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# Calculated electrostatic properties of ion traps

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A trap for charged particles is an important tool in many important experiments in atomic physics. The electrodes in many such traps are truncated hyperboloids. Data given here allow evaluation of the effects on trap performance of some of the details of the truncation. It is shown that if the truncation of the ring and endcap electrodes is done in a "balanced" way the performance degradation can be small. Some specific designs are suggested and some appropriate data reported.

## I. INTRODUCTION

For a variety of technical applications it is desirable to arrange electrostatic potentials to produce a local minimum (in one dimension), thereby providing a (conditional) trap for charged particles. Some quite remarkable measurements have been made on ions and elementary particles using traps of this sort. Electrons,[1-3] positrons,[4] and positive ions[5-8] have been trapped one at a time and some of their properties measured. Clouds of a small number of ions have been cooled with laser radiation to very low temperatures.[9-12] Several recent reviews are available giving both the techniques of trapping and the spectroscopic applications.[13-17]

For many of these applications the ideal potential distribution, expressed in cylindrical coordinates, is

$$V(r,z)=V_0+A(2z^2-r^2)\ , \tag{1}$$

where $V_0$ and $A$ are constants with $A$ having the same sign as the charge on the particle being trapped. The potential represented by Eq. (1) has a local minimum along the $z$ axis; however, that point is a local maximum in the $r$ direction. If charged particles are independently constrained to move near the $z$ axis they will be trapped in the neighborhood of $z=0$. A magnetic field can be used to effectively restrain motion in the $r$ direction (the Penning trap).[12-18] Also, a potential of the form of Eq. (1) can be used without a magnetic field to dynamically trap particles by making $A$ time dependent (the Paul trap).[13-15,19,20] In either case there is a need to build electrodes which cause the electrostatic potential to be a close approximation to that of Eq. (1). This paper examines some of the issues surrounding how to build electrodes which produce a potential function which is a good approximation to Eq. (1).

The straightforward way to generate a potential which approximates a function such as Eq. (1) is to choose a set of equipotentials which surround the volume of interest, build electrodes which conform to the shapes of the equipotentials, and set the electrode potentials to the appropriate values. In the particular case of Eq. (1) this prescription leads to three electrodes, each a hyperboloid, and each infinite in extent. Many traps of this type have been built by following this prescription near the center of the trap and at large distances arbitrarily truncating the hyperboloids and installing a suitable mounting arrangement. The electrode designs considered below are all variations on this prescription. While it is clear that the condition of the remote portions of the electrodes is unimportant, little information is available to allow a design decision on an appropriate shape at intermediate distances. A brief review of the solution to the ideal problem will provide a useful frame of reference.

For this discussion assume that the trapped particles have positive charge and that therefore $A$ in Eq. (1) is positive. The equipotential surface which goes through the center of the trap represented by Eq. (1) is a pair of cones defined by the equation

$$r^2=2z^2\ . \tag{2}$$

This surface divides space into three parts with the result that three electrodes are required. The equipotential surfaces are cylindrically symmetric and therefore the electrodes must have this symmetry. The electrode which intersects the $z=0$ plane is usually called the "ring." Let $r_0$ be the radius at which the surface of the ring intersects the $z=0$ plane. The ring must have a potential $V_0-Ar_0^2$, and its surface must conform to the shape

$$r^2-r_0^2=2z^2\ . \tag{3}$$

The other two electrodes each intersect the $z$ axis, have potentials higher than $V_0$, and are called "endcaps." Let

$z_0$ be the point of intersection of an endcap with the $z$ axis. The corresponding endcap must have a potential $V_0+2Az_0^2$, and the surface must conform to the shape

$$r^2=2(z^2-z_0^2)\ . \tag{4}$$

The other endcap is not required to have the same value of $z_0^2$; however, there is no apparent reason for making it different and in this paper it is assumed to be the same. In any case the shape of the surface and the potential are given by the same formula.

A design choice made for many of the traps which have been built as approximations to the above prescription is to have $r_0^2=2z_0^2$. This choice allows Eqs. (3) and (4) to be expressed more simply and causes the potential $V_0$ at the center of the trap to be the average of the potential of the ring and endcap electrodes. Because this choice has been so widely made, it has received special attention in the data reported below.

For any set of electrodes intended to approximate this ideal, it is possible to find a numerical representation of the potential function. However, it is difficult to judge from a table of potential values how adequate the configuration will be in practice. Indeed, different applications have different sensitivities to departures from the ideal. Nevertheless, for many experiments in atomic physics the needs are very similar. The measure of adequacy used here involves expansion of the potential function in spherical harmonics. The strategy adopted in this investigation was to pursue two separate objectives. A series of simple electrode shapes were examined to determine the trends as some of the dimensions were varied. The expectation is that some of these trends can be generalized to more complex and more practical electrode assemblies. The second part of the investigation was to examine the properties of some more complex designs which were chosen to be close to practical. In the latter cases a prescription is available for constructing relatively small electrodes which provide a good approximation to Eq. (1).

One of the ways used to build traps with potential functions approaching the ideal is to include "compensation" electrodes which can have their potentials adjusted to compensate for special errors. Van Dyck *et al.*[21] reported good success by placing extra electrodes in the gaps of a structure which was otherwise constructed to be nearly adequate. These extra electrodes can be arranged to compensate not only for design inadequacies but some of the errors of assembly as well. Gabrielse[22,23] has reported calculations giving some performance characteristics of the electrode designs similar to those considered here but with compensation electrodes added. In some circumstances compensation electrodes can be added with little penalty because there is an independent need to complete the electrostatic shielding. More generally, the extra electrodes represent a significant design complication which can be expected to relieve some problems while compounding others. In this paper only three electrode systems are explicitly considered. Nevertheless, by using a differencing technique it is possible to estimate some of the effects of adding additional electrodes. The additional information may be used in the design of compensation electrodes.

A general conclusion is that the errors caused by truncating the hyperbolic electrodes can be made small by (1) making the electrodes larger than necessary, (2) carefully choosing the shape (the option examined below), and (3) installing extra electrodes with potentials that can be adjusted. These solutions may be used in combination. An expectation is that it should be possible to make the truncation errors small relative to the errors associated with problems of construction and assembly. These laboratory errors will not all have the symmetries which go with the ideal. None of the three options listed above will help to correct those errors of construction and assembly which are not appropriately symmetric.

## II. DEFINITION OF THE PROBLEM

Equation (1) is considered an ideal potential for a trap because it leads to particularly simple equations of motion for the trapped particles. In the case of the Penning trap with a magnetic field along the $z$ axis, the $z$ component of motion is simple harmonic with the result that the frequency of oscillation is independent of the amplitude. This frequency is also independent of radial position. Thus a collection of ions with the same charge-to-mass ratio will all have (approximately) the same frequency. The result is a narrow linewidth for the driven resonance. In the case of the Paul trap the equations of motion have been worked out for the ideal potential.[19,20] This leads to secure knowledge of the stability conditions. In these applications the test of adequacy of an electrode design may be in how closely the equations of motion match those of the ideal potential. This test is too complex to apply in a general way; however, if errors in the potential are small and known, a perturbation analysis can be expected to provide a useful approximation in specific cases. Here we look for a measure of the departure of the potential function from the ideal which is applicable in a wide variety of situations.

The cylindrical symmetry of the ideal potential clearly calls for making the electrodes of this symmetry. This condition is well defined and can easily be approximately met by using rotating machinery in the fabrication of the electrodes. In this paper all electrodes are therefore assumed to be cylindrically symmetric. It is useful to make a power-series expansion of the potential about point $(z=0,r=0)$. With the restriction to cylindrical symmetry the series can be expressed as

$$V(r,z)=C_0+C_1H_1(r,z)+C_2H_2(z,r)+C_3H_3(r,z)+C_4H_4(r,z)+C_5H_5(r,z)+C_6H_6(r,z)+\cdots\ , \tag{5}$$

where

$$H_1(r,z)=z/s\ ,$$

$$H_2(r,z)=(2z^2-r^2)/s^2\ ,$$

$$H_3(r,z)=(2z^3-3zr^2)/s^3\ ,$$

$$H_4(r,z)=(8z^4-24z^2r^2+3z^4)/s^4\ ,$$

$$H_5(r,z)=(8z^5-40z^3r^2+15zr^4)/s^5\ ,$$

$$H_6(r,z)=(16z^6-120z^4r^2+90z^2r^4-5r^6)/s^6\ .$$

The $H_j$ functions (spherical harmonics) are homogeneous polynomials in $z/s, r/s$ with the numerical coefficients chosen so that each satisfies Laplace's equation. The $C_j$ are arbitrary constants with the units of potential and $s$ is a distance to be chosen to suit the circumstances of the problem. The choice of $s$ affects whether the series of Eq. (5) is convergent. In the present context the series is useful only in a volume about the origin in which all but the first few terms are negligible. (The spherical harmonics are defined only to an arbitrary factor and other authors have chosen a different normalization.[22-24])

For the present problem further restrictions simplify the situation. The terms involving $C_0$ and $C_2$ correspond to the ideal potential, Eq. (1). The odd-order terms might have been omitted because it is well known that the way to make them zero is to make both the electrodes and the applied potentials have reflection symmetry in the $z=0$ plane. In this paper the electrode structure is assumed to have reflection symmetry in the $z=0$ plane. The odd-order terms in Eq. (5) are included here because it is sometimes desirable to consider the consequences of a general set of applied potentials. The terms involving $C_4$ and $C_6$ are therefore the leading departures from the ideal. The ideal electrodes represented by Eqs. (3) and (4) have the special quality of yielding $C_j=0$ for all even $j$ greater than 2. If the odd-order terms are made small by making the applied potentials symmetric about $z=0$ then a good approximation to the ideal potential will always exist in a small volume around the origin. The problem then is to find ways to increase this volume without having unduly large electrodes.

The $C_j$ of Eq. (5) are linear functions of the potentials applied to the electrodes. It is useful to make some other definitions to separate the effects of applied potentials from the effects of electrode shape. It is necessary to choose an arbitrary zero for the potential. Here the potential is taken to be zero at large distances from the electrodes. To the extent that the electrodes form a nearly closed surface constituting an electrostatic shield the potential inside the trap is unaffected by the applied potentials on the outside. Accordingly, in this paper the ring electrode is used as a reference and is assumed to have zero potential. In view of the assumed mechanical symmetry of the endcaps, a general potential distribution can be expressed as a linear combination of a symmetric function $V_e(r,z)$ and an antisymmetric function $V_o(r,z)$. More specifically, a general solution is

$$V(r,z)=P_e V_e(r,z)+P_o V_o(r,z)\ . \tag{6}$$

If both the shielding is incomplete and the ring is not at zero potential, then Eq. (6) should have an additional term which is proportional to the potential of the ring electrode. Normalization of the two potential functions is arbitrary and our choice is that they have the value one on the endcap where $z$ is positive. It is convenient to express these functions in the form of Eq. (5), yielding

$$V_e(r,z)=A_0+A_2H_2(r,z)+A_4H_4(r,z)+A_6H_6(r,z)+\cdots, \tag{7}$$

$$V_o(r,z)=A_1H_1(r,z)+A_3H_3(r,z)+A_5H_5(r,z)+\cdots\ . \tag{8}$$

The coefficients $A_j$ are dimensionless and serve to describe the potential distribution without reference to the potentials applied to the electrodes. Some specific expressions in this language may be helpful. If $V^+$ is the potential (relative to the ring) of the endcap where $z$ is positive and $V^-$ is that of the other one, then $P_e=(V^++V^-)/2$ and $P_o=(V^+-V^-)/2$. The coefficient $A$ of Eq. (1) is $P_eA_2/s^2$ and the electric field at the center of the trap is $P_oA_1/s$.

In some atomic physics applications it is important to know the currents induced in the connection to one of the electrodes by a moving charge in the trap.[25] Shockley[26] has shown that the current to an electrode due to a moving charge at an arbitrary point is proportional to the electric field at the point due to having the test electrode at unit potential and all the other electrodes at zero potential. For the $+z$ endcap this potential distribution is achieved by having $P_e=P_o=0.5$, and the electric field at an arbitrary position is easily determined from Eqs. (6)—(8).

The objective of this report is to provide data and other information to allow the design of electrode systems which provide an adequate approximation to Eq (1). The general strategy is to calculate $A_j$, $j=0$—6, for a great many specific cases and to work out some general rules for how to achieve small $A_4$ and $A_6$. The shapes examined are restricted to those which are reasonably close to the ideal given by Eqs. (3) and (4) and which otherwise are judged to be either instructive or practical.

## III. METHODS

The numerical method used here involved first finding the surface charge density on the electrode surfaces. An approximate numerical representation was achieved by dividing the electrode surfaces into zones each of which had approximately uniform surface charge density on it. Then the effect of each zone was approximated by replacing it with a similar zone which had a uniform charge distribution. Numerical values of the charge densities were determined by choosing those values which yielded the correct potential at a grid of points across the electrode surfaces. Once the surface charge density was adequately represented it was possible to calculate either the potential distribution or the coefficients of the spherical harmonic expansion.

A more specific description of the procedure follows. First divide the electrode's surface into $N$ (cylindrically symmetric) zones each of which has approximately uniform surface charge density. Let $\sigma_j$ be the surface charge density of zone $j$. Choose a point $(r_j,z_j)$ on the surface in each zone and call its potential $V_j$. The potential is uniform across each of the electrodes so the values of $V_j$ corresponding to one electrode will all be identical. Define $B_{ij}$ such that the potential at the point $(r_i,z_i)$ due to a surface charge density $\sigma_j$ is $B_{ij}\sigma_j$.

The potential due to different charges combine linearly with the result

$$V_i=\sum_j^N B_{ij}\sigma_j \ . \tag{9}$$

The $V_i$ are known and the $B_{ij}$ can be evaluated by an application of Coulomb's law. The set of equations (9) can then be solved by standard numerical methods[27] to yield a set of $\sigma_j$. This procedure can yield a quite accurate solution if the division of the surfaces can be carried out to provide an adequately accurate representation of the surface charge density without making $N$ unduly large. The details of how the zones are created is unimportant as long as each of the zones is small enough. What one considers small enough depends on the specific problem. Here the maximum interest is the potential near the center of the trap, and in this case large zones are acceptable on the more remote portions of surfaces (indeed, the very remote surfaces may be ignored altogether).

The elementary surfaces used here were small segments of a cone. Such surfaces can conform exactly to those portions of the electrode surfaces which are represented by straight lines in Fig. 1. Those portions of the surface which are represented by curved lines in Fig. 1 are a more difficult problem. In this case two points were chosen on the curved surface and the conical segment was generated by connecting the points by a straight line in the $(r,z)$ plane. This system, while operationally simple, is problematic due to all of the approximating conical segments being on one side of the surface they were to represent. To reduce this and other problems, adjacent pairs of conical segments were assumed to have the same surface charge densities. Thus each of the zones (of uniform charge density) referred to above consisted of two conical segments. This doubling of the number of conical segments did not seriously complicate the numerical procedure and provided much better conformity to the hyperbolic surfaces. The point $(r_i,z_i)$ within the zone where the potential $V_i$ was specified was the vertex between the two segments. This system resulted in the point $(r_i,z_i)$ being exactly on the electrode surfaces; however, in the case of the hyperbolic surface the approximating surface charge density was on the average slightly displaced. To anticipate a result discussed below, this displacement was the dominant source of error in the calculation. Nevertheless, by making the zones small enough it was possible to make the error small.

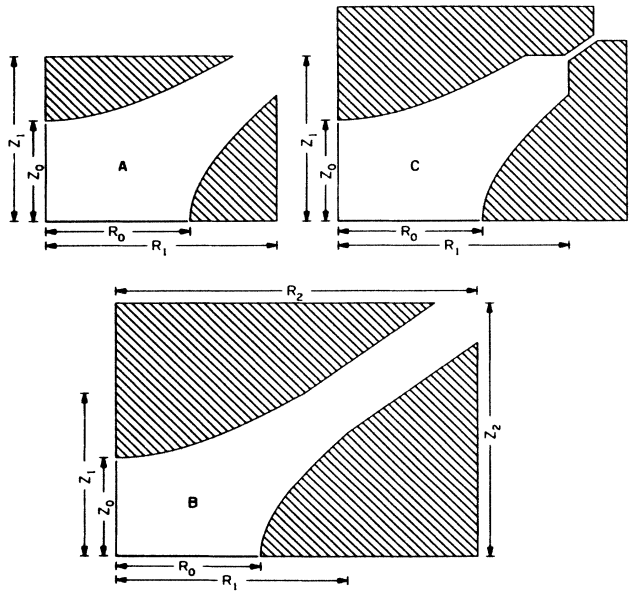


FIG. 1. Definitions of three electrode designs. Design A is simple and compact. Design B is fully shielded in such a way that a laser beam or other probe can be injected. Design C is also fully shielded and relatively compact.

Application of the above numerical procedure requires calculation of the potential due to a uniform surface charge on the conical ring. Let $\alpha$ be the angle of the conical segment with respect to the $z$ axis and let $(r_a,z_a)$ be a reference point on the surface. Further let $(r_1,z_1)$ be another point on the surface defined by

$$r_1=r_a+x\sin\alpha \ ,$$

$$z_1=z_a+x\cos\alpha \ ,$$

where $x$ is a measure of distance along the surface. The potential at an arbitrary point $(r,z)$ is given by

$$V(r,z)=\frac{\sigma}{2\pi\epsilon_0}\int_{x_1}^{x_2}\int_0^{\pi}\frac{r_1 dx\, d\theta}{X} \ , \tag{10}$$

where $X^2=(z-z_1)^2+r^2+r_1^2-2rr_1\cos\theta$, and $x_1$ and $x_2$ correspond to the edges of the conical segment. The integration over the $\theta$ variable can be carried out analytically in terms of the complete elliptic integral of the first kind, $K(k^2)$,[28]

$$V(r,z)=\frac{\sigma}{\pi\epsilon_0}\int_{x_1}^{x_2}\frac{r_a+x\sin\alpha}{R}K(k^2)dx \ , \tag{11}$$

where $R^2=(z-z_1)^2+(r+r_1)^2$,

$$k^2=\frac{4rr_1}{R^2} \ ,$$

$$K(k^2)=\int_0^{\pi/2}\frac{d\theta}{1-k^2\sin\theta} \ .$$

The integral of Eq. (11) can be evaluated by numerical quadrature. In one important circumstance this integration is difficult. The diagonal elements $B_{jj}$ involve evaluation of the potential due to a charged surface at a point on the surface. At this point the variable $k$ equals unity and the function $K$ is infinite. This problem is solved by breaking the integral into two parts one of which is a small region in the neighborhood of the singularity. Near the singularity the elliptic function $K$ can be well approximated by[28]

$$K(k^2)=0.5\ln\left[\frac{16}{1-k^2}\right] \ . \tag{12}$$

With this approximation the integration over the singularity can be done analytically,

Equation (11) served to determine the $B_{ij}$ needed for Eq. (9). Once the $\sigma_j$ were determined it also could be used to determine the potential at any point in the trap by summing the contributions of the elementary surfaces. The

$A_j$ were determined by expanding Eq. (11) in a Taylor series for the special case $r=0$, for which $k=0$ and $K(0)=\pi/2$. In this case the integral and the necessary derivatives can all be done analytically; however, the result is algebraically complicated. The method actually used was to do the derivatives analytically, then do the integral numerically. Thus the $A_j$ can be calculated directly from the charge distribution without first determining the potential.

A substantial effort was made to evaluate the reliability of these procedures as they apply to the present problem. The most important tests were specific to electrode shapes and are discussed below with the data. It is useful to anticipate one result. For several special cases a study was conducted of the effects on the $A_j$ of different ways of dividing the surfaces into zones. A prescription was adopted for distribution of the zones based on a characteristic zone width. As this zone width was varied the number of zones $N$ varied in a regular way with the form of the distribution remaining substantially constant. By extrapolating to zero width it was possible to estimate the error made by setting $N$ to a comfortable value rather than infinity. For a reference case (described below) the absolute errors in the $A_j$, $j=0-6$ were (1,28,22,22,0.6,0.9,$-0.2$) $\times 10^{-6}$, respectively. These values are representative of the expected error for other cases. By this standard other sources of error were made negligible. The computer used had a precision of approximately 14 digits and truncation error was a point of concern only in extreme cases. The expectation is that these errors are small compared to mechanical errors one would make in the assembly of a trap for atomic physics work.

The errors caused by failure to make the zones small enough were highly correlated. In comparing two similar electrode structures one expects both the results and the errors of computation to be similar. A result is that the relative errors are substantially smaller than the absolute errors. No citable measure of the relative errors is available.

A situation of particular interest involves two electrode structures one of which is an extension of the other. The extension can be considered to be a separate electrode at the same potential as the adjacent electrode. The difference of the resulting potential (or the $A_j$) can be attributed to the extra electrode. An extra electrode calls for an additional term in Eq. (6) which is linear in its potential. To determine the coefficient for this term it would be necessary to solve Eq. (9) one more time for each electrode. Even without this additional effort, the differencing technique referred to above allows an estimation of the coefficient with enough precision to be useful.

## IV. RESULTS

Using the methods of the preceding section, the first several $A_j$ were calculated for a large number of cases based on three general designs. Figure 1 shows a cross section in cylindrical coordinates of the three designs, called A, B, and C. Each of the drawings shows only the portion of the electrode assembly for positive $r$ and $z$. The full electrodes have reflection symmetry about the $z=0$ plane and cylindrical symmetry about the $r=0$ pole. Each of the designs includes a portion near the center of the trap with hyperbolic surfaces which match the ideal design. The differences in the three designs are in the way the hyperbolic surfaces are terminated. Design A is compact and simple. The principal inadequacy of this design is that for small electrodes the region in the center of the trap is not well shielded from disturbances outside the electrode structure. For instance, the trap potential could be affected significantly by the structure used to support the electrodes. Rather than include the supports and wires in the model, we use design A here to illustrate some interesting trends. Designs B and C are much better shielded and can be counted as practical designs. In both of these cases wires and mounting brackets outside the structure can be expected to have little effect on the potential near the trap center. Design B has an opening to admit a laser beam (or other probe) with the sides of the channel between the electrodes being parallel to the line $r=\sqrt{2}z$ [see Eq. (2)]. Variables $R_2$ and $Z_2$ were set to large values ($R_2=2.5$, $Z_2=R_2/\sqrt{2}$) such that they did not significantly affect the $A_j$. Design C is compact and closed, with the channel defined by two conditions: (1) The four points in Fig. 1 marking the ends of the channel form a rectangle with length $0.25R_0$ and width $0.05R_0$; (2) the sides are parallel to the line, $r=\sqrt{2}z$. In design B a reduction of $R_1$ simply removes some of the ring and leaves the endcap unchanged. For design C a similar reduction of $R_1$ leaves the size, shape, and orientation of the channel unchanged. In this case the location of the channel is changed requiring a change in the shape of each of the electrodes.

The surface charge densities were calculated for $R_0=1.0$ and a large variety of values for the variables $Z_0$, $Z_1$, and $R_1$ with both even and odd potential distributions. The results are reported by giving $A_j$, $j=0-6$, using the convention that $s=R_0$. A quality which these designs have in common is that if the hyperbolic part of the surface is made large the potential approaches the ideal. This feature permits a check of the procedures in circumstances in which some of the results are known in advance.

Tables I, II, and III display some of the results for cases A, B, and C, respectively. These data are all for the condition $Z_0=0.707\,11$. Table IV provides further data for design A with a range of values of $Z_0$, but only for values of $R_1$ and $Z_1$ which yield $A_4=0$. For each of the electrode structures it was necessary to choose a system for dividing the surfaces into zones in accordance with the general method described above. A great many systems were tried in an effort to find one such that the results were insensitive to the details of the zone division. The system used did not quite achieve this objective. Therefore, the data of Tables I–IV must be interpreted with these issues in mind.

The procedure used for dividing the electrode surfaces into zones involved choosing a sequence of points on the lines of Fig. 1 which represent the surfaces. These points were then connected by straight lines to form the small conical segments used to approximate the surface. For each electrode a nominal width was chosen. Then the

TABLE I. Computed coefficients $A_j$ in the spherical harmonic expansion of the potential for electrode design A. The variables of columns 1 and 2 are defined in Fig. 1. For these data $Z_0^2=0.5$, $R_0=1$.

| $Z_1$ | $R_1$ | $A_0$ | $A_2$ | $A_4$ | $A_6$ | $A_1$ | $A_3$ | $A_5$ |
|---|---|---|---|---|---|---|---|---|
| 1.800 00 | 1.237 09 | 0.513 373 77 | 0.493 327 29 | −0.005 218 11 | 0.001 824 66 | 1.150 441 45 | 0.260 243 58 | −0.004 747 44 |
| 1.800 00 | 1.283 30 | 0.508 656 32 | 0.495 781 03 | −0.003 438 51 | 0.001 161 86 | 1.145 392 61 | 0.266 096 07 | −0.004 767 87 |
| 1.800 00 | 1.341 32 | 0.504 982 24 | 0.497 610 65 | −0.002 006 50 | 0.000 659 24 | 1.141 257 34 | 0.270 816 23 | −0.004 723 93 |
| 1.800 00 | 1.414 21 | 0.502 448 08 | 0.498 830 60 | −0.000 995 69 | 0.000 320 63 | 1.138 274 83 | 0.274 181 08 | −0.004 660 06 |
| 1.800 00 | 1.489 95 | 0.501 140 23 | 0.499 446 77 | −0.000 466 79 | 0.000 148 71 | 1.136 678 92 | 0.275 968 49 | −0.004 615 39 |
| 1.800 00 | 1.565 71 | 0.500 514 64 | 0.499 738 29 | −0.000 212 09 | 0.000 067 18 | 1.135 895 21 | 0.276 843 03 | −0.004 590 90 |
| 1.800 00 | 1.670 66 | 0.500 160 59 | 0.499 902 44 | −0.000 067 53 | 0.000 021 27 | 1.135 442 14 | 0.277 347 69 | −0.004 576 02 |
| 1.800 00 | 1.758 16 | 0.500 056 67 | 0.499 950 41 | −0.000 024 96 | 0.000 007 80 | 1.135 306 59 | 0.277 498 52 | −0.004 571 44 |
| 1.800 00 | 1.852 46 | 0.500 016 30 | 0.499 968 96 | −0.000 008 36 | 0.000 002 56 | 1.135 253 21 | 0.277 557 88 | −0.004 569 60 |
| 1.800 00 | 1.946 00 | 0.500 002 76 | 0.499 975 39 | −0.000 002 96 | 0.000 000 85 | 1.135 235 42 | 0.277 577 73 | −0.004 569 06 |
| 1.800 00 | 2.050 00 | 0.499 998 67 | 0.499 977 16 | −0.000 001 19 | 0.000 000 29 | 1.135 229 76 | 0.277 583 99 | −0.004 568 83 |
| 1.800 00 | 2.202 91 | 0.499 997 25 | 0.499 977 90 | −0.000 000 67 | 0.000 000 13 | 1.135 227 96 | 0.277 586 02 | −0.004 568 79 |
| 1.800 00 | 2.360 29 | 0.499 997 34 | 0.499 977 73 | −0.000 000 62 | 0.000 000 11 | 1.135 227 92 | 0.277 586 03 | −0.004 568 74 |
| 1.800 00 | 2.500 00 | 0.499 997 04 | 0.499 978 02 | −0.000 000 61 | 0.000 000 11 | 1.135 227 72 | 0.277 586 29 | −0.004 568 79 |
| 1.000 00 | 2.500 00 | 0.492 244 85 | 0.503 411 22 | 0.003 248 19 | −0.000 966 47 | 1.124 165 18 | 0.289 727 44 | −0.004 063 84 |
| 1.028 47 | 2.500 00 | 0.494 749 03 | 0.502 334 44 | 0.002 182 18 | −0.000 662 55 | 1.127 769 77 | 0.285 807 45 | −0.004 256 88 |
| 1.064 59 | 2.500 00 | 0.496 841 47 | 0.501 411 67 | 0.001 303 78 | −0.000 402 39 | 1.130 759 52 | 0.282 531 94 | −0.004 395 91 |
| 1.110 48 | 2.500 00 | 0.498 379 98 | 0.500 718 08 | 0.000 664 53 | −0.000 207 53 | 1.132 942 90 | 0.280 119 95 | −0.004 485 76 |
| 1.158 53 | 2.500 00 | 0.499 216 50 | 0.500 336 91 | 0.000 319 67 | −0.000 100 49 | 1.134 126 34 | 0.278 809 72 | −0.004 530 09 |
| 1.207 02 | 2.500 00 | 0.499 633 86 | 0.500 145 62 | 0.000 148 30 | −0.000 046 81 | 1.134 715 80 | 0.278 156 17 | −0.004 551 09 |
| 1.274 75 | 2.500 00 | 0.499 878 28 | 0.500 032 72 | 0.000 048 00 | −0.000 015 23 | 1.135 060 12 | 0.277 772 35 | −0.004 563 09 |
| 1.331 69 | 2.500 00 | 0.499 952 78 | 0.499 998 42 | 0.000 017 50 | −0.000 005 61 | 1.135 165 24 | 0.277 655 67 | −0.004 566 64 |
| 1.393 28 | 2.500 00 | 0.499 982 50 | 0.499 984 67 | 0.000 005 32 | −0.000 001 77 | 1.135 207 12 | 0.277 608 97 | −0.004 568 11 |
| 1.456 02 | 2.500 00 | 0.499 992 56 | 0.499 979 99 | 0.000 001 19 | −0.000 000 46 | 1.135 221 27 | 0.277 593 12 | −0.004 568 56 |
| 1.505 76 | 2.500 00 | 0.499 995 51 | 0.499 978 92 | 0.000 000 07 | −0.000 000 11 | 1.135 225 77 | 0.277 589 07 | −0.004 568 67 |
| 1.610 16 | 2.500 00 | 0.499 996 82 | 0.499 978 07 | −0.000 000 54 | 0.000 000 08 | 1.135 227 35 | 0.277 586 53 | −0.004 568 76 |
| 1.718 09 | 2.500 00 | 0.499 996 84 | 0.499 977 82 | −0.000 000 61 | 0.000 000 11 | 1.135 227 09 | 0.277 586 03 | −0.004 568 82 |
| 1.800 00 | 2.500 00 | 0.499 997 04 | 0.499 978 02 | −0.000 000 61 | 0.000 000 11 | 1.135 227 72 | 0.277 586 29 | −0.004 568 79 |
| 1.000 00 | 1.237 09 | 0.500 475 86 | 0.499 391 23 | −0.000 000 66 | 0.000 140 13 | 1.133 085 12 | 0.279 608 84 | −0.004 207 92 |
| 1.028 47 | 1.283 30 | 0.500 203 86 | 0.499 723 24 | −0.000 000 77 | 0.000 061 95 | 1.133 857 18 | 0.278 949 63 | −0.004 394 56 |
| 1.064 59 | 1.341 32 | 0.500 067 79 | 0.499 891 23 | −0.000 000 56 | 0.000 021 54 | 1.134 460 46 | 0.278 385 20 | −0.004 498 02 |
| 1.110 45 | 1.414 21 | 0.500 014 50 | 0.499 955 91 | −0.000 000 65 | 0.000 005 54 | 1.134 865 85 | 0.277 973 35 | −0.004 545 22 |
| 1.158 53 | 1.489 95 | 0.500 000 93 | 0.499 973 05 | −0.000 000 57 | 0.000 001 31 | 1.135 067 60 | 0.277 760 96 | −0.004 560 75 |
| 1.207 02 | 1.565 71 | 0.499 998 12 | 0.499 977 03 | −0.000 000 59 | 0.000 000 35 | 1.135 159 65 | 0.277 662 04 | −0.004 565 88 |
| 1.274 75 | 1.670 66 | 0.499 997 04 | 0.499 977 84 | −0.000 000 60 | 0.000 000 13 | 1.135 207 32 | 0.277 608 56 | −0.004 568 07 |
| 1.331 69 | 1.758 16 | 0.499 997 16 | 0.499 977 84 | −0.000 000 61 | 0.000 000 11 | 1.135 220 81 | 0.277 593 83 | −0.004 568 52 |
| 1.393 28 | 1.852 46 | 0.499 997 26 | 0.499 977 67 | −0.000 000 59 | 0.000 000 11 | 1.135 225 61 | 0.277 588 35 | −0.004 568 68 |
| 1.456 02 | 1.946 00 | 0.499 997 02 | 0.499 977 93 | −0.000 000 64 | 0.000 000 12 | 1.135 226 99 | 0.277 586 76 | −0.004 568 77 |
| 1.503 42 | 2.046 57 | 0.499 996 85 | 0.499 978 27 | −0.000 000 39 | 0.000 000 03 | 1.135 227 42 | 0.277 587 38 | −0.004 568 70 |
| 1.556 63 | 2.124 59 | 0.499 996 81 | 0.499 977 80 | −0.000 000 55 | 0.000 000 09 | 1.135 226 93 | 0.277 586 26 | −0.004 568 80 |
| 1.610 16 | 2.202 91 | 0.499 997 00 | 0.499 977 97 | −0.000 000 59 | 0.000 000 10 | 1.135 227 55 | 0.277 586 30 | −0.004 568 76 |
| 1.718 09 | 2.360 29 | 0.499 997 14 | 0.499 977 53 | −0.000 000 62 | 0.000 000 12 | 1.135 227 29 | 0.277 585 77 | −0.004 568 77 |

TABLE II. Computed coefficients $A_j$ in the spherical harmonic expansion of the potential for electrode design B. The variables of columns 1 and 2 are defined in Fig. 1.

| $Z_1$ | $R_1$ | $A_0$ | $A_2$ | $A_4$ | $A_6$ | $A_1$ | $A_3$ | $A_5$ |
|---|---|---|---|---|---|---|---|---|
| 1.000 00 | 1.320 10 | 0.500 023 71 | 0.499 943 29 | −0.000 000 59 | 0.000 008 52 | 1.135 238 68 | 0.277 548 02 | −0.004 550 65 |
| 1.031 22 | 1.365 39 | 0.500 007 38 | 0.499 963 15 | −0.000 000 61 | 0.000 003 42 | 1.135 230 44 | 0.277 570 31 | −0.004 561 66 |
| 1.064 88 | 1.414 21 | 0.500 000 74 | 0.499 972 72 | −0.000 000 60 | 0.000 001 27 | 1.135 228 90 | 0.277 581 03 | −0.004 566 21 |
| 1.115 63 | 1.487 98 | 0.499 997 02 | 0.499 975 60 | −0.000 000 61 | 0.000 000 36 | 1.135 225 08 | 0.277 584 22 | −0.004 568 34 |
| 1.188 28 | 1.593 92 | 0.499 996 57 | 0.499 976 86 | −0.000 000 60 | 0.000 000 12 | 1.135 225 68 | 0.277 585 74 | −0.004 568 76 |
| 1.274 75 | 1.721 00 | 0.499 996 71 | 0.499 976 74 | −0.000 000 60 | 0.000 000 10 | 1.135 225 76 | 0.277 585 67 | −0.004 568 79 |
| 1.342 30 | 1.824 04 | 0.499 996 30 | 0.499 976 77 | −0.000 000 59 | 0.000 000 10 | 1.135 225 06 | 0.277 585 64 | −0.004 568 83 |
| 1.405 00 | 1.920 00 | 0.499 996 53 | 0.499 976 51 | −0.000 000 63 | 0.000 000 12 | 1.135 225 17 | 0.277 585 31 | −0.004 568 83 |
| 1.456 02 | 2.030 00 | 0.499 996 54 | 0.499 976 88 | −0.000 000 61 | 0.000 000 10 | 1.135 225 65 | 0.277 585 76 | −0.004 568 82 |

TABLE III. Computed coefficients $A_j$ in the spherical harmonic expansion of the potential for electrode design C. The variables of columns 1 and 2 are defined in Fig. 1.

| $Z_1$ | $R_1$ | $A_0$ | $A_2$ | $A_4$ | $A_6$ | $A_1$ | $A_3$ | $A_5$ |
|---|---|---|---|---|---|---|---|---|
| 0.980 23 | 1.336 46 | 0.499 811 93 | 0.500 204 91 | −0.000 000 68 | −0.000 058 20 | 1.135 136 17 | 0.277 843 46 | −0.004 693 62 |
| 1.026 91 | 1.404 61 | 0.499 941 67 | 0.500 042 98 | −0.000 000 50 | −0.000 016 76 | 1.135 197 03 | 0.277 661 59 | −0.004 604 97 |
| 1.075 54 | 1.475 09 | 0.499 981 85 | 0.499 993 70 | −0.000 000 57 | −0.000 004 29 | 1.135 217 04 | 0.277 605 64 | −0.004 578 24 |
| 1.124 21 | 1.545 43 | 0.499 992 70 | 0.499 980 85 | −0.000 000 59 | −0.000 001 00 | 1.135 223 03 | 0.277 590 91 | −0.004 571 13 |
| 1.174 52 | 1.617 95 | 0.499 996 25 | 0.499 977 13 | −0.000 000 61 | −0.000 000 15 | 1.135 225 44 | 0.277 586 73 | −0.004 569 24 |
| 1.225 49 | 1.691 28 | 0.499 996 15 | 0.499 975 99 | −0.000 000 59 | 0.000 000 06 | 1.135 223 97 | 0.277 585 30 | −0.004 568 95 |
| 1.277 14 | 1.765 52 | 0.499 996 46 | 0.499 976 19 | −0.000 000 60 | 0.000 000 09 | 1.135 224 78 | 0.277 585 40 | −0.004 568 82 |
| 1.329 37 | 1.840 42 | 0.499 996 66 | 0.499 976 48 | −0.000 000 62 | 0.000 000 10 | 1.135 225 49 | 0.277 585 59 | −0.004 568 80 |
| 1.382 19 | 1.915 81 | 0.499 996 73 | 0.499 976 81 | −0.000 000 60 | 0.000 000 09 | 1.135 225 99 | 0.277 585 94 | −0.004 568 76 |
| 1.435 04 | 1.992 32 | 0.499 996 07 | 0.499 976 53 | −0.000 000 60 | 0.000 000 10 | 1.135 224 57 | 0.277 585 43 | −0.004 568 83 |
| 1.488 40 | 2.069 10 | 0.499 996 21 | 0.499 976 83 | −0.000 000 60 | 0.000 000 10 | 1.135 225 18 | 0.277 585 73 | −0.004 568 82 |

widths of the zones were adjusted slightly to allow an integral number to be smoothly distributed over the curved surface. On the more remote portions of the surface larger zones were assigned, and at the sensitive places, such as near the center and near sharp corners, zones were divided in half enough times that the charge distribution could be reasonably represented. The details of the algorithm for division is not important; however, it is important that a regular system was used. As the nominal width of the segments was varied the total number of segments varied in a regular way. The $A_j$ also varied in a regular way and it was possible to extrapolate reasonably to zero width. The largest number of zones which could be easily handled was about 250; however, in this case the time required to solve the linear equations was excessive. Also the larger number of arithmetic operations produced more "random" noise from numerical truncation error. The "standard" width of a segment on the hyperbolic surface of the endcaps was such that the component in the $r$ direction was 0.05. Similarly the width of a segment on the ring had a $z$ component of 0.035. These widths resulted in about 60—100 zones.

The extrapolation to zero-width zones was done for many electrode structures. The most intense application of this test was for design A with moderate-sized electrodes. This yielded errors in the $A_j$ of $(1\pm2,\ 28\pm6,\ 22\pm4,\ 22\pm5,\ 0.6\pm0.6,\ 0.9\pm0.9,\ -0.2\pm0.4)\times10^{-6}$ for $j=0-6$, respectively. The stated uncertainties are based on estimates of the reliability of the extrapolation procedure. Within the tolerances cited the errors were insensitive to electrode size and the data of Table I would be a little more accurate with these corrections. The data of Figs. 2 and 3 were adjusted to compensate for these errors. These corrections were not applied to the data in the tables because the improvement was small and they could not be applied to other geometries. These extrapolations were too difficult to carry out for every case; however, a few checks indicated that all the data of Tables I—IV are subject to similar errors. The expectation is that this level of uncertainty is acceptable for most applications in atomic physics.

In the limit of large $R_1$, $Z_1$ the even $A_j$ are known to approach 0.5, 0.5, 0.0, 0.0 for all three designs. When the corrections discussed above are applied to the data of Table I these limits are approached well within the estimated uncertainties in the corrections. When the odd $A_j$ are similarly corrected the values for $j=1,3,5$ approach 1.135 253, 0.277 608, −0.004 570. Gabrielse[23] has examined this limit in another context and has reported corresponding values (translated to the language used here) of 1.13, 0.28, and −0.0045.

The absolute uncertainties implied by the above discussion are too large to justify the number of digits given in these tables. The extra digits are given to permit examination of the changes which result from small changes in the sizes of the electrodes. The errors of computation are highly correlated such that for two similar models the errors are similar.

An interesting design option is to use design A but to surround the trap by another electrode with adjustable potential. Such an electrode can serve as a shield but can

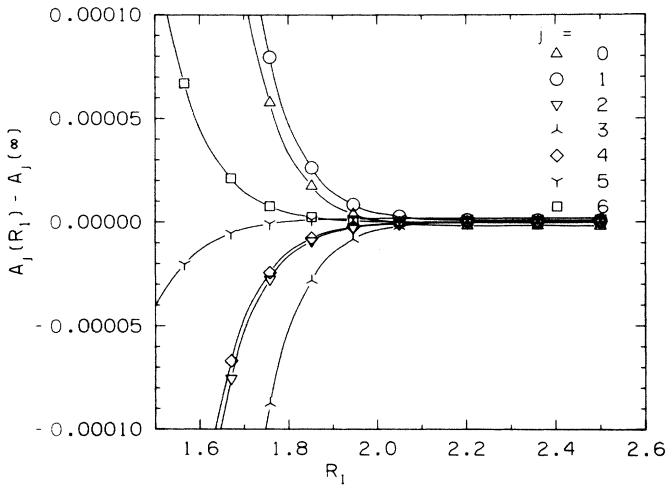


FIG. 2. Variation of the $A_j$ with ring-electrode truncation for design A with $Z_0=0.707\,11$. The endcap electrode was large and constant ($Z_1=1.8$).

TABLE IV. Computed coefficients $A_j$ in the spherical harmonic expansion of the potential for electrode design A. The variables of columns 1–3 are defined in Fig. 1. The data presented here are selected for the condition that $A_4=0$.

| $Z_1$ | $R_1$ | $Z_0$ | $A_0$ | $A_2$ | $A_6$ | $A_1$ | $A_3$ | $A_5$ |
|---|---|---|---|---|---|---|---|---|
| 0.871 00 | 1.300 00 | 0.300 00 | 0.847 518 87 | 0.846 976 67 | 0.000 522 59 | 2.908 538 91 | 1.902 843 95 | 0.946 593 43 |
| 0.886 91 | 1.300 00 | 0.400 00 | 0.757 573 53 | 0.757 495 54 | 0.000 042 72 | 2.160 888 61 | 0.887 474 59 | 0.193 379 41 |
| 0.928 10 | 1.300 00 | 0.500 00 | 0.666 685 39 | 0.666 575 77 | 0.000 008 33 | 1.699 942 22 | 0.530 399 84 | 0.045 034 00 |
| 0.977 71 | 1.300 00 | 0.600 00 | 0.581 463 56 | 0.581 261 35 | 0.000 016 72 | 1.382 462 68 | 0.370 126 49 | 0.006 824 08 |
| 1.034 52 | 1.300 00 | 0.700 00 | 0.505 195 49 | 0.504 849 55 | 0.000 043 12 | 1.148 409 84 | 0.283 420 78 | −0.004 069 87 |
| 1.097 44 | 1.300 00 | 0.800 00 | 0.438 820 17 | 0.438 290 12 | 0.000 087 70 | 0.968 181 91 | 0.229 128 25 | −0.006 891 27 |
| 1.036 56 | 1.500 00 | 0.400 00 | 0.757 563 65 | 0.757 528 53 | 0.000 038 54 | 2.160 972 42 | 0.887 167 16 | 0.193 473 00 |
| 1.069 95 | 1.500 00 | 0.500 00 | 0.666 656 12 | 0.666 636 37 | 0.000 003 24 | 1.700 164 83 | 0.529 859 47 | 0.045 101 82 |
| 1.112 55 | 1.500 00 | 0.600 00 | 0.581 389 75 | 0.581 368 65 | 0.000 000 45 | 1.382 960 68 | 0.369 328 78 | 0.006 822 88 |
| 1.161 12 | 1.500 00 | 0.700 00 | 0.505 051 09 | 0.505 026 94 | 0.000 000 82 | 1.149 383 92 | 0.282 414 62 | −0.004 187 37 |
| 1.215 74 | 1.500 00 | 0.800 00 | 0.438 603 84 | 0.438 571 81 | 0.000 002 87 | 0.969 891 70 | 0.228 028 51 | −0.007 161 34 |
| 1.239 20 | 1.500 00 | 0.840 00 | 0.414 741 65 | 0.414 703 97 | 0.000 004 32 | 0.909 292 88 | 0.211 426 75 | −0.007 486 29 |
| 1.251 24 | 1.500 00 | 0.860 00 | 0.403 367 46 | 0.403 326 68 | 0.000 005 19 | 0.880 993 65 | 0.203 918 13 | −0.007 550 55 |
| 1.263 49 | 1.500 00 | 0.880 00 | 0.392 354 06 | 0.392 309 75 | 0.000 006 18 | 0.853 920 79 | 0.196 865 66 | −0.007 566 42 |
| 1.275 93 | 1.500 00 | 0.900 00 | 0.381 692 40 | 0.381 644 47 | 0.000 007 27 | 0.828 001 26 | 0.190 225 07 | −0.007 543 57 |
| 1.340 94 | 1.500 00 | 1.000 00 | 0.333 339 64 | 0.333 267 88 | 0.000 014 60 | 0.713 543 94 | 0.161 997 91 | −0.007 079 83 |
| 1.410 00 | 1.500 00 | 1.100 00 | 0.292 363 59 | 0.292 261 35 | 0.000 024 73 | 0.619 749 96 | 0.139 890 99 | −0.006 374 41 |
| 1.482 32 | 1.500 00 | 1.200 00 | 0.257 585 37 | 0.257 450 32 | 0.000 036 71 | 0.541 840 56 | 0.122 002 95 | −0.005 643 42 |
| 1.109 10 | 1.600 00 | 0.400 00 | 0.757 563 67 | 0.757 527 56 | 0.000 039 99 | 2.160 973 65 | 0.887 149 93 | 0.193 476 78 |
| 1.136 51 | 1.600 00 | 0.500 00 | 0.666 655 67 | 0.666 636 88 | 0.000 003 32 | 1.700 176 29 | 0.529 828 43 | 0.451 106 09 |
| 1.179 29 | 1.600 00 | 0.600 00 | 0.581 389 46 | 0.581 370 84 | 0.000 000 13 | 1.383 000 32 | 0.369 269 74 | 0.006 825 64 |
| 1.225 26 | 1.600 00 | 0.700 00 | 0.505 048 16 | 0.505 030 22 | 0.000 000 04 | 1.149 474 85 | 0.282 312 16 | −0.004 191 82 |
| 1.276 54 | 1.600 00 | 0.800 00 | 0.438 597 73 | 0.438 579 80 | 0.000 000 42 | 0.970 083 59 | 0.227 882 65 | −0.007 179 28 |
| 1.298 59 | 1.600 00 | 0.840 00 | 0.414 733 74 | 0.414 715 27 | 0.000 000 73 | 0.909 540 99 | 0.211 266 64 | −0.007 511 43 |
| 1.309 94 | 1.600 00 | 0.860 00 | 0.403 359 02 | 0.403 340 06 | 0.000 000 93 | 0.881 274 31 | 0.203 751 64 | −0.007 579 72 |
| 1.321 49 | 1.600 00 | 0.880 00 | 0.392 345 07 | 0.392 325 59 | 0.000 001 16 | 0.854 236 57 | 0.196 693 85 | −0.007 599 79 |
| 1.333 25 | 1.600 00 | 0.900 00 | 0.381 683 44 | 0.381 663 22 | 0.000 001 43 | 0.828 355 74 | 0.190 048 57 | −0.007 581 43 |
| 1.394 93 | 1.600 00 | 1.000 00 | 0.333 337 05 | 0.333 311 13 | 0.000 003 48 | 0.714 141 34 | 0.161 817 44 | −0.007 142 08 |
| 1.460 96 | 1.600 00 | 1.100 00 | 0.292 390 19 | 0.292 354 62 | 0.000 006 87 | 0.620 687 64 | 0.139 743 97 | −0.006 462 57 |
| 1.530 65 | 1.600 00 | 1.200 00 | 0.257 687 04 | 0.257 638 75 | 0.000 011 57 | 0.543 244 76 | 0.121 933 95 | −0.005 756 70 |
| 1.166 22 | 1.700 00 | 0.400 00 | 0.757 563 32 | 0.757 529 87 | 0.000 038 09 | 2.160 976 81 | 0.887 153 40 | 0.193 477 60 |
| 1.192 50 | 1.700 00 | 0.500 00 | 0.666 655 68 | 0.666 637 54 | 0.000 003 13 | 1.700 180 33 | 0.529 822 56 | 0.045 107 34 |
| 1.243 56 | 1.700 00 | 0.600 00 | 0.581 388 98 | 0.581 371 25 | 0.000 000 10 | 1.383 010 11 | 0.369 252 79 | 0.006 826 32 |
| 1.289 51 | 1.700 00 | 0.700 00 | 0.505 048 20 | 0.505 031 38 | −0.000 000 08 | 1.149 506 49 | 0.282 279 03 | −0.004 192 89 |
| 1.338 18 | 1.700 00 | 0.800 00 | 0.438 596 49 | 0.438 581 14 | 0.000 000 03 | 0.970 154 99 | 0.227 826 58 | −0.007 185 04 |
| 1.359 01 | 1.700 00 | 0.840 00 | 0.414 732 16 | 0.414 717 30 | 0.000 000 09 | 0.909 637 82 | 0.211 201 40 | −0.007 520 05 |
| 1.369 73 | 1.700 00 | 0.860 00 | 0.403 357 23 | 0.403 342 57 | 0.000 000 14 | 0.881 386 01 | 0.203 682 02 | −0.007 589 89 |
| 1.380 66 | 1.700 00 | 0.880 00 | 0.392 343 22 | 0.392 328 68 | 0.000 000 19 | 0.854 364 95 | 0.196 619 91 | −0.000 611 73 |
| 1.391 78 | 1.700 00 | 0.900 00 | 0.381 681 43 | 0.381 667 03 | 0.000 000 25 | 0.828 502 29 | 0.189 970 70 | −0.007 595 22 |
| 1.450 31 | 1.700 00 | 1.000 00 | 0.333 336 26 | 0.333 321 58 | 0.000 000 78 | 0.714 407 71 | 0.161 726 05 | −0.007 167 10 |
| 1.513 34 | 1.700 00 | 1.100 00 | 0.292 398 04 | 0.292 381 27 | 0.000 001 81 | 0.621 130 13 | 0.139 655 69 | −0.006 501 01 |
| 1.580 30 | 1.700 00 | 1.200 00 | 0.257 720 85 | 0.257 700 14 | 0.000 003 48 | 0.543 933 66 | 0.121 872 13 | −0.005 809 50 |
| 1.242 50 | 2.000 00 | 0.500 00 | 0.666 655 45 | 0.666 636 45 | 0.000 003 58 | 1.700 178 48 | 0.529 817 49 | 0.045 107 19 |
| 1.356 65 | 2.000 00 | 0.600 00 | 0.581 388 88 | 0.581 370 35 | 0.000 000 21 | 1.383 012 21 | 0.369 245 29 | 0.006 826 42 |
| 1.457 00 | 2.000 00 | 0.700 00 | 0.505 047 99 | 0.505 030 92 | −0.000 000 09 | 1.149 518 27 | 0.282 264 04 | −0.004 193 29 |
| 1.522 62 | 2.000 00 | 0.800 00 | 0.438 596 88 | 0.438 581 50 | −0.000 000 04 | 0.970 193 11 | 0.227 797 29 | −0.007 187 71 |
| 1.543 05 | 2.000 00 | 0.840 00 | 0.414 732 40 | 0.414 717 76 | −0.000 000 02 | 0.909 692 55 | 0.211 164 40 | −0.007 524 33 |
| 1.553 00 | 2.000 00 | 0.860 00 | 0.403 357 41 | 0.403 343 13 | −0.000 000 01 | 0.881 451 12 | 0.203 640 91 | −0.007 595 23 |
| 1.562 85 | 2.000 00 | 0.880 00 | 0.392 343 28 | 0.392 329 37 | −0.000 000 01 | 0.854 441 88 | 0.196 574 65 | −0.007 618 20 |
| 1.572 72 | 2.000 00 | 0.900 00 | 0.381 681 13 | 0.381 667 57 | 0.000 000 00 | 0.828 591 99 | 0.189 920 91 | −0.007 602 96 |
| 1.623 86 | 2.000 00 | 1.000 00 | 0.333 336 22 | 0.333 324 27 | 0.000 000 02 | 0.714 595 29 | 0.161 655 93 | −0.007 183 30 |
| 1.678 75 | 2.000 00 | 1.100 00 | 0.292 401 16 | 0.292 390 60 | 0.000 000 04 | 0.621 478 42 | 0.139 573 40 | −0.006 529 12 |
| 1.737 70 | 2.000 00 | 1.200 00 | 0.257 735 52 | 0.257 726 05 | 0.000 000 09 | 0.544 522 54 | 0.121 795 08 | −0.005 852 29 |

also serve as a means to make small adjustments to the $C_j$. Traps using this effect have been built, and Gabrielese[22,23] has calculated properties of some specific designs. The potential near the center of the trap and all the $C_j$ are linear functions of the potential of the shielded electrode. There are many cases in Tables I and II of pairs of electrode structures in which one is an extension of the other. In such cases the extension may be regarded

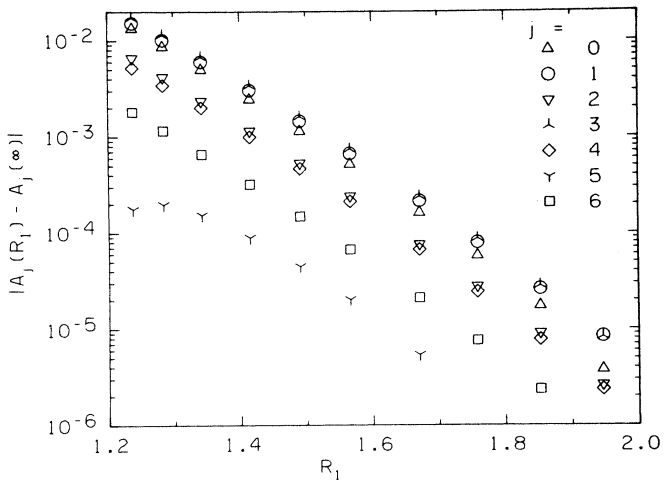


FIG. 3. Logarithmic plot of the variation of the $A_j$ with ring truncation for design A with $Z_0=0.707\,11$. The endcap electrode was large and constant ($Z_1=1.8$). Only absolute values are displayed and the curves are truncated on the right to avoid changes of sign.

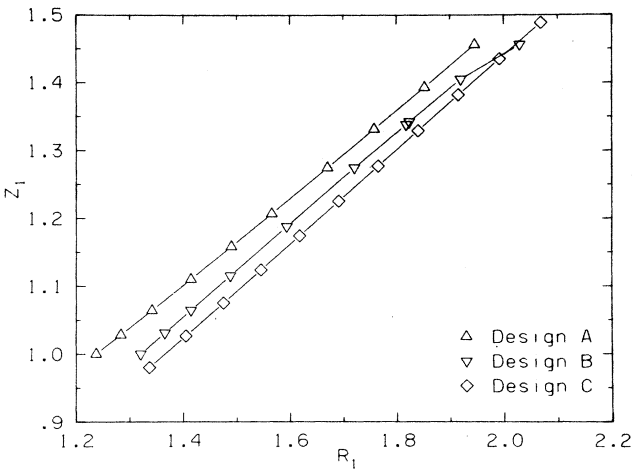


FIG. 4. Truncation conditions to produce $A_4=0$ for designs A, B, and C, with $Z_0=0.707\,11$.

as an extra electrode. The data of Tables I yield the effect of having the extra electrode at only one potential (that of the adjacent electrode). While this does not provide a general solution it is a useful basis for estimation. In reviewing these data some generalizations of this type are pointed out.

Figure 2 shows some data based on design A which illustrates some important effects. In this case the endcap electrodes are large ($Z_1=1.8$) and positioned with $Z_0=0.707\,11$. The outer radius $R_1$ of the ring electrode is varied and the response of the $A_j$ is plotted. In the limit of large $R_1$ the even $A_j$ are known to approach ideal and the odd $A_j$ are reported above to approach 1.352 53, 0.277 608, −0.004 570. The displayed ordinate is the difference between $A_j(R_1)$ and these asymptotic values. The displayed curves do not go to zero at large $R_1$ because of errors in the computation. The absolute values of some of the same data are shown in Fig. 3 in logarithmic form. In this instance a broader range of data is covered and it is apparent that the variation of each of the $A_j$ is approximately exponential with increasing electrode size. Other data from Table I allow a similar plot depicting the results of changing the endcap size. The results are quite similar except for a change in sign of each of the $A_j$. From Figs. 2 and 3 we see that if $R_1>2.0$ the shielding is effectively complete and there is little change in any of the $A_j$ produced by changes in the size of the ring. Other data in Table I show that if $Z_1>1.6$ the size of the endcaps are of little practical consequence.

Truncation of the hyperbolic part of the surfaces on the ring and endcap electrodes produce effects of different sign with the result that it is possible to truncate both and keep $A_4=0$. This condition is referred to below as "balanced" truncation. Tables I, II, and III include such data for each of the three electrode designs. (In Table I these data have $A_4=0.6\times10^{-6}$ in anticipation of the correction discussed above.) Figure 4 displays the electrode sizes implied by these data. At the upper right of the graph the curves are somewhat irregular because $A_4$ is insensitive to the electrode sizes and errors in the calculation have an enhanced effect. Figure 5 displays data from Table I for those cases which have balanced truncation. Figure 5 is similar enough to Fig. 2 that it may be helpful to remark on some of the differences. In Fig. 2, $Z_1$ was held constant as $R_1$ was varied, whereas in Fig. 5, $Z_1$ was varied to keep $A_4$ constant. Also the data of Fig. 2 were corrected for an error (discussed above) but the comparable errors for much of the data of Fig. 5 were not evaluated. Without this correction the data of Fig. 5 do not approach the known asymptotic values as closely. Note that it is the odd-order coefficients in Fig. 5 which show the largest departure from the asymptotic values. Also note that to the tolerance of the calculation the asymptotic values of

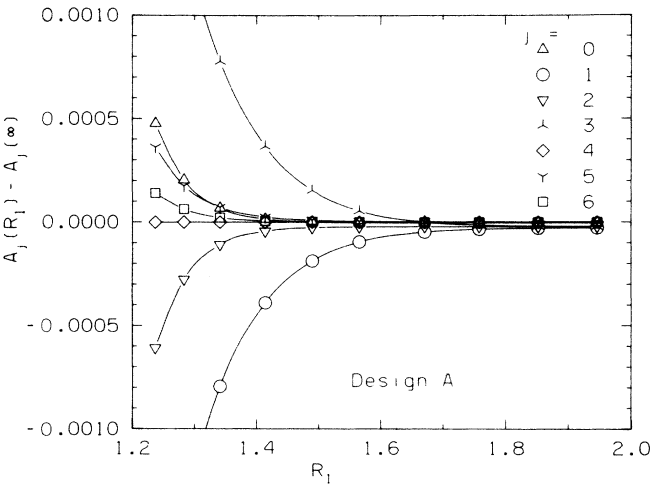


FIG. 5. Variation of $A_j$ with $R_1$, with $Z_1$ related to $R_1$ by the condition that $A_4=0$.

the even order $A_i$ in Fig. 5 are achieved for all values of $R_1 > 1.5$. It is a reasonable conjecture that the next few terms in the even series are also small for this circumstance. However, for values of $R_1$ as small as 1.5 the gap between the electrodes is large enough that the shielding is poor. Thus the potentials of other objects in the neighborhood will be rather important. Also it may be important to have the ring electrode at "ground" potential.

A small increase in the endcap size causes a decrease in $A_4$, and a small increase in the ring causes an increase. In either case there was a change in each of the $A_j$ approximately proportional to the change in surface area. Figure 6 shows the partial derivative of $A_4$ with respect to $R_1$ holding $Z_1$ constant. These data were generated as a by-product of the trial-and-error procedures used in finding the conditions of balanced truncation for design A. These data provide a way to estimate the effect on $A_4$ of some flaws in construction of electrodes. If the change in the ring is about 1.6 of the change in the endcap then $A_4$ is changed little. A small increase in $Z_1$ produces more surface area than the same size increase in $R_1$.

The data presented in Tables I–III and Figs. 1–6 are all for electrodes with $Z_0 = 0.707\,11$, this being the case of maximum interest. Most of the general conclusions from above can be expected to apply for other values of $Z_0$. Table IV contains data on design A with a variety of values for $Z_0$ and with $Z_1$ and $R_1$ chosen to yield $A_4 = 0$. A change in $Z_0$ involves more than a displacement of the endcaps; the shape of the hyperbolic surfaces change as well. Figure 7 is a plot showing some of the geometrical conditions required to produce $A_4 = 0$. The curve for $R_1 = 2.0$ has an irregularity on the left which is the result of errors in the calculation. In this circumstance both the ring and the endcaps are rather large and the electrodes are rather close together with the result that $A_4$ is small and nearly independent of all of the variables. The data displayed at the left end of the curve for $R_1 = 2.0$ are included only because they help to define the range of validity of the data.

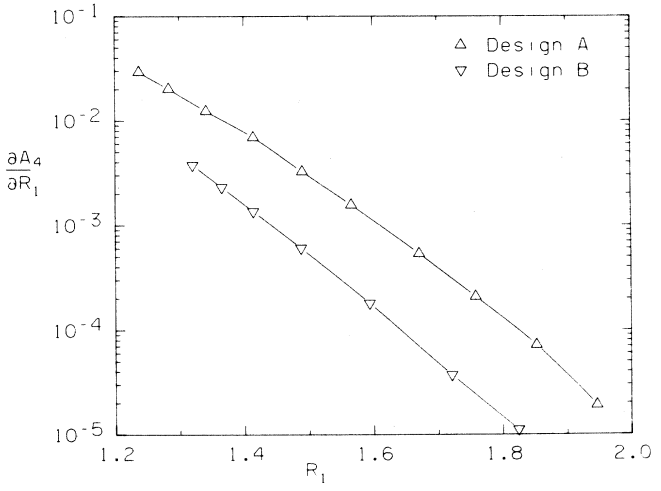


FIG. 6. The partial derivative of $A_4$ with respect to $R_1$ (constant $Z_1$) plotted vs $R_1$ for designs A and B with balanced truncation.

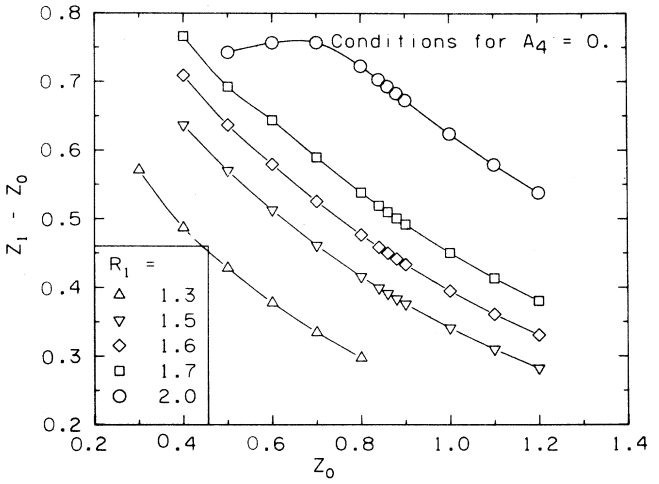


FIG. 7. Conditions for $A_4 = 0$ for design A. These data are an extension of those displayed in Fig. 4 to include variation of $Z_0$.

Figure 8 shows the sensitivity of $A_4$ to small changes in $Z_1$ with everything else constant. These data are not derived from Table IV but are a byproduct of the search procedures used in finding the values of $Z_1$ which yield $A_4 = 0$. Figure 9 reports data on the partial derivative of $A_2$ with respect to $Z_1$ with all other dimensions constant. These data allow an estimate of the effect on $A_4$ and $A_2$ of adding a small electrode at the endcap potential. From Table IV it is possible to determine the change in ring-electrode size which is required to balance a change in the size of the endcap electrodes. From the combination of these results it is possible to deduce the effect on $A_4$ of a small extra electrode at the ring potential. The small electrodes referred to here must be in very special places for the results to apply literally. Nevertheless it should be possible to make useful estimates of the effects of a larger electrode positioned to function as a shield. An interesting observation from Fig. 9 is that if $Z_0 = 0.860$ then $A_2$

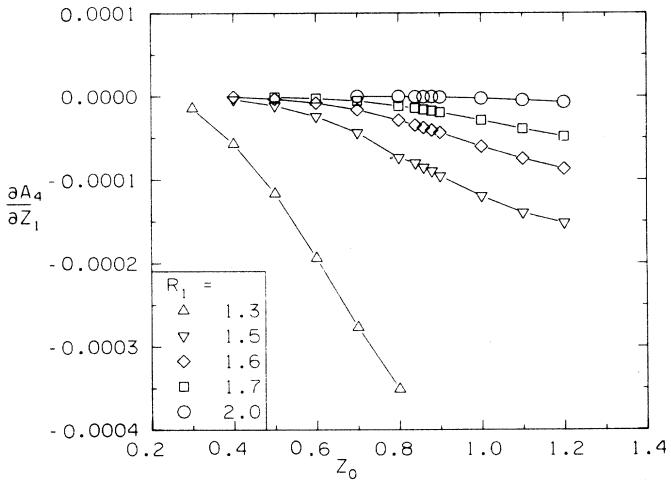


FIG. 8. Derivative of $A_4$ with respect to $Z_1$ (constant $R_1$) for the conditions of balanced truncation using design A.

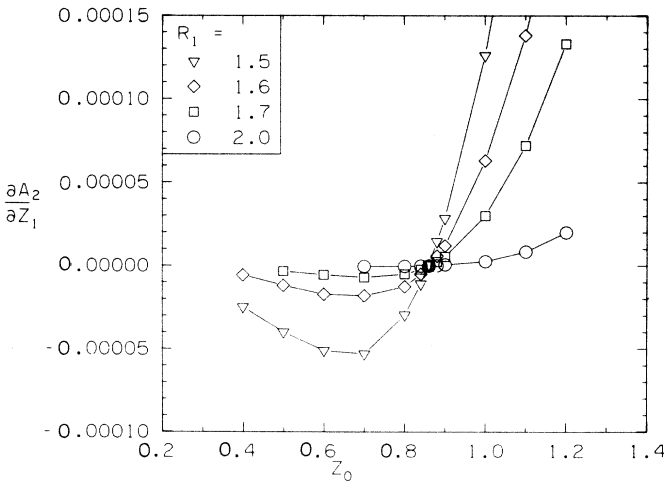


FIG. 9. Derivative of $A_2$ with respect to $Z_1$ (constant $R_1$) for the conditions of balanced truncation using design A.

is independent of $Z_1$ for all values of $R_1$. This implies that for these conditions $A_2$ will also be approximately independent of the potential of the shield electrode referred to above. This confirms a conclusion of Gabrielse[22] who explicitly examined a similar electrode structure but with compensation electrodes included. He discovered that if $Z_0/R_0=0.86$ then $A_2$ is independent of the potential of the compensation electrode.

## V. CONCLUSIONS

In the design of charged-particle traps for precise atomic physics work the equations of motion for the trapped particles can be made relatively simple by using hyperbolic electrode surfaces near the center of the trap. Some calculated data have been presented which permit a prediction of the effects of other shapes at moderate distances. The curved surfaces may be terminated quite quickly and still preserve the essential harmonic character of the trap.

Some qualitative conclusions may be reached which are beyond the direct numerical results. Design A is attractive mostly because the electrodes are small implying good space utilization and low capacitance. If these qualities are pressed the shielding will be poor and the potentials in the trap will be affected by the potentials outside. This problem can be solved by building a shield electrode which is electrically insulated from the other electrodes. If the shield is placed at the endcap potential it will be effectively an extension of the endcap and $A_4$ will be driven in a negative and $A_6$ in a positive direction. Similarly if the shield has the potential of the ring electrode $A_4$ and $A_6$ will be pulled in the opposite directions. It follows that there is an intermediate potential which makes either $A_4$ or $A_6$ have the value it would have had without the shield. Also it is a reasonable inference that under these circumstances the other coefficient will be relatively small. These general conclusions are independent of the details of construction of the shield. Some of the data allow a reasonable estimate of the magnitude of the effect of adding another electrode.

With any of the designs discussed it is interesting to consider dividing one or more of the three electrodes into two pieces. If the gap is kept small and the two pieces have the same potential the data reported above will be reasonably applicable. By adjusting the potential of the outermost piece it is possible to compensate for other problems which may have caused the trap to become anharmonic. Another reason for making such a division might be to reduce the capacitance of the electrode nearest the center of the trap. If each of the electrodes is appropriately divided, designs B and C become examples of design A with shield (and compensation) electrodes added.

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