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Influence of white noise on delayed bifurcations

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We analyze the time-dependent solution of the simplest Fokker-Planck equation which describes the action of Gaussian white noise in a single-mode laser. The pump parameter is linearly swept in time. Because of the critical slowing down, the time evolution of the probability distribution exhibits a delay with respect to the instantaneous value of the pump parameter. The delay turns out to depend crucially on the sweeping rate and to be essentially independent of the initial value of the swept parameter.

The electric field amplitude E in a tuned single-mode, homogeneously broadened ring laser in the good-cavity limit can be described by the semiclassical equation

$$\dot{E}=E\left(-1+\frac{A}{1+E^2}\right) , \tag{1}$$

where A is the pump parameter and time is normalized to the cavity buildup time. This equation has two steady solutions ($E=0$, $E^2=A-1$) and a single bifurcation point at $A=1$. A linear-stability analysis shows that $E=0$ is the stable solution for $A<1$ and $E^2=A-1$ is the stable solution for $A>1$. A common way to study experimentally a steady bifurcation point like $A=1$ for Eq. (1) is to sweep A across $A=1$:

$$A=A(t)=A_0+vt,\quad A_0<1,\quad v>0 . \tag{2}$$

This dynamic procedure differs significantly from the static which assumes that A is time independent. To examine this difference, we analyze the (dynamic) stability of the trivial solution $E=0$ with A given by Eq. (2). Linearizing Eq. (1) around $E=0$ gives $\dot{E}=E(-1+A)$. The solution of this equation is

$$E(t)=E(0)\exp[\lambda(t)] ,\qquad \lambda(t)=t\left(A_0+\frac{vt}{2}-1\right) . \tag{3}$$

As in the static case the instability condition is $\lambda(t^*)=0$, which defines the critical time t^* at which the solution of the linearized equation begins to diverge. Let $A^*=A(t^*)$ and $\bar{A}=A(\bar{t})=1$. Hence $\bar{t}$ is the time at which the static bifurcation point $A=1$ is reached. One easily finds

$$t^*=2\bar{t} , \tag{4a}$$

$$A^*-\bar{A}=\bar{A}-A_0 . \tag{4b}$$

These results hold if v is nonzero but otherwise arbitrary. When $v=0$, then $\lambda(t)=t(A_0-1)$ and the static result is recovered. Equation (4a) shows that t^* is exactly twice the time required to reach the static bifurcation. A salient property of Eq. (4b) is that A^* is independent of the sweeping rate v but depends only on the initial value A_0. These results are not limited to the good cavity case. Similar conclusions have been drawn from the laser-Lorenz equations with arbitrary atomic and cavity decay rates.[1]

It has been shown[2,3] that the critical property which is responsible for large delays in steady bifurcations is the existence of a steady solution ($E=0$ in this case) which is independent of the control parameter A. This naturally raises the question as to how fluctuations may modify the results in Eq. (4). One way to model the various causes which may invalidate Eq. (1), especially below $A=1$, is to add a phenomenological constant δ to the right-hand side of Eq. (1). One may then study the delay as a function of δ. It has been shown in Ref. 2 that for small sweeping rates ($0<v\ll 1$) the delay is $O(1)$ if $0<\delta\ll v$ but is vanishingly small if $\delta=O(1)$. The transition between these two domains occurs for $\delta=O(v)$. Although this approach indicates clearly some of the constraints under which a deterministic steady bifurcation may display a large delay, it fails to include all relevant features of a truly stochastic source of perturbation. Therefore we analyze in this paper the simplest Fokker-Planck equation which can be associated with Eqs. (1) and (2):

$$\frac{\partial}{\partial t}P(E,t)=\frac{\partial}{\partial E}\left[E\left(1-\frac{A(t)}{1+E^2}\right)+q\frac{\partial}{\partial E}\right]P(E,t) , \tag{5}$$

where q is the amplitude of a (additive) Gaussian white noise. We integrated numerically Eq. (5) following the Crank-Nicolson discretization method, for several values of A_0, q, v, and with the initial condition $P(E,0)=\delta(E)$, using a numerical code essentially identical to that utilized in Ref. 4.

A first result is that the dependence of the evolution of the mean electric field intensity $\langle E^2(A(t))\rangle$ on the initial value $A(0)=A_0$, given in the deterministic case by Eq. (4b), disappears in the presence of a white noise. In fact, if one compares the evolution of $\langle E^2(A(t))\rangle$ for the same values of the noise amplitude q and the sweep velocity v and for two different values $A_0^{(2)}>A_0^{(1)}$, one finds that in both cases the evolution appears completely identical, apart from a short transient starting from $A_0^{(2)}$. For example, curve c in Fig. 1 was obtained both for $A_0=0$ and for

$A_0 = -0.5$. This difference from the deterministic result Eq. (4) can be easily understood on examining the facts that give rise to the delay in the deterministic case. The first fact is the critical slowing down at $A \simeq 1$. The second fact is that, starting from an initial value $E(0)$ with $A_0 < 1$, the value of E decreases exponentially for $t < \bar{t}$ [see Eq. (3)]. Hence when $A(t) = \bar{A} = 1$, E is smaller the larger $\bar{A} - A_0$ is, which implies that the delay increases with $\bar{A} - A_0$. The dependence on A_0 disappears in the case of white noise because the fluctuations continuously restore the value of E, i.e., they destroy the exponential decrease of E for $t < \bar{t}$. The same argument suggests that the dependence on A_0 can be recovered in the case of colored noise, when the rate of fluctuations becomes comparable to or smaller than the sweeping rate.

On the other hand, the critical slowing down is still valid in the presence of noise and gives rise to a delay in the bifurcation, as illustrated by Fig. 1 which shows the time evolution of the mean value $\langle E^2(A(t))\rangle$ of the electric intensity for $q = 10^{-3}$ and $v = 10^{-1}$, 10^{-2}, and 10^{-3}. It also exhibits the stationary electric field intensity $\langle E^2(A(t))\rangle_{st}$ calculated from the stationary solution of the Fokker-Planck equation obtained by replacing $A(t)$ by A in Eq. (5):

$$\begin{aligned} \langle E^2(A(t))\rangle_{st} &= \int_{-\infty}^{+\infty} dE\, E^2 P_{st}(E, A(t)) \ , \\ P_{st}(E,A) &= \mathcal{N} \exp\left[\frac{-U_A(E)}{q}\right] \ , \\ U_A(E) &= \tfrac{1}{2}[E^2 - A \ln(1+E^2)] \ , \\ \mathcal{N}^{-1} &= \int_{-\infty}^{+\infty} dE \exp\left[\frac{-U_A(E)}{q}\right] \ . \end{aligned} \tag{6}$$

As expected, the presence of white noise decreases the delay with respect to the deterministic value given by Eq. (4), especially when the difference $\bar{A} - A_0$ becomes large. For $q = v = 10^{-3}$ the delay is negligible, and it becomes sizeable when $v \gg q$. Hence in the case of white noise the behavior is basically complementary to the deterministic case, in which the delay is independent of the velocity v and depends on the initial condition A_0.[5] Here the delay becomes appreciable only when the sweep rate becomes suitably larger than the noise parameter q. We note that, if the velocity v is large enough, the mean value $\langle E^2(A(t))\rangle$ does not reach exactly, even after very long times, the corresponding stationary value $\langle E^2(A(t))\rangle_{st}$. This means that the system is swept so fast that it cannot relax to the stationary probability distribution corresponding to $A = A(t)$. In Fig. 2 the sweep velocity is positive until the control parameter attains the value $A = A_M$ (curve a). The sign of v is then inverted and A is swept back to zero (curve b). Since during the reversed evolution the mean electric field intensity becomes larger than the corresponding stationary value, a dynamically induced hysteresis cycle is generated.

Let us now consider the time evolution of the full probability distribution $P(E,t)$. For the values of v considered in Fig. 1 the evolution is as follows. Within a short transient, $P(E,t)$ approaches a configuration which practically coincides with $P_{st}(E, A(t))$. When A approaches unity, due to the critical slowing down, the probability distribution no longer follows the value of A adiabatically and becomes quite different from $P_{st}(E, A(t))$ (see Fig. 3). Only after $A(t)$ exceeds unity by a suitable amount dependent on v, does $P(E,t)$ return to a configuration close to $P_{st}(E, A(t))$.

Another aspect of interest in this problem is the bifurcation time distribution. In the deterministic theory, the bifurcation time is defined as the time value such that the output intensity E^2 reaches a prefixed value E_{th}^2. The choice of E_{th}^2 is arbitrary provided $E_{th}^2 > E^2(0)$; we take $E_{th}^2 = 0.1$. In the presence of noise the bifurcation time becomes sto-

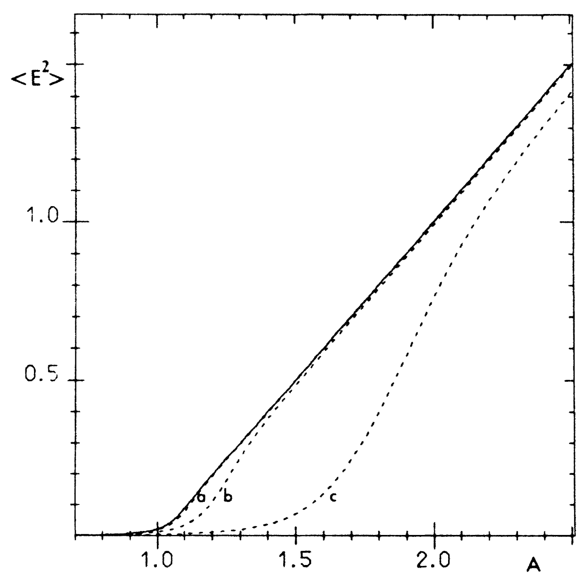

FIG. 1. Evolution of the mean value of the electric field intensity $\langle E^2(A(t))\rangle$ for $q = 10^{-3}$ and a, $v = 10^{-3}$; b, $v = 10^{-2}$; c, $v = 10^{-1}$. The full line depicts the stationary mean value $\langle E^2(A(t))\rangle_{st}$ of the electric field intensity for $q = 10^{-3}$.

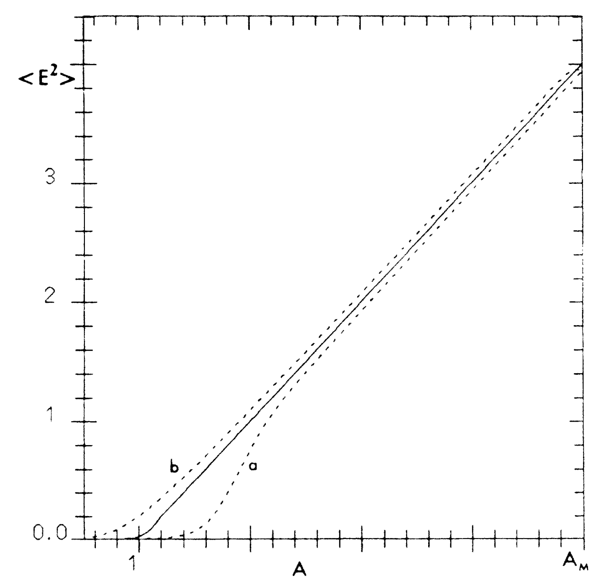

FIG. 2. Evolution of the mean value of the electric field intensity $\langle E^2(A(t))\rangle$ for $q = 10^{-3}$ and a, $v = 10^{-1}$; b, $v = -10^{-1}$; $A_0 = A_M = 4.999$. At $A(t) = A_M$ the sign of the sweep velocity v is reversed, giving rise to a dynamically induced hysteresis cycle. Full line as in Fig. 1.

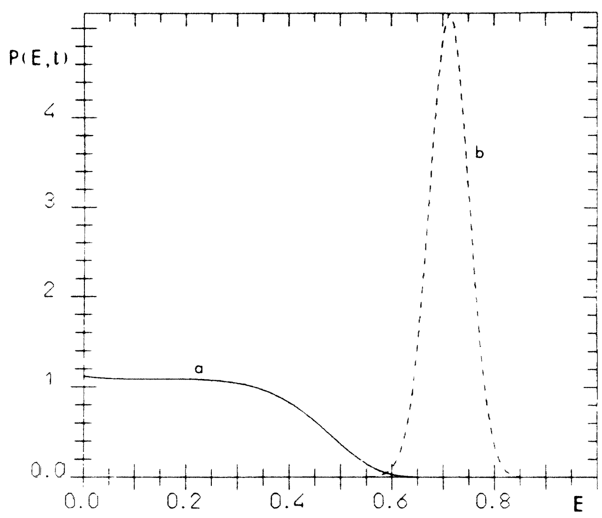

FIG. 3. Curve a shows the probability distribution $P(E,t)$ for $q=10^{-3}$, $v=10^{-1}$, $A_0=0$, and $t=15.095$. Curve b is the distribution $P_{st}(E,A(t))$ for the same values of the parameters. Note that the distributions are even functions of E.

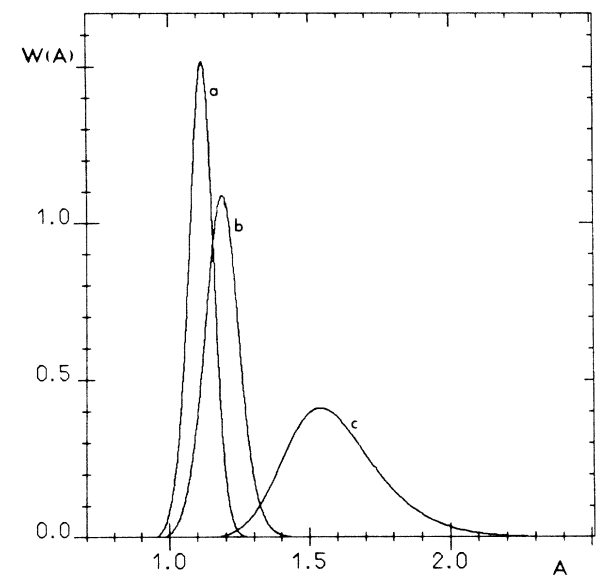

FIG. 4. Dynamical bifurcation probability distributions $W(A)$ (see text) for $q=10^{-3}$ and a, $v=10^{-3}$; b, $v=10^{-2}$; c, $v=10^{-1}$.

chastic. The distribution of bifurcation times normalized to unity $\tilde{W}(t)$, can be defined as

$$\tilde{W}(t)=\frac{d\bar{P}(t)}{dt},\quad \bar{P}(t)=2\int_{E_{th}}^{+\infty} dE\, P(E,t) \ . \tag{7}$$

As before, it is more convenient to use the variable A instead of time, which allows us to compare directly the distributions for different values of rates v. From Eq. (2) the bifurcation probability distribution, when expressed in terms of A, is given by

$$W(A)=\frac{1}{v}\tilde{W}\left(\frac{A-A_0}{v}\right) \ . \tag{8}$$

Figure 4 shows the distribution of $W(A)$ for $q=10^{-3}$ and three values of v. Clearly, the width of the distribution increases with the velocity. It would be interesting to measure the bifurcation time distribution experimentally.

In conclusion, we have shown that, even if the critical slowing down is intrinsically accompanied by an enhanced sensitivity to noise, the phenomenon of delayed bifurcation persists in the presence of white noise and, as argued above, the situation should be even more favorable in the case of colored noise.

We mention finally that a recent work by Kondepudi, Moss, and McClintock[6] studies the time evolution of the probability distribution as a control parameter is swept through a bifurcation point in the presence of additive Gaussian white noise. Reference 6 does not analyze the delay phenomenon, but the problem of branch selectivity when the symmetry with respect to the transformation $E \leftrightarrow -E$ is broken.

Recent experimental investigations on the switching in a laser with a saturable absorber with a swept pump parameter[7] seem to confirm the predictions of our paper.

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[5]Here, we mean the delay with respect to the variable A, expressed by Eq. (4b). The delay with respect to time depends on v only via the dependence of A on v, i.e., $t^*=2(1-A_0)/v$ as obtained from Eqs. (4a) and (2).
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