tag{4.14}$$

or in terms of the transmission coefficient T for the mirrors

$$\left|\frac{d\phi}{d\theta}\right|_{\max} = \frac{4(1-T)^{1/2}}{T} . \tag{4.15}$$

Hence the smallest rms fluctuations in $\Delta\theta$ achieved by a Fabry-Perot is

$$\Delta\theta_{\min} = \frac{\Delta\phi}{\left|\frac{d\phi}{d\theta}\right|_{\max}} = \frac{T\Delta\phi}{4(1-T)^{1/2}} . \tag{4.16}$$

For mirrors with a small transmission coefficient T, and using (4.9)

$$\Delta\theta_{\min} \simeq \frac{T}{4n^{1/2}} . \tag{4.17}$$

Hence, as with the Mach-Zehnder, the sensivity of the interferometer scales as $n^{-1/2}$ where n is the total number of photons entering the interferometer. As with the derivation of (3.17), Eq. (4.17) is based on the assumption that light enters only one port of the interferometer.

In the next section it is shown that the sensitivity of an interferometer can be greatly enhanced if light, prepared in a suitable quantum state, is allowed to enter both ports of the interferometer. Although the arguments will be applied to the Mach-Zehnder, with the tools developed in this section, they can be applied to the Fabry-Perot interferometer as well.

V. SURPASSING THE STANDARD NOISE LIMIT

In the last two sections it was shown that an interferometer can be regarded as a device which performs rotations on the operators (J_x, J_y, J_z) defined by Eq. (2.2). Photodetectors placed in the output beam of the interferometer measure the operator J_z. Hence the overall rotation must lie in a plane containing the z axis. For the choice of beam splitters used in the Mach-Zehnder of Sec. III and the mirrors used in the Fabry-Perot of Sec. IV this rotation was in the x-z plane. [See Eqs. (3.8) and (4.7)].

For the Mach-Zehnder the sequence of rotations performed is depicted in Fig. 2. A cone was used to represent a J_z eigenstate. Based on how such an object transforms under the rotations performed by the interferometer a minimum detectable phase shift of order $n^{-1/2}$ was derived. As will be shown, a J_z eigenstate is not the optimum eigenstate for interferometry. In particular, by forming a linear superposition of J_z eigenstates near $m = 0$, one might imagine constructing a squashed cone or "fan-shaped" state lying in the x-y plane as depicted in Fig. 4(a). Such a state constructed from a superposition of J_z eigenstates near $m = 1$ would have an extent along the z axis of order unity. Figure 4 indicates how such a geometrical object would transform under the rotations performed by the interferometer. Since the extent of the state along the z axis is ~ 1 and the distance from the origin to the edge of the cone is $\sim j$, Fig. 4(d) would indicate that the minimal detectable ϕ is $\phi_{\min} \sim 1/j$. Or, from Eq. (3.6),

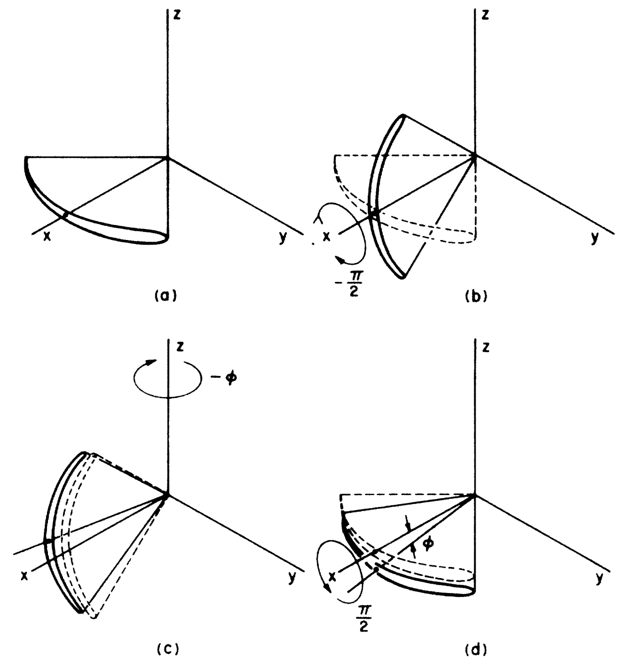

FIG. 4. The performance of a Mach-Zehnder interferometer in which an input state, of length j when depicted in the (J_x,J_y,J_z) space, is a flattened cone whose width along the z axis is of order unity. The sequence of rotations performed by the interferometer is the same as that of Fig. 2. In contrast to the state depicted in Fig. 2, an overall rotation $\phi\sim 1/j$ can be resolved with the state depicted here.

$$\phi_{\min}\sim 1/n \ . \tag{5.1}$$

Hence, by choosing the appropriate incoming state $|\,\text{in}\rangle$, an interferometer's sensitivity can be greatly improved over the $n^{-1/2}$ sensitivity of Eq. (3.17) or Eq. (4.9).

The above discussion is now made rigorous by explicitly exhibiting a state with the properties described above. Consider the state

$$|\,\text{in}\rangle=\frac{1}{\sqrt{2}}\,|\,j,0\rangle+\frac{1}{\sqrt{2}}\,|\,j,1\rangle \ . \tag{5.2}$$

From this equation one can immediately show

$$\langle \text{in}\,|\,J_z^k\,|\,\text{in}\rangle=\tfrac{1}{2} \ . \tag{5.3}$$

Hence this state lies close to the x-y plane and has a mean-square height of order unity,

$$(\Delta J_z)^2=\tfrac{1}{4} \ . \tag{5.4}$$

It is also straightforward to show that

$$\begin{aligned}&\langle \text{in}\,|\,J_x \text{in}\rangle=\tfrac{1}{2}[j(j+1)]^{1/2}\ ,\\ &\langle \text{in}\,|\,J_y\,|\,\text{in}\rangle=0\ ,\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}&\langle \text{in}\,|\,J_x^2\,|\,\text{in}\rangle=\tfrac{1}{2}[j(j+1)-\tfrac{1}{2}]\ ,\\ &\langle \text{in}\,|\,J_y^2\,|\,\text{in}\rangle=\tfrac{1}{2}[j(j+1)-\tfrac{1}{2}]\ .\end{aligned} \tag{5.6}$$

So the mean-square uncertainties in J_x and J_y are

$$\begin{aligned}&(\Delta J_x)^2=\tfrac{1}{4}[j(j+1)-1]\ ,\\ &(\Delta J_y)^2=\tfrac{1}{2}[j(j+1)-\tfrac{1}{2}]\ .\end{aligned} \tag{5.7}$$

Equations (5.5) and (5.7) indicate that the state is oriented along the x axis and is very broad along the x and y axes. This state could thus be represented by a geometrical object similar to that depicted in Fig. 4(a). One can also show that

$$\langle \text{in}\,|\,J_xJ_z\,|\,\text{in}\rangle=\tfrac{1}{4}[j(j+1)]^{1/2} \tag{5.8}$$

and consequently

$$\langle \text{in}\,|\,J_xJ_z+J_zJ_x\,|\,\text{in}\rangle=\tfrac{1}{2}[j(j+1)]^{1/2} \ . \tag{5.9}$$

The rms fluctuations in ϕ for this state will now be determined. Substituting Eqs. (5.3) and (5.5) into Eq. (3.9), $\bar{J}_z$, half the mean differenced photocurrent, is given by

$$\bar{J}_z=-\tfrac{1}{2}[j(j+1)]^{1/2}\sin\phi+\tfrac{1}{2}\cos\phi \ . \tag{5.10}$$

Substituting Eqs. (5.6) and (5.9) into Eq. (3.10) one has

$$\overline{J_z^2}=\tfrac{1}{2}[j(j+1)-\tfrac{1}{2}]\sin^2\phi+\tfrac{1}{2}\cos^2\phi \ . \tag{5.11}$$

The mean-square fluctuation in J_z is then

$$(\Delta J_z)^2=\tfrac{1}{4}[j(j+1)-1]\sin^2\phi+\tfrac{1}{4}\cos^2\phi \ . \tag{5.12}$$

The mean-square fluctuation in ϕ is given by

$$(\Delta\phi)^2=\frac{(\Delta J_z)^2}{\left[\dfrac{d\bar{J}_z}{d\phi}\right]^2} \tag{5.13}$$

or

$$(\Delta\phi)^2=\frac{[j(j+1)-1]\sin^2\phi+\cos^2\phi}{\{[j(j+1)]^{1/2}\cos\phi+\sin\phi\}^2} \ . \tag{5.14}$$

This quantity has its minimum value when $\sin\phi=0$, then

$$(\Delta\phi)^2_{\min}=\frac{1}{j(j+1)} \tag{5.15}$$

or in terms of the number of photons passing through the interferometer, since $j=n/2$,

$$(\Delta\phi)^2_{\min}=\frac{4}{n(n+2)} \ . \tag{5.16}$$

Hence when the state Eq. (5.2) is fed into the input ports of an interferometer a minimum rms fluctuation $\Delta\phi_{\min}$ in the phase of order n^{-1} can be achieved:

$$\Delta\phi_{\min}\simeq\frac{2}{n} \ . \tag{5.17}$$

This maximum sensitivity is however achieved only at particular values of ϕ satisfying $\sin\phi=0$. For other values of ϕ the sensitivity of the interferometer is degraded. Since $\phi=\phi_1-\phi_2$, ϕ_1 may be tracked as a function of time with the precision Eq. (5.17) by controlling ϕ_2 with a feedback loop which maintains $\phi_1-\phi_2$ at zero. The error signal for this loop is the differenced photodetector current $2J_z$. The use of feedback loops with be further discussed in Sec. VII.

A state $|\text{in}\rangle$ which allows an interferometer to achieve phase uncertainty of order n^{-1} has now been presented. How one prepares light in such a state, or a state similar to it, is the topic of the next section. Here we simply point out some properties of the state $|\text{in}\rangle$ of Eq. (5.2). It is a superposition of the states $|j,0\rangle$ and $|j,1\rangle$. For the state $|j,0\rangle$, N has the eigenvalue $n=2j$ and J_z has the eigenvalue $m=0$. Equations (2.2) and (2.3) allow one to recognize this state as one in which exactly j photons enter each of the two input ports of the interferometer. For the state $|j,1\rangle$, N has the eigenvalue $n=2j$ and J_z has the eigenvalue $m=1$. This state can be recognized as one in which exactly $j+1$ photons enter the input port $a_{1\,\text{in}}$ while exactly $j-1$ photons enter the input port $a_{2\,\text{in}}$.

VI. THE TWO-MODE FOUR-WAVE MIXER

In the last section it was shown that the sensitivity of an interferometer could be greatly improved provided one could prepare the light delivered to the input ports of the interferometer in a state which consists of a superposition of two states, one in which exactly j photons enter each of the two input ports in the interferometer and a state in which $j+1$ photons enter one port while $j-1$ photons enter the other port. In this section it is shown that states similar to this can be generated with two-mode four-wave mixers. For the analysis of such a device it will be convenient to introduce a set of operators whose commutation relations are those for the generators of the group SU(1,1).

In particular we introduce the Hermitian operators

$$\begin{aligned} K_x &= \tfrac{1}{2}(a_1^\dagger a_2^\dagger + a_1 a_2) , \\ K_y &= -\frac{i}{2}(a_1^\dagger a_2^\dagger - a_1 a_2) , \\ K_z &= \tfrac{1}{2}(a_1^\dagger a_1 + a_2 a_2^\dagger) . \end{aligned} \tag{6.1}$$

The commutation relations for these operators,

$$\begin{aligned} [K_x,K_y] &= -iK_z , \\ [K_y,K_z] &= iK_x , \\ [K_z,K_x] &= iK_y , \end{aligned} \tag{6.2}$$

can be recognized as those belonging to the group[6,8] SU(1,1). It is also useful to introduce the raising and lowering operators

$$\begin{aligned} K_+ &= K_x + iK_y = a_1^\dagger a_2^\dagger , \\ K_- &= K_x - iK_y = a_1 a_2 \end{aligned} \tag{6.3}$$

which satisfy the commutation relations[9]

$$\begin{aligned} [K_-,K_+] &= 2K_z , \\ [K_z,K_\pm] &= \pm K_\pm . \end{aligned} \tag{6.4}$$

The Casimir invariant K^2 is

$$K^2 = K_z^2 - K_x^2 - K_y^2 \tag{6.5}$$

which upon the substitution of Eq. (6.1) becomes

$$K^2 = J_z(J_z+1) , \tag{6.6}$$

where J_z is given in Eq. (2.2). In fact, the operator J_z commutes with all the K_i.

There has been a considerable amount of theoretical work, beginning with Yuen and Shapiro,[10] on four-wave mixers as possible sources of squeezed states. The reader is directed to Reid and Walls[11] and references therein for work that has been done on four-wave mixers. For the purposes of this paper, a four-wave mixer will be regarded as a device with two input ports $a_{1\,\text{in}}, a_{2\,\text{in}}$ and two output ports $a_{1\,\text{out}}, a_{2\,\text{out}}$ which performs the mode transformation of the form[12,13]

$$\begin{bmatrix} a_{1\,\text{out}} \\ a_{2\,\text{out}}^\dagger \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_{1\,\text{in}} \\ a_{2\,\text{in}}^\dagger \end{bmatrix} . \tag{6.7}$$

Both backward degenerate four-wave mixing in which two counter propagating pump beams pass through the nonlinear medium, and forward four-wave mixing, in which the pump beam propagates in only one direction through the nonlinear medium, perform mode transformations[14] of the form (6.7). Since the incoming and outgoing creation and annihilation operators must satisfy (2.1), the following restrictions are placed on the S_{ij}:

$$\begin{aligned} &|S_{11}|^2 - |S_{12}|^2 = 1 , \\ &|S_{22}|^2 - |S_{21}|^2 = 1 , \\ &S_{11}S_{21}^* = S_{12}S_{22}^* . \end{aligned} \tag{6.8}$$

From these relationships one can show

$$\begin{aligned} |S_{11}|^2 &= |S_{22}|^2 , \\ |S_{12}|^2 &= |S_{21}|^2 . \end{aligned} \tag{6.9}$$

The phases of the S_{ij} are controlled by the pump phase.

How the operators (6.1) transform under the scattering matrix (6.7)

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{6.10}$$

will now be determined for some particular examples. A possible realization of S is

$$S = \begin{bmatrix} \cosh(\tfrac{1}{2}\beta) & e^{-i\delta}\sinh(\tfrac{1}{2}\beta) \\ e^{i\delta}\sinh(\tfrac{1}{2}\beta) & \cosh(\tfrac{1}{2}\beta) \end{bmatrix} , \tag{6.11}$$

where δ is controlled by the phase of the pump light relative to some master clock and β is related to the reflectivity R of the four-wave mixer (when it is used as a phase-conjugating mirror) via $\sinh^2(\tfrac{1}{2}\beta)=R$.

When the pump phase is set such that $\delta=\pi/2$ Eq. (6.11) becomes

$$S = \begin{bmatrix} \cosh(\tfrac{1}{2}\beta) & -i\sinh(\tfrac{1}{2}\beta) \\ i\sinh(\tfrac{1}{2}\beta) & \cosh(\tfrac{1}{2}\beta) \end{bmatrix} . \tag{6.12}$$

Under this transformation, the vector $\mathbf{K}=(K_x,K_y,K_z)$ transforms as

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & \sinh\beta \\ 0 & \sinh\beta & \cosh\beta \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{in}} \tag{6.13}$$

which represents a Lorentz boost along the y axis, where z transforms as time. This transformation can be expressed in the form

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{out}} = e^{i\beta K_x} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} e^{-i\beta K_x} . \tag{6.14}$$

Equivalently, in the Schrödinger picture where $\mathbf{K}$ remains fixed the state vector transforms as

$$|\,\text{out}\,\rangle = e^{i\beta K_x}\,|\,\text{in}\,\rangle . \tag{6.15}$$

When the pump phase is set at $\delta=0$, the scattering matrix (6.11) becomes

$$S = \begin{bmatrix} \cosh(\frac{1}{2}\beta) & \sinh(\frac{1}{2}\beta) \\ \sinh(\frac{1}{2}\beta) & \cosh(\frac{1}{2}\beta) \end{bmatrix} . \tag{6.16}$$

Under this transformation $\mathbf{K}$ transforms as

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{out}} = \begin{bmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{in}} . \tag{6.17}$$

This transformation has the form of a Lorentz boost along the x axis and can be expressed in the form

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{out}} = e^{-i\beta K_y} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} e^{i\beta K_y} . \tag{6.18}$$

In the Schrödinger picture the state vector is transformed as

$$|\,\text{out}\,\rangle = e^{i\beta K_y}\,|\,\text{in}\,\rangle . \tag{6.19}$$

The operators performing the transformations of Eqs. (6.15) and (6.19) are two-mode squeeze operators.[6,12,13]

At this point it will be useful to determine how $\mathbf{K}$ transforms when the two input light beams sustain phase shifts. Letting $a_{1\,\text{in}}$ undergo a phase shift of ϕ_1 and $a_{2\,\text{in}}$ undergo a phase shift of ϕ_2, then

$$S = \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_2} \end{bmatrix} . \tag{6.20}$$

Under this transformation, $\mathbf{K}$ transforms as

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{in}} = \begin{bmatrix} \cos(\phi_1+\phi_2) & \sin(\phi_1+\phi_2) & 0 \\ -\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{out}} \tag{6.21}$$

which can be recognized as a rotation about the z axis by an angle $\phi=-(\phi_1+\phi_2)$. This transformation may be expressed as

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}_{\text{out}} = e^{-i(\phi_1+\phi_2)K_z} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} e^{i(\phi_1+\phi_2)K_z} . \tag{6.22}$$

In the Schrödinger picture the state vector is transformed as

$$|\,\text{out}\,\rangle = e^{i(\phi_1+\phi_2)K_z}\,|\,\text{in}\,\rangle . \tag{6.23}$$

The transformation (6.11) can be factorized into the form

$$S(\delta)S(\beta)S(-\delta) = \begin{bmatrix} \cosh(\frac{1}{2}\beta) & e^{-i\delta}\sinh(\frac{1}{2}\beta) \\ e^{i\delta}\sinh(\frac{1}{2}\beta) & \cosh(\frac{1}{2}\beta) \end{bmatrix} , \tag{6.24}$$

where

$$S(\delta) = \begin{bmatrix} e^{-i\delta} & 0 \\ 0 & 1 \end{bmatrix} , \tag{6.25}$$

$$S(\beta) = \begin{bmatrix} \cosh(\frac{1}{2}\beta) & \sinh(\frac{1}{2}\beta) \\ \sinh(\frac{1}{2}\beta) & \cosh(\frac{1}{2}\beta) \end{bmatrix} , \tag{6.26}$$

$$S(-\delta) = \begin{bmatrix} e^{i\delta} & 0 \\ 0 & 1 \end{bmatrix} . \tag{6.27}$$

From (6.20) the transformation $S(-\delta)$ can be recognized as a rotation about the z axis by an angle $-\delta$. $S(\beta)$ represents a Lorentz boost along the x axis, and $S(\delta)$ represents a rotation about the z axis by the angle δ. The product of transformations Eq. (6.24) thus represent a Lorentz transformation along a direction making an angle δ with respect to the x axis. Hence, in the Schrödinger picture, after the incoming light $|\,\text{in}\,\rangle$ has passed through a four-wave mixer, it will be in the state

$$|\,\text{out}\,\rangle = e^{-i\delta K_z} e^{i\beta K_y} e^{i\delta K_z}\,|\,\text{in}\,\rangle . \tag{6.28}$$

It has now been demonstrated that a four-wave mixer performs Lorentz transformations on the vector $\mathbf{K}$, the direction of the Lorentz boost being determined by the pump phase which is at the experimenter's control. Since J_z commutes with $\mathbf{K}$, it remains unchanged under the transformations performed by the four-wave mixer. From Eq. (2.2) one sees that this invariant is equal to half the difference in the number of photons entering the input port of the four-wave mixer. This invariant has been noted by Graham[15] and Reid and Walls.[16]

Let us now consider the case when no light enters the input ports of the four-wave mixer. The state delivered to the output is then given by Eq. (6.28) where $|\,\text{in}\,\rangle$ is the vacuum state $|\,0\,\rangle$.

The probability amplitude that n_1 photons will appear in the output beam $a_{1\,\text{out}}$ and n_2 photons in the beam $a_{2\,\text{out}}$ is

$$\langle\, n_1,n_2\,|\,\text{out}\,\rangle = \langle\, n_1,n_2\,|\, e^{-i\delta K_z} e^{i\beta K_y} e^{i\delta K_z}\,|\,0\,\rangle , \tag{6.29}$$

where the state $|\,n_1,n_2\,\rangle$ is

$$|n_1,n_2\rangle=\frac{(a_{1\,\mathrm{out}}^{\dagger})^{n_1}(a_{2\,\mathrm{out}}^{\dagger})^{n_2}}{\sqrt{n_1!n_2!}}|0\rangle \;. \tag{6.30}$$

From (6.1) one sees that K_z can be put in the form

$$K_z=\tfrac{1}{2}(N_1+N_2+1)\;, \tag{6.31}$$

where $N_1=a_{1\,\mathrm{out}}^{\dagger}a_{1\,\mathrm{out}}$ and $N_2=a_{2\,\mathrm{out}}^{\dagger}a_{2\,\mathrm{out}}$ are the number operators for output beams 1 and 2, respectively. With this equation it is readily apparent that

$$e^{i\delta K_z}|0\rangle=e^{i\delta/2}|0\rangle \tag{6.32}$$

and

$$e^{i\delta K_z}|n_1,n_2\rangle=e^{i(\delta/2)(n_1+n_2+1)}\;. \tag{6.33}$$

So Eq. (6.29) simplifies to

$$\langle n_1,n_2|\mathrm{out}\rangle=e^{-i(\delta/2)(n_1+n_2)}\langle n_1,n_2|e^{i\beta K_y}|0\rangle\;. \tag{6.34}$$

In order to simplify things further we make use of the identity[9]

$$\exp(\tau K_+-\tau^*K_-)=\exp\left[\left[\frac{\tau}{|\tau|}\tanh|\tau|\right]K_+\right]\times\exp[-2(\ln\cosh|\tau|K_z]\times\exp\left[-\left[\frac{\tau^*}{|\tau|}\tanh|\tau|\right]K_-\right]\;. \tag{6.35}$$

Hence, noting (6.1) and (6.3), $e^{i\beta K_y}$ can be put into the form

$$e^{i\beta K_y}=\exp[i\tanh(\tfrac{1}{2}\beta)K_+]\exp[-2\ln\cosh(\tfrac{1}{2}\beta)K_z]\times\exp[i\tanh(\tfrac{1}{2}\beta)K_-]\;. \tag{6.36}$$

From (6.3) $K_-=a_1a_2$, hence

$$\exp[i\tanh(\tfrac{1}{2}\beta)K_-]|0\rangle=|0\rangle\;. \tag{6.37}$$

Making use of Eq. (6.31) one has

$$\exp[-2\ln\cosh(\tfrac{1}{2}\beta)K_z]|0\rangle=\mathrm{sech}(\tfrac{1}{2}\beta)|0\rangle\;. \tag{6.38}$$

Finally, using (6.3)

$$\exp[i\tanh(\tfrac{1}{2}\beta)K_+]|0\rangle=\sum_{n=0}^{\infty}[i\tanh(\tfrac{1}{2}\beta)]^n|n,n\rangle\;, \tag{6.39}$$

where $|n,n\rangle$ is defined by (6.30).

Collecting the results, Eq. (6.34), (6.37), (6.38), and (6.39), one has

$$\langle n_1,n_2|\mathrm{out}\rangle=\delta_{n_1,n_2}e^{-i(\delta/2)(n_1+n_2)}\times\mathrm{sech}(\tfrac{1}{2}\beta)[i\tanh(\tfrac{1}{2}\beta)]^{n_1}\;. \tag{6.40}$$

The probability $P(n_1,n_2)$ that the beam $a_{1\,\mathrm{out}}$ will contain n_1 photons and the beam $a_{2\,\mathrm{out}}$ n_2 photons is thus

$$P(n_1,n_2)=\delta_{n_1,n_2}\mathrm{sech}^2(\tfrac{1}{2}\beta)[\tanh^2(\tfrac{1}{2}\beta)]^{n_1}\;. \tag{6.41}$$

From this equation one sees that $P(n_1,n_2)$ is zero if $n_1\neq n_2$. For the vacuum state one has

$$J_z|0\rangle=0\;. \tag{6.42}$$

Since J_z is an invariant for the four-wave mixing process, when there are n_1 photons in beam 1 there must be n_1 photons in the second beam as well, that is, the photons are emitted in correlated pairs. These photons are in fact more highly correlated than allowed classically.[15,16]

From (6.41) the mean $\langle n\rangle$ and the mean-square $\langle n^2\rangle$ number of photons emitted by the four-wave mixer can be computed:

$$\langle n\rangle=\sum_{n_1n_2}^{\infty}(n_1+n_2)P(n_1,n_2)=2\sinh^2(\tfrac{1}{2}\beta)\;, \tag{6.43}$$

$$\langle n^2\rangle=\sum_{n_1n_2}^{\infty}(n_1+n_2)^2P(n_1,n_2)=4[\sinh^2(\tfrac{1}{2}\beta)\cosh^2(\tfrac{1}{2}\beta)+\sinh^4(\tfrac{1}{2}\beta)]\;. \tag{6.44}$$

The mean-square fluctuation in n is rather large:

$$(\Delta n)^2=4\sinh^2(\tfrac{1}{2}\beta)\cosh^2(\tfrac{1}{2}\beta) \tag{6.45}$$

so

$$\frac{(\Delta n)^2}{\langle n\rangle^2}=\coth^2(\tfrac{1}{2}\beta)\geq 1\;. \tag{6.46}$$

Hence the light emitted is super-Poissonian but approaches Poissonian as β becomes large.

We are now in a position to argue that a four-wave mixer can generate states similar to the one described in the last section, Eq. (5.2). The case when no photons enter the input port of the four-wave mixer has already been discussed. The eigenvalue m of J_z is 0. Hence if a total of n photons were measured coming out of the four-wave mixer one can infer that the light leaving the four-wave mixer was in the state $|j,0\rangle$ where, from Eq. (2.5), $j=n/2$.

If instead the state

$$|2,0\rangle=\frac{a_{1\,\mathrm{in}}^{\dagger 2}}{\sqrt{2}}|0\rangle \tag{6.47}$$

is fed into the input ports, that is, if two photons are forced to enter the input port $a_{1\,\mathrm{in}}$ of the four-wave mixer, then the eigenvalue m for J_z is 1. If one thus measures a total of n photons leaving the four-wave mixer one can infer that the light leaving the four-wave mixer is in the state $|j,1\rangle$ where again $j=n/2$.

If now the light entering the input port is a superposition of a vacuum state and the $|2,0\rangle$ state of Eq. (6.47), say,

$$|\mathrm{in}\rangle=\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|2,0\rangle\;, \tag{6.48}$$

then upon measuring n photons leaving the four-wave mixer all one can infer is that the state leaving the four-mixer is in a superposition of the states $|j,0\rangle$ and $|j,1\rangle$. Hence a four-wave mixer can generate states of the form (5.2) provided the state (6.48) is fed into its input.

From a practical point of view the state Eq. (6.48) may be hard to generate. It would be more practical to attenuate laser light until on average there is one photon per unit coherence time propagating along the beam. The input state generated by feeding this coherent state into the input port $a_{1\,\rm in}$ of the four-wave mixer will have a strong overlap with states $|n_1,n_2\rangle$ only when n_1 is small and n_2 is 0. Consequently, $\langle J_z\rangle$ and ΔJ_z for the light fed into the four-wave mixer will still be of order unity. Such light when passed through the four-wave mixer should still produce "fan-shaped" states that will allow an interferometer to reach a phase sensitivity $\Delta\phi$ of order $1/n$. Interferometry, using the light coming from a four-wave mixer fed with coherent states, will be discussed in detail in the next section.

VII. ACHIEVING A PHASE SENSITIVITY OF $1/N$

The device to be considered in this section is depicted in Fig. 5. It consists of a Mach-Zehnder interferometer whose input ports are fed by the output beams b_1 and b_2 of a four-wave mixer. The four-wave mixing medium is pumped with a laser. Part of the laser light is split off of the main beam phase shifted, attenuated and then fed into the input port a_1 of the four-wave mixer. The other input port a_2 is terminated with a cold blackbody absorber so that no light enters the four-wave mixer from this port. The pump light's phase is controlled with the phase shifter δ.

Letting a_1 and a_2 denote the creation operators for the light beams fed into the input ports of the four-wave mixer, the state vector for this light $|\alpha\rangle$ is defined by

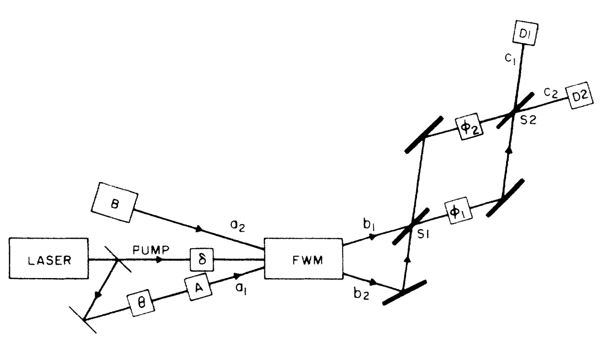

FIG. 5. A method by which the state depicted in Fig. 4 can be generated and fed into an interferometer. The state is generated via a degenerate four-wave mixer (FWM) pumped via a laser. A small fraction of the pump light is split off of the pump beam, phase shifted by θ, attenuated by A and then fed into one of the FWM inputs, a_1. The input port a_2 is terminated with a cold blackbody absorber B. The two output ports b_1 and b_2 of the four-wave mixer are fed into the input ports of the Mach-Zehnder interferometer. δ is a phase shifter for the pump light before it enters FWM1.

$$a_1|\alpha\rangle=\alpha|\alpha\rangle \tag{7.1}$$

and

$$a_2|\alpha\rangle=0\;, \tag{7.2}$$

that is, $|\alpha\rangle$ is a coherent state for a_1 and a vacuum state for a_2.

Since J_z is an invariant under the transformation (6.7) performed by the four-wave mixer, its expectation values can be computed at the input port,

$$J_z=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2)\;. \tag{7.3}$$

One can readily show

$$\langle J_z\rangle=\tfrac{1}{2}|\alpha|^2 \tag{7.4}$$

and

$$(\Delta J_z)^2=\tfrac{1}{4}|\alpha|^2\;. \tag{7.5}$$

For $|\alpha|^2$ of order unity such a state will lie near the x-y plane and have a spread along the z axis of order unity.

In order to compute the expectation values of J_x and J_y at the output of the four-wave mixer it is necessary to express J_x and J_y in terms of the operators a_1 and a_2. Again we choose the scattering matrix for the four-wave mixer to be given by Eq. (6.11),

$$\begin{bmatrix} b_1 \\ b_2^\dagger \end{bmatrix}=\begin{bmatrix} \cos(\tfrac{1}{2}\beta) & e^{-i\delta}\sinh(\tfrac{1}{2}\beta) \\ e^{i\delta}\sinh(\tfrac{1}{2}\beta) & \cosh(\tfrac{1}{2}\beta) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2^\dagger \end{bmatrix}\;. \tag{7.6}$$

Using this transformation, the output J_x and J_y expressed in terms of the input creation and annihilation operators are

$$\begin{aligned} J_x={}&\tfrac{1}{4}\cos\delta\,\sinh\beta(a_1^\dagger a_1^\dagger+a_1a_1+a_2^\dagger a_2^\dagger+a_2a_2)\\ &-\frac{i}{4}\sin\delta\,\sinh\beta(a_1^\dagger a_1^\dagger-a_1a_1+a_2^\dagger a_2^\dagger-a_2a_2)\\ &+\tfrac{1}{2}\cosh\beta(a_1^\dagger a_2+a_2^\dagger a_1)\;, \end{aligned} \tag{7.7}$$

$$\begin{aligned} J_y={}&-\frac{i}{4}\cos\delta\,\sinh\beta(a_1^\dagger a_1^\dagger-a_1a_1-a_2^\dagger a_2^\dagger+a_2a_2)\\ &-\tfrac{1}{4}\sin\delta\,\sinh\beta(a_1^\dagger a_1^\dagger+a_1a_1-a_2^\dagger a_2^\dagger-a_2a_2)\\ &-\frac{i}{2}\cosh\beta(a_1^\dagger a_2-a_2^\dagger a_1)\;. \end{aligned} \tag{7.8}$$

The expectation values of J_x and J_y for the state $|\alpha\rangle$ can now be readily evaluated. Writing

$$\alpha=|\alpha|e^{-i\theta} \tag{7.9}$$

one has

$$\begin{aligned} \langle\alpha|J_x|\alpha\rangle&=\tfrac{1}{2}|\alpha|^2\sinh\beta\cos(2\theta-\delta)\;,\\ \langle\alpha|J_y|\alpha\rangle&=\tfrac{1}{2}|\alpha|^2\sinh\beta\sin(2\theta-\delta)\;. \end{aligned} \tag{7.10}$$

The mean-square fluctuation in J_x and J_y is independent of ϕ and δ:

$$\begin{aligned} (\Delta J_x)^2&=(\Delta J_y)^2\\ &=\frac{|\alpha|^2}{2}(\sinh^2\beta+\tfrac{1}{2})+\tfrac{1}{8}\sinh^2\beta\;. \end{aligned} \tag{7.11}$$

One also has

$$\langle \alpha | J_x J_z + J_z J_x | \alpha \rangle = \frac{|\alpha|^2}{2}(|\alpha|^2+1)\sinh\beta\cos(2\theta-\delta) . \quad (7.12)$$

Since from Eq. (3.8) $J_{z\,\mathrm{out}}$, measured at the output of the interferometer, is

$$J_{z\,\mathrm{out}} = -(\sin\phi)J_x + (\cos\phi)J_z , \quad (7.13)$$

one can now evaluate

$$\langle J_{z\,\mathrm{out}} \rangle = -\tfrac{1}{2}|\alpha|^2 \sin\phi\sinh\beta\cos(2\theta-\delta) + \tfrac{1}{2}|\alpha|^2\cos\phi \quad (7.14)$$

and

$$(\Delta J_{z\,\mathrm{out}})^2 = \frac{|\alpha|^2}{2}\sin^2\phi\sinh^2\beta - \frac{|\alpha|^2}{2}\sin\phi\cos\phi\sinh\beta\cos(2\theta-\delta) + \frac{1}{8}\sin^2\phi\sinh^2\beta + \frac{|\alpha|^2}{4} . \quad (7.15)$$

The mean-square phase uncertainty $\Delta\phi$ can be evaluated via

$$(\Delta\phi)^2 = \frac{(\Delta J_{z\,\mathrm{out}})^2}{\left|\dfrac{\partial\langle J_{z\,\mathrm{out}}\rangle}{\partial\phi}\right|^2} . \quad (7.16)$$

This quantity is minimized with respect to the pump phase δ when $2\theta-\delta=0$, that is, $\cos(2\phi-\delta)=1$. This quantity is also near its minimum value when $\phi=0$. When ϕ is set to zero Eq. (7.16) reduces to

$$(\Delta\phi)^2 = \frac{1}{|\alpha|^2\sinh^2\beta} . \quad (7.17)$$

The parameter β will now be expressed in terms of the mean number of photons $\langle N \rangle$ leaving the four-wave mixer and $|\alpha|$. This will allow us to optimize $\Delta\phi$ holding $\langle N \rangle$ fixed. From Eq. (6.31) the number operator N can be expressed in terms of K_z,

$$N = 2K_z - 1 . \quad (7.18)$$

Upon leaving the interferometer the state $|\alpha\rangle$ has been transformed, according to Eq. (6.28), into the state $|\psi\rangle$,

$$|\psi\rangle = e^{-i\delta K_z} e^{i\beta K_y} e^{i\delta K_z} |\alpha\rangle . \quad (7.19)$$

From Eq. (6.18) and Eq. (6.22) one has

$$e^{-i\delta K_z} e^{-i\beta K_y} e^{i\delta K_z} K_z e^{-i\delta K_z} e^{i\beta K_y} e^{i\delta K_z} = (\cos\phi)(\sinh\beta)K_x + (\sin\phi)(\sinh\beta)K_y + (\cosh\beta)K_z . \quad (7.20)$$

The mean number of photons leaving the four-wave mixer can then easily be shown to be

$$\overline{N} = (|\alpha|^2+1)\cosh\beta - 1 . \quad (7.21)$$

This equation is easily solved for $\sinh^2\beta$. Equation (7.17) then becomes

$$(\Delta\phi)^2 = \frac{(|\alpha|^2+1)^2}{|\alpha|^2[(\overline{N}+1)^2 - (|\alpha|^2-1)^2]} . \quad (7.22)$$

This expression can be optimized for $|\alpha|$ holding N fixed. One finds that for large N $(\Delta\phi)^2$ is smallest when $|\alpha|^2$ is close to 1, hence

$$\Delta\phi \simeq \frac{2}{\overline{N}} . \quad (7.23)$$

This equation implies that the interferometer of Fig. 5 can achieve a phase sensitivity approaching $1/N$, and that photons are most economically used by the interferometer when the coherent state $|\alpha\rangle$ fed into the input port a_1 has its intensity reduced to $|\alpha|^2 \simeq 1$, that is, on average only one photon per unit coherence time of the four-wave mixer enters the input port a_1.

The sensitivity $\Delta\phi$ of Eq. (7.23) with the particular numerical coefficient 2 cannot be achieved in practice. The reason for this is now indicated. Equation (7.23) holds only for $\phi \lesssim 1/\overline{N}$. Hence a practical interferometer employed to measure ϕ_1 must incorporate a feedback loop which adjusts ϕ_2 to follow ϕ_1 such that $\phi=\phi_2-\phi_1=0$. However, for angles ϕ outside the narrow range $|\phi| \lesssim 1/\overline{N}$ the uncertainty in ϕ, defined by Eq. (7.16), becomes

$$\Delta\phi = \left[\frac{4|\alpha|^2+1}{2|\alpha|^4}\right]^{1/2} \tan\phi \quad (7.24)$$

which for small ϕ and $|\alpha|^2=1$ becomes

$$\Delta\phi \simeq (\tfrac{5}{2})^{1/2}\phi . \quad (7.25)$$

That is, the uncertainty in ϕ is greater than ϕ itself.

A feedback loop presented with a measurement of ϕ whose uncertainty is greater than ϕ will generally not be able to adjust ϕ_2 properly to drive ϕ to zero. The uncertainty $\Delta\phi$ can be decreased by increasing $|\alpha|$ or by averaging several[4] successive measurements of ϕ. In the next section the problem of locking an interferometer to $\phi=0$ will be discussed in more detail and the maximum uncertainty in $\Delta\phi$ that a feedback loop can tolerate will be determined.

VIII. TRACKING THE PHASE

In the last section an interferometer capable of achieving a phase sensitivity $\Delta\phi$ approaching $1/N$, where N is the number of photons passing through the interferometer per unit measurement time, was discussed. This sensitivity is only achieved, however, for a small range of angles within $1/N$ of $\phi=0$. The interferometer can be made to follow the phase ϕ_1 with a sensitivity approaching $1/N$ provided a feedback loop is employed to adjust a controllable phase shifter ϕ_2 such that $\phi=\phi_2-\phi_1$ is maintained at zero. The operation of the interferometer of the last section with a feedback loop will now be discussed in more detail.

The parameters θ, δ, $|\alpha|$, and β are under the control of the experimenter. It will be assumed that $2\theta-\delta=0$ and that $|\alpha|$ and β are known. Then the quantum statistics of the light entering the interferometer is well characterized and in particular one knows the numbers $\langle J_z\rangle$ and $\langle J_x\rangle$, which according to Eq. (7.4) and (7.10) are

$$\begin{aligned}\langle J_z\rangle&=\tfrac{1}{2}|\alpha|^2 ,\\ \langle J_x\rangle&=\tfrac{1}{2}|\alpha|^2\sinh\beta .\end{aligned} \tag{8.1}$$

The differenced photocurrent is measured at the output of the interferometer, that is, the photodetectors measure $2J_z$. A sequence of measurements will generate a string of numbers, each of which is an eigenvalue of $2J_z$. One is free to process these numbers and in particular one can subtract $\langle J_z\rangle$ from them and divide them by $-2\langle J_x\rangle$. Then the sequence of numbers $\{d_1,d_2,\ldots\}$ are eigenstates of the operator D

$$D=\frac{(\sin\phi)J_x}{\langle J_x\rangle}-\frac{(\cos\phi)J_z-\langle J_z\rangle}{\langle J_x\rangle} . \tag{8.2}$$

For simplicity it will be assumed that ϕ is small so that the approximations $\sin\phi\simeq\phi$ and $\cos\phi\simeq 1$ can be made. Then one can write

$$D=\phi A-B , \tag{8.3}$$

where

$$\begin{aligned}A&=\frac{J_x}{\langle J_x\rangle} ,\\ B&=\frac{J_z-\langle J_z\rangle}{\langle J_z\rangle} .\end{aligned} \tag{8.4}$$

Since

$$\begin{aligned}\langle A\rangle&=1 ,\\ \langle B\rangle&=0\end{aligned} \tag{8.5}$$

it is immediately evident that

$$\langle D\rangle=\phi , \tag{8.6}$$

that is, the sequence of numbers $\{d_1,d_2,\ldots\}$ are estimates of ϕ.

The phase shifter ϕ_2 of Fig. 5 will be taken to be controllable. A feedback algorithm that will track ϕ_1 maintaining $\phi=\phi_2-\phi_1$ at zero will now be described. Let $\phi_2(i)$ be the setting of ϕ_2 during the ith measurement. The measurement provides the estimate of $\phi(i)=\phi_2(i)-\phi_1(i)$, d_i, which is an eigenvalue of

$$D_i=\phi(i)A_i-B_i . \tag{8.7}$$

The feedback loop then adjusts ϕ_2 to the new setting

$$\phi_2(i+1)=\phi_2(i)-\lambda d_i ,$$

or in operator form

$$\phi_2(i+1)=\phi_2(i)-\lambda A_i[\phi_2(i)-\phi_1(i)]+\lambda B_i , \tag{8.8}$$

where λ is a feedback parameter.

It is now assumed that the successive measurements are performed on a time scale equal to the characteristic coherence time of the four-wave mixer so that the ith operators A_i and B_i are independent of the j operators A_j and B_j. Then Eq. (8.8) can readily be iteratively substituted into itself. For convenience $\phi_1(i)$ will be held fixed to $\phi_1(0)$. It will be determined how rapidly ϕ_2 approached $\phi_1(0)$ given the feedback algorithm (8.8). Equation (8.8) iteratively substituted into itself yields

$$\phi(n)=\sum_{k=0}^{n-1}(1-\lambda A_k)\phi(0)+\lambda\sum_{k=0}^{n-1}B_K\prod_{m=k+1}^{n-1}(1-\lambda A_m) , \tag{8.9}$$

where the product is defined in the usual way, except that

$$\prod_{m=n}^{n-1}F(m)\equiv 1 . \tag{8.10}$$

The mean value $\langle\phi(n)\rangle$ is, using Eq. (8.5),

$$\langle\phi(n)\rangle=(1-\lambda)^n\langle\phi(0)\rangle . \tag{8.11}$$

It is apparent that the mean value of $\phi(n)$ will converge to zero only if $|1-\lambda|<1$. Hence the feedback parameter is restricted to the range

$$0<\lambda<2 . \tag{8.12}$$

The mean-square value of $\phi(n)$ can be obtained by squaring (8.9) and then taking the expectation value. One obtains

$$\begin{aligned}\langle[\phi(n)]^2\rangle=&\langle(1-\lambda A)^2\rangle^n[\phi(0)]^2\\ &+\lambda^2\langle B^2\rangle\sum_{k=0}^{n-1}\langle(1-\lambda A)^2\rangle^k\\ &-\lambda^2\langle AB+BA\rangle\phi(0)\\ &\times\sum_{k=0}^{n-1}\langle(1-\lambda A)^2\rangle^{n-1-k}\langle 1-\lambda A\rangle^k .\end{aligned} \tag{8.13}$$

The sums can be evaluated to yield

$$\begin{aligned}\langle[\phi(n)]^2\rangle=&\langle(1-\lambda A)^2\rangle^n[\phi(0)]^2\\ &+\lambda^2\langle B^2\rangle\frac{1-\langle(1-\lambda A^2)\rangle^n}{1-\langle(1-\lambda A)^2\rangle}\\ &-\lambda^2\langle AB+BA\rangle\frac{\langle(1-\lambda A)^2\rangle^n-\langle 1-\lambda A\rangle^n}{\langle(1-\lambda A)^2\rangle-\langle 1-\lambda A\rangle} .\end{aligned} \tag{8.14}$$

The expectation values $\langle(1-\lambda A)^2\rangle$ and $\langle 1-\lambda A\rangle$ can be written, keeping in mind Eq. (8.5), as

$$\begin{aligned}\langle 1-\lambda A\rangle&=1-\lambda ,\\ \langle(1-\lambda A)^2\rangle&=(1-\lambda)^2+\lambda^2(\Delta A)^2 .\end{aligned} \tag{8.15}$$

Substituting these expressions into (8.14) one finally has

$$\begin{aligned}&\langle[\phi(n)]^2\rangle\\ &=[(1-\lambda)^2+\lambda^2(\Delta A)^2]^n[\phi(0)]^2\\ &\quad+\lambda^2\langle B^2\rangle\frac{1-[(1-\lambda)^2+\lambda^2(\Delta A)^2]^n}{1-[(1-\lambda)^2+\lambda^2(\Delta A)^2]}\\ &\quad-\lambda^2\langle AB+BA\rangle\frac{[(1-\lambda)^2+\lambda^2(\Delta A)^2]^n-(1-\lambda)^n}{[(1-\lambda)^2+\lambda^2(\Delta A)^2]-(1-\lambda)} .\end{aligned} \tag{8.16}$$

From this equation one sees that in order for $\langle[\phi(n)]^2\rangle$ to converge one must, in addition to (8.12), have $[(1-\lambda)^2+\lambda^2(\Delta A)^2]^n<1$. This expression yields the restriction

$$\lambda\langle A^2\rangle<2 . \tag{8.17}$$

As a particular example, consider the case when $\lambda=1$, then

$$\langle[\phi(n)]^2\rangle=(\Delta A)^{2n}[\phi(0)]^2+\langle B^2\rangle\frac{1+(\Delta A)^{2n}}{1-(\Delta A)^2} -\langle AB+BA\rangle(\Delta A)^{2n-1} . \tag{8.18}$$

The mean-square value of ϕ converges to

$$\langle\phi^2\rangle_{\min}=\frac{\langle B^2\rangle}{1-(\Delta A)^2} . \tag{8.19}$$

As ϕ is driven to zero, the characteristic number of measurements $\bar{n}$ that must be made to reduce ϕ^2 to $1/e$ of its original value is

$$\bar{n}=-\frac{1}{\ln(\Delta A)^2} . \tag{8.20}$$

We now substitute the results of Sec. VII into these expressions. One has

$$(\Delta A)^2=\frac{2|\alpha|^2(\sinh^2\beta+\frac{1}{2})+\frac{1}{2}\sinh^2\beta}{|\alpha|^4\sinh^2\beta} , \tag{8.21}$$

$$\langle B^2\rangle=\frac{1}{|\alpha|^2\sinh^2\beta} , \tag{8.22}$$

and

$$\langle AB+BA\rangle=\frac{2}{|\alpha|^2\sinh\beta} . \tag{8.23}$$

In the large-β limit Eq. (8.21) reduces to

$$(\Delta A)^2=\frac{2|\alpha|^2+\frac{1}{2}}{|\alpha|^4} . \tag{8.24}$$

In order that the sensitivity $(\Delta\phi)^2$ not be degraded too much from its minimum value $(\Delta\phi)^2=\langle B^2\rangle$ [see Eq. (8.19)], let us choose $(\Delta A)^2=\frac{1}{2}$. Then the characteristic number of measurements necessary to reduce ϕ^2 to $1/e$ of its original value is $\bar{n}=1.44$. From (8.24) one sees that $|\alpha|^2$ has the value $|\alpha|^2=2+\sqrt{5}$. Using (7.21), Eq. (8.19) becomes, for large $\bar{N}$,

$$\langle\phi^2\rangle_{\min}\simeq\frac{2(|\alpha|^2+1)^2}{|\alpha|^2\bar{N}^2} \simeq\frac{12.9}{\bar{N}^2} . \tag{8.25}$$

Suppose ϕ_1 is stationary so that ϕ_2 has settled down and ϕ fluctuates with the mean-square value of (8.25), i.e.,

$$\Delta\phi\simeq\frac{3.6}{\bar{N}} . \tag{8.26}$$

Then if a small disturbance should come along to displace ϕ_1 by an amount $\Delta\phi$ from its quiescent value it will take approximately 1.44 measurements in order for ϕ_2 to adjust itself to the new ϕ_1. Hence, on the average, the total number of photons $\bar{N}_T$ used to detect this displacement is of the order $\bar{N}_T=1.44\bar{N}$ and $\Delta\phi$ in terms of the total number of photons used is

$$\Delta\phi\simeq\frac{5}{\bar{N}_T} . \tag{8.27}$$

Hence, by increasing the number of photons fed into the four-wave mixer from 1 to $|\alpha|^2\simeq2+\sqrt{5}\simeq4.24$, the interferometer can be operated stably in a feedback mode and ϕ_2 can detect changes in ϕ_1 as small as that given by (8.27).

Consider now the case where, instead of choosing $|\alpha|^2$ large enough so that the feedback loop would be stable with the feedback parameter λ set to unity, one chose $|\alpha|^2=1$ as was done in Sec. VII in order to optimize the sensitivity (7.22) with $\bar{N}$ fixed. In this case

$$(\Delta A)^2=\tfrac{5}{2} ,\quad \langle B^2\rangle=\frac{1}{\sinh^2\beta} ,\quad \langle AB+BA\rangle=\frac{2}{\sinh\beta} . \tag{8.28}$$

Since $(\Delta A)^2>1$ it is apparent from (8.18) that the feedback loop cannot be operated stably with the feedback parameter set to unity. In fact, from Eq. (8.17) it follows that λ must be less than $\frac{4}{7}$ if the feedback loop is to be operated stably. In order to make the e-folding time for ϕ^2 as short as possible we choose the value of λ which minimizes

$$[(1-\lambda)^2+\lambda^2(\Delta A)^2]$$

of Eq. (8.16), that is,

$$\lambda=\frac{1}{1+(\Delta A)^2}=\frac{2}{7} . \tag{8.29}$$

If $\phi_1(0)$ is held fixed $\langle\phi^2\rangle$ settles to a steady-state value

$$\langle\phi^2\rangle_{\min}=\frac{\lambda^2\langle B^2\rangle}{1-[(1-\lambda)^2+\lambda^2(\Delta A)^2]} . \tag{8.30}$$

The characteristic number of measurements which must be made to reduce ϕ^2 to $1/e$ of its initial value is

$$\bar{n}=-\frac{1}{\ln[(1-\lambda)^2+\lambda^2(\Delta A)^2]} . \tag{8.31}$$

For $\lambda=\frac{2}{7}$ and $(\Delta A)^2=\frac{5}{2}$ one has, upon using Eq. (7.21),

$$\langle\phi^2\rangle_{\min}=\frac{1.14}{\bar{N}^2} \tag{8.32}$$

and

$$\bar{n}=3.0 . \tag{8.33}$$

Again consider the case where ϕ_1 has remained constant for a long time so that the rms fluctuations in ϕ have settled down to the value determined by (8.32), $\Delta\phi\simeq1.07/\bar{N}$. Suppose now that ϕ_1 is displaced instantaneously to a new value a distance $\Delta\phi$ from its old value. It takes characteristically three measurements for ϕ_2 to adjust itself to the new ϕ_1. Hence, on the average, the total number of photons N_T used to detect the change in ϕ_1 is $N_T = 3\bar{N}$. The sensitivity of the interferometer, operated with $|\alpha|^2 = 1$ and $\lambda = \frac{2}{7}$, is thus expressed in terms of the total number of photons needed to observe the change as

$$\Delta\phi = \frac{3.2}{N_T} , \tag{8.34}$$

a number that is somewhat better than Eq. (8.27).

In this section it has been shown that by using suitable feedback loops the interferometer of Sec. VII can track changes in ϕ_1 in a stable manner and can achieve a phase sensitivity of order $1/N$. Hence the two problems encountered in Sec. VII, namely the fact that the interferometer achieves its optimum sensitivity only for a small range of phases, $\phi \lesssim 1/N$, and that the fluctuations in $J_{z\,\text{out}}$, the interferometer's output, are greater than $\langle J_{z\,\text{out}} \rangle$ for $|\alpha|^2$ set at its optimum value, can be overcome be operating the interferometer with $|\alpha|^2$ slightly degraded or by choosing the response of the feedback loop to be such that it averages enough successive measurements of ϕ that a useful error signal can be generated.

In the literature[2-4] a number of schemes for achieving interferometer sensitivities of $1/N$ have been described. All of these schemes employ standard interferometers into which light from degenerate-parametric amplifiers or four-wave mixers is injected. In the next section we will describe a novel set of interferometers which dispense with beam splitters and use the SU(1,1) boosts to convert phase shifts into light amplitude changes rather than the SU(2) rotations employed by a conventional interferometer.

IX. AN SU(1,1) MACH-ZEHNDER INTERFEROMETER

In Sec. III it was shown how the operation of a Mach-Zehnder interferometer could be viewed in terms of rotations of the vector **J** under the rotation group SU(2).

In this picture relative phase shifts between two light beams correspond to rotations about the z axis while photodetectors are sensitive to rotations in a plane containing the z axis. The function of the beam splitters was to convert a rotation about the z axis into one perpendicular to the z axis.

In this section an interferometer whose operation can be viewed in terms of transformations of the vector **K** Eq. (6.1), under the Lorentz group SU(1,1) is considered. From Eq (6.21) one sees that the common mode phase shift of two light beams corresponds to a rotation of **K** about the z axis. But from Eq. (6.1) one sees that photodetectors placed in the two light beams will be sensitive only to transformations perpendicular to the z axis.

Again, a device is required which will convert rotations about the z axis into transformations perpendicular to this axis. The four-wave mixers described in Sec. VI can carry out such transformations. These transformations consist of Lorentz boosts.

As a specific example, consider the device of Fig. 6. The phase shifter ψ in the pump beam is adjusted such that four-wave mixer FWM2 performs the inverse of the transformation performed by four-wave mixer FWM1. In

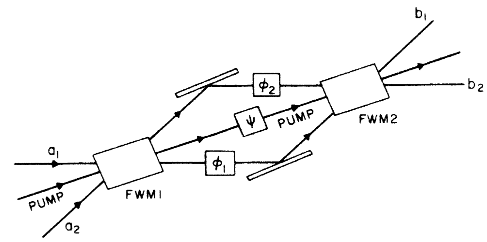

FIG. 6. An SU(1,1) interferometer. The beam splitters of a conventional interferometer have been replaced by the four-wave mixers FWM1 and FWM2. The light pumping FWM2 is phase shifted from the light pumping FWM1 by the angle ψ.

particular let FWM1 have the scattering matrix

$$S(-\beta) = \begin{bmatrix} \cosh(\frac{1}{2}\beta) & +i\sinh(\frac{1}{2}\beta) \\ -i\sinh(\frac{1}{2}\beta) & \cosh(\frac{1}{2}\beta) \end{bmatrix} . \tag{9.1}$$

As can be seen from Eq. (6.13), **K** transforms as a Lorentz boost $L(-\beta, y)$ along the $-y$ axis under this scattering matrix:

$$L(-\beta, y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & -\sinh\beta \\ 0 & -\sinh\beta & \cosh\beta \end{bmatrix} . \tag{9.2}$$

The scattering matrix for FWM2 is

$$S(\beta) = \begin{bmatrix} \cosh(\frac{1}{2}\beta) & -i\sinh(\frac{1}{2}\beta) \\ i\sinh(\frac{1}{2}\beta) & \cosh(\frac{1}{2}\beta) \end{bmatrix} . \tag{9.3}$$

This scattering matrix transforms **K** as a Lorentz boost $L(\beta, y)$ along the $+y$ axis. The transformation performed by the phase shifters ϕ_1 and ϕ_2 is, from Eq. (6.20),

$$S(\phi) = \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_2} \end{bmatrix} . \tag{9.4}$$

Under this scattering matrix **K** transforms as a rotation $R(\phi, z)$ about the z axis by an angle $\phi = -(\phi_1 + \phi_2)$,

$$R(\phi, z) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{9.5}$$

The overall scattering matrix for the device of Fig. 6 is

$$S = S(\beta) S(\phi) S(-\beta) , \tag{9.6}$$

and the overall transformation performed on **K** is

$$\mathbf{K}_{\text{out}} = L(\beta, y) R(\phi, z) L(-\beta, y) \mathbf{K}_{\text{in}} . \tag{9.7}$$

It will be useful to reexpress this transformation as follows:

$$L(\beta, y) R(\phi, z) L(-\beta, y) = R(\theta, z) L(\gamma x) R(\theta, z) , \tag{9.8}$$

where $L(\gamma, x)$ denotes a Lorentz boost along the x axis,

$$L(\gamma,x)=\begin{bmatrix}\cosh\gamma & 0 & \sinh\gamma\\ 0 & 1 & 0\\ \sinh\gamma & 0 & \cosh\gamma\end{bmatrix}. \tag{9.9}$$

Equation (9.8) holds when θ and γ are chosen such that

$$\sin\theta=\frac{(1-\cos\phi)\cosh\beta}{[\sin^2\phi+(1-\cos\phi)^2\cosh^2\beta]^{1/2}}, \tag{9.10}$$

$$\cos\theta=\frac{\sin\phi}{[\sin^2\phi+(1-\cos\phi)^2\cosh^2\beta]^{1/2}}, \tag{9.11}$$

$$\cosh\gamma=(1-\cos\phi)\cosh^2\beta+\cos\phi, \tag{9.12}$$

$$\sinh\gamma=\sinh\beta[\sin^2\phi+(1-\cos\phi)^2\cosh^2\beta]^{1/2}. \tag{9.13}$$

Hence the transformation performed by the device of Fig. 6 on $\mathbf{K}$ can be regarded as a rotation θ about the z axis, followed by a Lorentz boost along the x axis, followed by a second rotation θ about the z axis.

Let us now consider the operation of this device when no light enters the input ports, that is when $|\text{in}\rangle$ is the vacuum state $|0\rangle$. The vacuum state is both an eigenstate of K_z and J_z:

$$K_z|0\rangle=\tfrac{1}{2}|0\rangle, \tag{9.14}$$

$$J_z|0\rangle=0. \tag{9.15}$$

Consequently, from Eq. (6.6) the invariant K^2 is zero, that is, we can think of $\mathbf{K}$ for the vacuum state as lying on the light cone. In the spirit of Fig. 2, the vacuum state is depicted in Fig. 7(a) as a cone whose base intersects the z axis at $\frac{1}{2}$. The Lorentz boost $L(-\beta,y)$ is equivalent, in the Schrödinger picture, to a boost of the state vector in the opposite direction. The Lorentz boost performed by the first four-wave mixer is depicted in Fig. 7(b). The mean value of K_z in terms of the mean number of photons $\langle N\rangle$ emitted by the four-wave mixer is, from Eq. (6.31),

$$\langle K_z\rangle=\tfrac{1}{2}(\langle N\rangle+1).$$

The phase shifts ϕ_1 and ϕ_2 encountered by the two light beams leaving the four-wave mixer then rotate the state vector about the z axis by an angle $-\phi=\phi_1+\phi_2$. This is depicted in Fig. 7(c). A second Lorentz boost with the same rapidity, but in the opposite direction, is then performed. If $\phi=0$ the final state will be a vacuum state and no photons will be detected by the photodetectors in the output beams. If ϕ is nonzero the state of the light delivered to the photodetectors will be a Lorentz-boosted vacuum, the rapidity parameter being determined by Eq. (9.12) or (9.13).

In Fig. 7(b) the projected ellipse lying in the x-y plane has a width of $\frac{1}{2}$ and the distance from the origin to its center is $\langle K_z\rangle=\frac{1}{2}(\langle N\rangle+1)$. Hence Fig. 7(c) suggests that the minimum detectable phase $\phi_{\min}$ is of the order

$$\phi_{\min}=\frac{1}{\langle N\rangle+1}, \tag{9.16}$$

that is, this detector can achieve a phase sensitivity approaching $1/N$.

That this is the case will now be demonstrated with an explicit calculation. From Eq. (9.6) and Eqs. (6.15) and (6.23) the incoming state vector $|\text{in}\rangle$ is transformed as

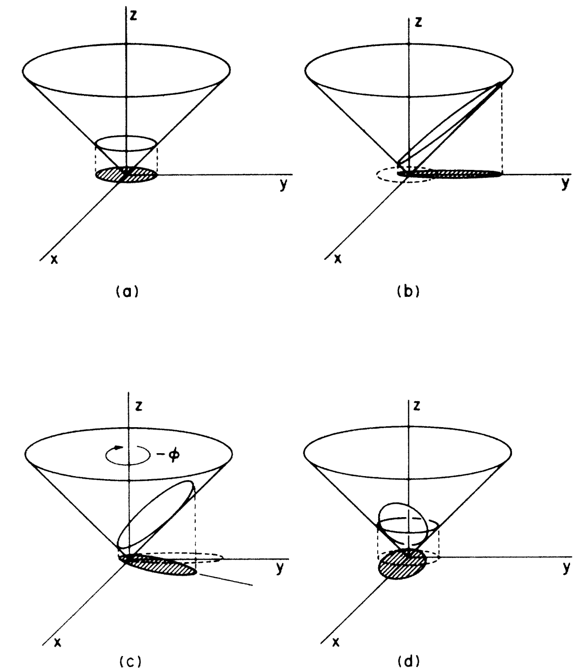

FIG. 7. A geometrical view of the performance of an SU(1,1) interferometer. (a) The input state consisting of the vacuum state is depicted in the (K_x,K_y,K_z) space, where K_x and K_y are regarded as space coordinates and K_z as a time coordinate, as a circle on the light cone. (b) The first four-wave mixer performs a Lorentz boost along the positive y axis. (c) The phase shifts accumulated by the light beams propagating in the interferometer result in a rotation in the xy plane. (d) The second four-wave mixer performs a Lorentz transformation along the negative y axis. The total number of photons leaving the interferometer is a linear function of K_z.

$$|\text{out}\rangle=e^{-i\beta K_x}e^{-i\phi K_z}e^{i\beta K_x}|\text{in}\rangle, \tag{9.17}$$

but from (9.8) this is equivalent to the transformation

$$|\text{out}\rangle=e^{i\theta K_z}e^{i\gamma K_y}e^{-i\theta K_z}|\text{in}\rangle. \tag{9.18}$$

The operator N_d for the total number of photons detected by the photodetectors placed in the output beam is from Eq. (6.31)

$$N_d=2K_z-1. \tag{9.19}$$

Hence in order to evaluate $\langle N_d\rangle$ and ΔN_d one needs to evaluate $\langle\text{out}|K_z|\text{out}\rangle$ and $\langle\text{out}|K_z^2|\text{out}\rangle$. From Eq. (9.18) one has

$$\langle\text{out}|K_z|\text{out}\rangle=\langle\text{in}|e^{i\theta K_z}e^{-i\gamma K_y}K_z e^{i\gamma K_y}e^{-i\theta K_z}|\text{in}\rangle. \tag{9.20}$$

Since $|\text{in}\rangle$ is the vacuum state, one has

$$e^{-i\theta K_z}|\text{in}\rangle=e^{-i\theta/2}|\text{in}\rangle. \tag{9.21}$$

Equation (9.20) thus simplifies to

$$\langle \text{out} \mid K_z \mid \text{out} \rangle = \langle 0 \mid e^{-\gamma K_y} K_z e^{i\gamma K_y} \mid 0 \rangle \ . \tag{9.22}$$

From Eqs. (6.18) and (6.17),

$$e^{-i\gamma K_y} K_z e^{i\gamma K_y} = (\sinh\gamma) K_x + (\cosh\gamma) K_z \ , \tag{9.23}$$

but

$$\langle 0 \mid K_z \mid 0 \rangle = 0 \tag{9.24}$$

so

$$\langle \text{out} \mid K_z \mid \text{out} \rangle = \tfrac{1}{2}\cosh\beta \ . \tag{9.25}$$

From Eq. (9.19), the mean number of photons $\langle N_d \rangle$ detected by the photodetectors is

$$\langle N_d \rangle = \cosh\gamma - 1 \ . \tag{9.26}$$

In a similar manner one can show

$$(\Delta N_d)^2 = \sinh^2\gamma \ . \tag{9.27}$$

The dependence of γ on ϕ and β is given by Eqs. (9.12) and (9.13). Hence Eqs. (9.26) and (9.27) can be rewritten as

$$\begin{aligned} \langle N_d \rangle &= (1-\cos\phi)\sinh^2\beta \ , \\ (\Delta N_d)^2 &= [\sin^2\phi + (1-\cos\phi)^2\cosh^2\beta]\sinh^2\beta \ . \end{aligned} \tag{9.28}$$

The mean-square fluctuation in ϕ due to the photon statistics is thus

$$\begin{aligned} (\Delta\phi)^2 &= \frac{(\Delta N_d)^2}{\left[\dfrac{\partial \langle N_d \rangle}{\partial \phi} \right]^2} \\ &= \frac{\sin^2\phi + (1-\cos\phi)^2\cosh^2\beta}{\sin^2\phi \sinh^2\beta} \ . \end{aligned} \tag{9.29}$$

The quantity $(\Delta\phi)^2$ is minimized when $\phi=0$, then

$$(\Delta\phi)^2_{\min} = \frac{1}{\sinh^2\beta} \ . \tag{9.30}$$

Expressed in terms of the mean number of photons $\langle N \rangle$ emitted by the first four-wave mixer, $\sin^2\beta = \langle N \rangle (\langle N \rangle - 2)$, so

$$(\Delta\phi)^2_{\min} = \frac{1}{\langle N \rangle (\langle N \rangle - 2)} \ . \tag{9.31}$$

Hence it has now been shown that the SU(1,1) interferometer depicted in Fig. 6 can achieve a phase sensitivity approaching $1/N$. This sensitivity is achieved when no light is fed into the input ports. A comparison with Fig. 5 shows that SU(1,1) interferometers achieving a sensitivity of $1/N$ require fewer optical elements than an SU(2) interferometer achieving the same sensitivity. Further, at $\phi=0$ no light is delivered to the photodetectors, that is, the pairs of pump photons converted into pairs of four-wave-mixer output photons are absorbed by the second four-wave mixer and converted back into pump photons. Hence an SU(1,1) interferometer can be very economical with photons. It will absorb pump power only when ϕ is nonzero. It is also worth pointing out that the beams 1 and 2 need not be at the same frequency, as long as they are placed symmetrically about the pump frequency, ω_0, that is, with beam 1 at the frequency $\omega_0+\Delta\omega$ and beam 2 at the frequency $\omega_0-\Delta\omega$; the scattering matrix[6,12,13] for the four-wave mixer will still have the form (6.11).

By using techniques similar to those used in deriving Eq. (6.41) one can show that the probability $P(N)$ of detecting a total of N photons leaving the output ports of the interferometer is

$$P(N) = \begin{cases} 0 \ \ \text{if } N \text{ is odd} \\ \dfrac{2}{(1-\cos\phi)\sinh^2\beta+2} \left[\dfrac{(1-\cos\phi)\sinh^2\beta}{(1-\cos\phi)\sinh^2\beta+2} \right]^{N/2} & \text{if } N \text{ is even} \ . \end{cases} \tag{9.32}$$

Let N_i denote the number of photons counted during the ith measurement in a sequence of measurements. One is free to take the square root of each of these numbers. Hence it is meaningful to talk about the average and rms value of $\sqrt{N}$. The motivation for investigating the statistics of $\sqrt{N}$ stems from the fact that $\langle N \rangle$ is an even function of ϕ. Now

$$\langle \sqrt{N} \rangle = \sum_{N \text{ even}} \sqrt{N} P(N) \tag{9.33}$$

can be put into the form

$$\langle \sqrt{N} \rangle = \frac{2^{3/2}}{(1-\cos\phi)\sinh^2\beta+2} \times \sum_{k=0}^{\infty} \sqrt{k} \left[\frac{(1-\cos\phi)\sinh^2\beta}{(1-\cos\phi)\sinh^2\beta+2} \right]^k . \tag{9.34}$$

The sum has the form

$$\sum_{k=0}^{\infty} \sqrt{k} \left[\frac{1}{a} \right]^k$$

which can be approximated by the integral

$$\int_0^{\infty} x^{1/2} \left[\frac{1}{a} \right]^x dx = \frac{\sqrt{\pi}}{2(\ln a)^{3/2}} \ . \tag{9.35}$$

Hence

$$\langle \sqrt{N} \rangle \simeq \frac{\sqrt{2\pi}}{(1-\cos\phi)\sinh^2\beta+2} \times \left[\ln \left[\frac{(1-\cos\phi)\sinh^2\beta+2}{(1-\cos\phi)\sinh^2\beta} \right] \right]^{-3/2} . \tag{9.36}$$

Since this approximation holds reasonably well for

$$(1-\cos\beta)\sinh^2\beta \gtrsim 4 \tag{9.37}$$

the logarithm can be approximated via $\ln(1+x)\simeq x$ and one has

$$\langle\sqrt{N}\,\rangle \simeq \frac{\sqrt{\pi}}{2}(1-\cos\phi)^{1/2}\sinh\beta\ . \tag{9.38}$$

Approximating $(1-\cos\phi)$ as $\phi^2/2$ one finally has

$$\langle\sqrt{N}\,\rangle = \frac{1}{2}\left[\frac{\pi}{2}\right]^{1/2} |\phi|\sinh\beta\ . \tag{9.39}$$

Hence the norm of the phase $\tilde{\phi}$ inferred from a measurement of N is

$$|\tilde{\phi}| = \frac{2\left[\frac{2}{\pi}\right]^{1/2}\sqrt{N}}{\sinh\beta}\ . \tag{9.40}$$

From (9.39) one has

$$\langle\,|\tilde{\phi}|\,\rangle = |\phi|\ , \tag{9.41}$$

that is, the mean value of $|\tilde{\phi}|$ inferred from a measurement of N is equal to the norm of the actual phase setting ϕ of the interferometer. Now

$$\langle\,|\tilde{\phi}|^2\rangle = \frac{8}{\pi}(1-\cos\phi) \simeq \frac{4}{\pi}\phi^2\ , \tag{9.42}$$

so

$$(\Delta|\tilde{\phi}|^2) = \frac{4-\pi}{\pi}\phi^2 \simeq 0.273\phi^2\ . \tag{9.43}$$

Hence for $N \gtrsim 4$ it has been shown that the uncertainty in the inferred norm of the phase $\tilde{\phi}$ is to a good approximation

$$\Delta|\tilde{\phi}| = 0.522|\tilde{\phi}|\ . \tag{9.44}$$

It is also instructive to ask what the probability $P(|\tilde{\phi}|: (1-\alpha)|\phi| < |\tilde{\phi}| < (1+\alpha)|\phi|)$ is that a measured $|\tilde{\phi}|$ will lie in the range

$$(1-\alpha)|\phi| < |\tilde{\phi}| < (1+\alpha)|\phi|\ .$$

From (9.40) this is equivalent to determining the probability $P(N\colon N_1 < N < N_2)$ that N lies in the range $N_1 < N < N_2$ where

$$\begin{aligned} N_1 &= \frac{\pi}{8}(1-\alpha)^2\phi^2\sinh^2\beta\ , \\ N_2 &= \frac{\pi}{8}(1+\alpha)^2\phi^2\sinh^2\beta\ . \end{aligned} \tag{9.45}$$

One can show rigorously

$$P(N\colon N_1 < N < N_2) = \left[\left[\frac{(1-\cos\phi)\sinh^2\beta}{(1-\cos\phi)\sinh^2\beta+2}\right]^{N_1/2} - \left[\frac{(1-\cos\phi)\sinh^2\beta}{(1-\cos\phi)\sinh^2\beta+2}\right]^{N_2/2}\right]\ . \tag{9.46}$$

So from Eq. (9.45)

$$P(|\tilde{\phi}|: (1-\alpha)|\phi| < |\tilde{\phi}| < (1+\alpha)|\phi|) = \left[\left[\frac{(1-\cos\phi)\sinh^2\beta}{(1-\cos\phi)\sinh^2\beta+2}\right]^{(\pi/16)(1-\alpha)^2\phi^2\sinh^2\beta} - \left[\frac{(1-\cos\phi)\sinh^2\beta}{(1-\cos\phi)\sinh^2\beta+2}\right]^{(\pi/16)(1+\alpha)^2\phi^2\sinh^2\beta}\right]\ . \tag{9.47}$$

Approximating $1-\cos\phi$ by $\phi^2/2$, this expression can be put into the form

$$P(|\tilde{\phi}|: (1-\alpha)|\phi| < |\tilde{\phi}| < (1+\alpha)|\phi|) = \left[\left[\frac{x}{x+4}\right]^{(\pi/16)(1-\alpha)^2 x} - \left[\frac{x}{x+4}\right]^{(\pi/16)(1+\alpha)^2 x}\right]\ , \tag{9.48}$$

where $x = \phi^2\sinh^2\beta$.

Now

$$\lim_{x\to\infty}\left[\frac{x}{x+4}\right]^x = e^{-4}\ . \tag{9.49}$$

This limiting value is not a bad approximation for $[x/(x+4)]^x$ even for x as low as 10, the level at which on average five photons are counted in the interferometer output beams. Hence

$$P(|\tilde{\phi}|: (1-\alpha)|\phi| < |\tilde{\phi}| < (1+\alpha)|\phi|) \simeq e^{-(\pi/2)(1-\alpha)^2} - e^{-(\pi/2)(1+\alpha)^2}\ . \tag{9.50}$$

As an example, let $\alpha = \frac{1}{2}$, then

$$P(|\tilde{\phi}|: |\phi|/2 < |\tilde{\phi}| < 3|\phi|/2) \simeq e^{-\pi/8} - e^{-9\pi/8} = 0.646\ . \tag{9.51}$$

Hence from a single measurement of $|\tilde{\phi}|$ one has 65% confidence that $|\phi|/2 < |\tilde{\phi}| < 3|\phi|/2$.

The interferometer described here suffers from drawbacks similar to those of the SU(2) interferometer of Sec. VII. Maximum sensitivity occurs at $\phi=0$ and the sensitivity rapidly degrades as ϕ is adjusted away from zero. It was shown in Sec. VIII that such drawbacks can be overcome with feedback. However, implementing a feedback algorithm for the SU(1,1) interferometer described here is complicated by the fact that $\langle N\rangle$ is an even function of ϕ

and hence the sign of the error signal cannot be determined from the number of photons counted by the photodetector during a single measurement.

The sign of the error signal can be generated by changing (dithering) ϕ_2 between successive measurements and constructing the derivative signal $(N_{i+1}-N_i)/\Delta\phi_2$.

Alternatively, one could implement the feedback algorithm which will now be described. Make repeated measurements of $|\tilde{\phi}|$ until $|\phi|$ is determined to some predetermined precision: $\Delta|\phi|=\alpha|\phi|$ where α is a constant. Then move ϕ_2 according to

$$\phi_2(\text{new})=\phi_2(\text{old})+|\tilde{\phi}| \ . \tag{9.52}$$

Make repeated measurements of $|\tilde{\phi}|$ at this new setting so that a new $|\phi|$ can be inferred with the precision $\Delta|\phi|=\alpha|\phi|$. If $|\phi|$ inferred for the new setting of ϕ_2 is less than $|\phi|$ inferred for the old setting one assumes that one has moved in the right direction. If, on the other hand, the inferred value of $|\phi|$ for the new setting of ϕ_2 is greater than the inferred $|\phi|$ for the old setting one assumes that one has moved ϕ_2 in the wrong direction and ϕ_2 is then readjusted so that

$$\phi_2(\text{new})=\phi_2(\text{old})-|\tilde{\phi}| \ . \tag{9.53}$$

The process is then repeated.

If $|\phi|$ is determined to sufficient precision this algorithm will move one closer to $\phi=0$ most of the time. On the occasions when this algorithm moves one in the wrong direction it generally does not move ϕ_2 very far in the wrong direction and the lost ground is regained during the next few iterations of the feedback procedure. Further, since a single measurement already determines $|\phi|$ with a precision $\Delta\phi\approx 0.5|\phi|$ at a 65% confidence level, one does not have to make very many repeated measurements of $|\phi|$ in order for the feedback algorithm to work.

X. SINGLE-MODE SU(1,1) INTERFEROMETERS

In this section interferometers based on devices having the scattering matrix

$$a_{\text{out}}=\cosh(\tfrac{1}{2}\beta)a_{\text{in}}+e^{-i\delta}\sinh(\tfrac{1}{2}\beta)a_{\text{in}}^{\dagger} \tag{10.1}$$

will be described. Such a single-mode device can be regarded as a limiting case of the four-wave mixers of Sec. VI in which the two input and the two output beams are made collinear and are sufficiently close in frequency that they cannot be resolved during the coherence time of the device. Both four-wave mixers and parametric amplifiers configured properly[17,18] are capable of performing the mode transformation Eq. (10.1). Connected with Eq. (10.1) it is convenient to introduce the operators

$$\begin{aligned} L_x&=\tfrac{1}{4}(a^{\dagger}a^{\dagger}+aa) \ , \\ L_y&=-\frac{i}{4}(a^{\dagger}a^{\dagger}-aa) \ , \\ L_z&=\tfrac{1}{4}(a^{\dagger}a+aa^{\dagger}) \ . \end{aligned} \tag{10.2}$$

These operators behave as generators[6,9] of the group SU(1,1) satisfying commutation relations identical with Eq. (6.2):

$$\begin{aligned} [L_x,L_y]&=-iL_z \ , \\ [L_yL_z]&=iL_x \ , \\ [L_z,L_x]&=iL_y \ . \end{aligned} \tag{10.3}$$

Again, it is useful to introduce the raising and lowering operators

$$\begin{aligned} L_+&=L_x+iL_y=\tfrac{1}{2}a^{\dagger}a^{\dagger} \ , \\ L_-&=L_x-iL_y=\tfrac{1}{2}aa \end{aligned} \tag{10.4}$$

which satisfy the commutation relations

$$\begin{aligned} [L_-,L_+]&=2L_z \ , \\ [L_z,L_\pm]&=\pm L_\pm \ . \end{aligned} \tag{10.5}$$

The Casimir invariant

$$L^2=L_z^2-L_x^2-L_y^2 \ , \tag{10.6}$$

when expressed in terms of the operators a and $a^{\dagger}$, reduces to the number

$$L^2=-\tfrac{3}{16} \ . \tag{10.7}$$

It is useful to determine how $\mathbf{L}=(L_x,L_y,L_z)$ transforms under specific cases of Eq. (10.1). Under the mode transformation

$$a_{\text{out}}=\cosh(\tfrac{1}{2}\beta)a_{\text{in}}+\sinh(\tfrac{1}{2}\beta)a_{\text{in}}^{\dagger} \ , \tag{10.8}$$

$\mathbf{L}$ transforms as a boost along the x axis:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}_{\text{out}} = \begin{bmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{bmatrix} \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}_{\text{in}} . \tag{10.9}$$

Under the mode transformation

$$a_{\text{out}}=\cosh(\tfrac{1}{2}\beta)a_{\text{in}}-i\sinh(\tfrac{1}{2}\beta)a_{\text{in}}^{\dagger} \tag{10.10}$$

$\mathbf{L}$ transforms as a boost along the y axis:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & \sinh\beta \\ 0 & \sinh\beta & \cosh\beta \end{bmatrix} \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} . \tag{10.11}$$

A phase shift

$$a_{\text{out}}=e^{-i\phi}a_{\text{in}} \tag{10.12}$$

transforms $\mathbf{K}$ as a rotation about the z axis by the amount 2ϕ:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}_{\text{out}} = \begin{bmatrix} \cos2\phi & -\sin2\phi & 0 \\ \sin2\phi & \cos2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_y \\ L_y \\ L_z \end{bmatrix}_{\text{in}} . \tag{10.13}$$

It can be shown that $\mathbf{K}$ transforms under Eq. (10.1) as a rotation of $\mathbf{K}$ about the z axis by the angle $-\delta/2$ followed by a boost β along the x axis followed by a rotation about the z axis by the angle $\delta/2$. Equivalently, in the Schrödinger picture the state vector transforms according to

$$|\,\text{out}\rangle = e^{-i(\delta/2)L_x} e^{i\beta L_x} e^{i(\delta/2)L_z}\,|\,\text{in}\rangle\;, \tag{10.14}$$

where $e^{i\beta L_x}$ is a single-mode squeeze operator[19] and $e^{i(\delta/2)L_z}$ has been called a single-mode rotation operator.[6,12,13]

More generally one could consider a device which transforms a state vector according to

$$|\,\text{out}\rangle = e^{i\phi L_z} e^{i\beta L_x} e^{i\theta L_z}\,|\,\text{in}\rangle\;. \tag{10.15}$$

The probability distribution for the number n of photons in the output beam will now be determined for the case when the input consists of vacuum fluctuations. A more general case, when the input consists of coherent states, has been treated by Yuen.[20] A photodetector in the output beam measures $N = a^\dagger a$, which can, from Eq. (10.2), be written in the form

$$N = 2L_z - \tfrac{1}{2}\;. \tag{10.16}$$

The amplitude that n photons will be counted in the output beam is $\langle n\,|\,\text{out}\rangle$, hence the probability $P(n)$ that n photons will be counted is

$$P(n) = |\,\langle n\,|\,\text{out}\rangle\,|^2\;. \tag{10.17}$$

Now for an n-photon state $|\,n\,\rangle$ one has

$$L_z\,|\,n\,\rangle = \left[\frac{n}{2} + \frac{1}{4}\right]|\,n\,\rangle\;. \tag{10.18}$$

Hence one has

$$e^{i\theta L_z}\,|\,0\rangle = e^{i\theta/4}\,|\,0\rangle \tag{10.19}$$

and

$$e^{-i\phi L_z}\,|\,n\,\rangle = e^{-i\phi(n/2+1/4)}\,|\,n\,\rangle\;. \tag{10.20}$$

The probability distribution (10.17) thus reduces to

$$P(n) = |\,\langle n\,|\,e^{i\beta L_x}\,|\,0\rangle\,|^2\;. \tag{10.21}$$

Hence $P(n)$ is independent of the phase angles ϕ and θ, i.e., $P(n)$ depends only on the magnitude of the boost. Again, using (6.35) one has

$$e^{i\beta L_x} = \exp[i\tanh(\tfrac{1}{2}\beta)L_+]\exp[-2\ln\cosh(\tfrac{1}{2}\beta)L_z] \times\exp[i\tanh(\tfrac{1}{2}\beta)L_-]\;. \tag{10.22}$$

Hence by using the techniques used to arrive at Eq. (6.41) one can show

$$P(n) = \begin{cases} 0, & n \text{ odd} \\ \dfrac{1}{2^n}\begin{bmatrix} n \\ \dfrac{n}{2}\end{bmatrix} \dfrac{\tanh^2(\frac{1}{2}\beta)}{\cosh(\frac{1}{2}\beta)}, & n \text{ even}\end{cases} \tag{10.23}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{n!}{(n-m)!m!}\;. \tag{10.24}$$

It is useful to compare this probability distribution with distribution $P_T(n)$ for the probability that a total of n photons will be counted leaving the two-port four-mixer of Sec. VI. From Eq. (6.41)

$$P_T(n) = \begin{cases} 0, & n \text{ odd} \\ \operatorname{sech}^2(\frac{1}{2}\beta)[\tanh^2(\frac{1}{2}\beta)]^2, & n \text{ even}\;.\end{cases} \tag{10.25}$$

With the identity

$$\sum_{k=0}^{n}\begin{bmatrix} 2k \\ k\end{bmatrix}\begin{bmatrix} 2(n-k) \\ n-k\end{bmatrix} = 2^{2n} \tag{10.26}$$

one can show

$$P_T(n) = \sum_{\substack{n_1,n_2 \\ n_1+n_2=n}} P(n_1)P(n_2)\;. \tag{10.27}$$

This equation implies that the statistics of the total number of photons leaving the two-mode four-wave mixer is the same as the statistics of the total number of photons coming from two independent single-mode devices. This observation will allow a simplification of the discussion of feedback loops for the interferometer discussed in this section, since the results of Sec. IX can be made to apply by pairwise averaging successive measurements made with a single-mode device.

The interferometer to be considered in this section is depicted in Fig. 8 where for the sake of definiteness, degenerate-parametric amplifiers DPA1 and DPA2 are used to perform the Lorentz boost. DPA1 will be taken to perform the boost $e^{i\beta L_x}$ on the incoming state vector. The phase shifter is taken to perform a phase shift $e^{-i\phi L_z}$ on the light beam. The last degenerate-parametric amplifier is taken to perform the boost

$$e^{-i\delta L_z} e^{-i\beta L_x} e^{i\delta L_z}\;,$$

where δ is proportional to the phase of the pump light entering DPA2. Letting $|\,\text{in}\rangle$ denote the state vector for the incoming light, the state vector $|\,\text{out}\rangle$ for light leaving the interferometer is

$$|\,\text{out}\rangle = e^{-i\delta L_z} e^{-i\beta L_x} e^{i\delta L_z} e^{-i\phi L_z} e^{i\beta L_x}\,|\,\text{in}\rangle\;. \tag{10.28}$$

The behavior of this device when $|\,\text{in}\rangle$ is the vacuum state will now be considered. Figures analogous to Fig. 7 can be drawn to illustrate the behavior of the interferometer. However, in this case the Casimir invariant Eq. (10.6) has the numerical value $-\frac{3}{16}$. Hence $\mathbf{L}$ lies on a spacelike hyperboloid instead of the light cone of Fig. 7. The vacuum state is an eigenstate of L_z

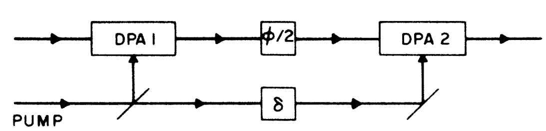

FIG. 8. A single-mode SU(1,1) interferometer. The device employs two degenerate-parametric amplifiers DPA1 and DPA2. The output of the device is sensitive to the difference between the phases ϕ and δ accumulated by the signal and pump beam, respectively.

$$L_z|0\rangle=\tfrac{1}{4}|0\rangle \tag{10.29}$$

and hence could be represented as a circle drawn around the hyperboloid at a height along the z axis of $\frac{1}{4}$.

We now determine the mean and variance in the number of photons counted at the output of the interferometer. Since the number operator N for the total number of photons counted at the output of the detector is linear in L_z, Eq. (10.16), one would like to determine $\langle \text{out}|L_z^k|\text{out}\rangle$. One can readily show from Eq. (10.28) that

$$\langle \text{out}|L_z^k|\text{out}\rangle=\langle \text{in}|e^{-i\beta L_x}e^{i(\phi-\delta)L_z}e^{i\beta L_x}L_z^k e^{-i\beta L_x}e^{-i(\phi-\delta)L_z}e^{i\beta L_x}|\text{in}\rangle . \tag{10.30}$$

Using the techniques of Sec. IX this can be further reduced to

$$\langle \text{out}|L_z^k|\text{out}\rangle=\langle \text{in}|e^{i\theta L_z}e^{-i\gamma L_y}e^{i\theta L_z}L_z^k e^{-i\theta L_z}e^{i\gamma L_y}e^{-i\theta L_z}|\text{in}\rangle=\langle \text{in}|e^{-i\gamma L_y}L_z^k e^{i\gamma L_y}|\text{in}\rangle , \tag{10.31}$$

where in analogy with Eqs. (9.12) and (9.13),

$$\cosh\gamma=[1-\cos(\phi-\delta)]\cosh^2\beta+\cos(\phi-\delta) , \tag{10.32}$$
$$\sinh\gamma=\sinh\beta\{\sin^2(\phi-\delta)+[1-\cos(\phi-\delta)]^2\cosh^2\beta\}^{1/2} .$$

It is straightforward then to show that

$$\langle N\rangle\equiv\langle \text{out}|N|\text{out}\rangle=\tfrac{1}{4}\cosh\gamma , \tag{10.33}$$

and

$$\langle N^2\rangle\equiv\langle \text{out}|N^2|\text{out}\rangle =\tfrac{1}{2}\sinh^2\gamma+\tfrac{1}{4}(\cosh\gamma-1)^2 \tag{10.34}$$

or, using Eq. (10.32)

$$\langle N\rangle=\tfrac{1}{2}[1-\cos(\phi-\delta)](\cosh^2\beta-1) , \tag{10.35}$$
$$(\Delta N)^2=\tfrac{1}{2}\sinh^2\beta\{\sin^2(\phi-\delta)+[1-\cos(\phi-\delta)]^2\cosh^2\beta\} .$$

The mean-square fluctuation in the readings for ϕ, given by

$$(\Delta\phi)^2=\frac{\langle \Delta N\rangle^2}{\left|\dfrac{\partial\langle N\rangle}{\partial\phi}\right|^2} , \tag{10.36}$$

is readily evaluated and has a minimum given by

$$(\Delta\phi_{\min})^2=\frac{1}{2\sinh^2\beta} . \tag{10.37}$$

The mean number of photons $\langle N_I\rangle$ in the light beam passing through the phase shifter ϕ is

$$\langle N_I\rangle=\langle 0|e^{-i\beta L_x}Ne^{i\beta L_x}|0\rangle =\tfrac{1}{2}(\cosh\beta-1) . \tag{10.38}$$

Solving this equation for $\sinh^2\beta$ one finally has

$$(\Delta\phi_{\min})^2=\frac{1}{8\langle N_I\rangle(\langle N_I\rangle-1)} . \tag{10.39}$$

Hence it has been shown that the device of Fig. 8 can indeed achieve a phase sensitivity approaching $1/n$. This minimum sensitivity is achieved when $\phi-\delta=0$. Hence by implementing a feedback loop one can track ϕ maintaining $\phi-\delta=0$ by using the error-correcting signal to adjust the phase shifter δ in the pump beam delivered to DPA2. As was mentioned earlier the statistics of the total number of photons counted in two successive measurements of ϕ are the same as for the total number of photons leaving the interferometer of Sec. IX. Hence the feedback algorithms discussed in Sec. IX will also work for the single-mode device discussed here.

XI. CONCLUSION

A geometric or Lie-group-theoretical approach to the analysis of interferometers was presented. Such an approach facilitates identifying the input states which optimize the interferometer's sensitivity. It was shown that ordinary interferometers are characterized by the group SU(2) which is equivalent to the group of rotations in three dimensions. With suitable input states such an interferometer can achieve a phase sensitivity $\Delta\phi$ approaching $1/N$ where N is the total number of photons passing through the phase-shifting element ϕ of the interferometer. Although this sensitivity can only be achieved for ϕ within $1/N$ of $\phi=0$, it was shown that by employing a feedback loop the interferometer could track phase as a function of time with a precision of $1/N$.

A class of interferometers in which four-wave mixers serve as active analogs of beam splitters was also presented. Such interferometers are characterized by the group SU(1,1) and have the virtue of being able to achieve a phase sensitivity approaching $1/N$ with only vacuum fluctuations entering the input port. SU(1,1) interferometers can consequently achieve the $1/N$ phase sensitivity much more readily and in fact have a simpler construction than SU(2) interferometers. The output of a four-wave mixer depends on the relative phase between the pump and the incoming signal. It is this phase sensitivity which the SU(1,1) interferometers employ. In fact, it was shown that degenerate-parametric amplifiers which are also phase-sensitive devices can be used to construct SU(1,1) interferometers.

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