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Coalescence of Saffman-Taylor fingers: A new global instability

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We study the stability of a parallel array of Saffman-Taylor fingers in the limit of infinite viscosity contrast. We discover a modulatory instability which prevents the system from remaining in steady-state motion. We discuss how this instability signals the beginning of the coalescence of the finger array to the final single-finger steady-state configuration. Finally, we speculate on the importance of this global instability for the understanding of patterns in diffusion-limited growth.

In the past year, it has become clear that the single Saffman-Taylor finger[1] in a Hele-Shaw cell[2] is stable with respect to the small perturbations.[3-5] These results are in agreement with recent experiments.[6] It is fair to say we now understand in some detail the steady-state solution which the system approaches at large time. On the other hand, much less is understood about the actual process whereby the final state is approached.

In this paper, we present one mechanism which drives the system towards a single finger. Specifically, we are interested in how the many-fingered array usually seen at early times[7] collapses towards the above final state. We study this question by extending the stability analysis from one finger to an array of fingers. We will discover that there exists an instability which causes the breakdown of the finger array; schematically, the unstable mode corrresponds to a modulation of the translation zero mode of each separate finger. Physically, this mode causes one finger to move slightly ahead of the neighboring fingers and the instability means that it will then grow faster and eventually "win."

The equations for flow in a Hele-Shaw cell take the form

$$\nabla^2 p = 0 \ ,$$
$$p(x(s)) = -\gamma\kappa(s) \ ,$$
$$-\hat{\mathbf{n}}\cdot\frac{d\mathbf{x}(s)}{dt} = \frac{\partial p}{\partial \hat{n}} \ ,$$

where the interface is given by $\mathbf{x}(s)$ with normal vector $\hat{\mathbf{n}}$ and curvature κ. The pressure p obeys the additional boundary condition $\partial p/\partial y = 0$, $y = \pm L$, $p \sim -x$ as $x \to \infty$. With our scaling, the dimensionless parameter γ is given by $\gamma = (\sigma/12\mu v)(b/a)^2$, where μ is the fluid viscosity, σ the interfacial tension, v the imposed flow at infinity, b the gap thickness, and $2aL$ the channel width. The above equations strictly apply to the case of infinite viscosity contrast between the two fluids.

The Saffman-Taylor single-finger solution at $\gamma = 0$ can be expressed as the mapping

$$z = \rho + \frac{2}{\pi}(1-\lambda)\ln\left|\frac{1+e^{-\pi\rho}}{2}\right| \ , \tag{1}$$

where $z = x + iy$ and $\rho = \phi + i\psi$, where the velocity potential is ϕ and the stream function ψ. It is obvious that this solution also can be used to describe L parallel fingers, each of width λ. We can now proceed to compute the stability operator by considering a new interface parametrized by $\phi = \delta(\psi)$, including the effects of nonzero surface tension. For simplicity, we will neglect the "shape-correction" contributions and consider perturbing the original $\gamma = 0$ shape. The justification for this approach has been given elsewhere.[3,8]

We assume that

$$p = \sum_{n=-\infty}^{\infty} a_n \cos[(n\pi + \tfrac{1}{2}k)\psi]e^{-(n\pi + k/2)\phi} \ ,$$

where the boundary condition is satisfied if $kL = 2m\pi$ for integral m. Applying the two interfacial conditions leads to the equations

$$\delta - \gamma\kappa^1[\delta] = \sum a_n \cos[(n\pi + \tfrac{1}{2}k)\psi] \ , \tag{2a}$$

$$\sum (\pi n + \tfrac{1}{2}k) a_n \cos[(n\pi + \tfrac{1}{2}k)\psi] = L_0[\delta] \ , \tag{2b}$$

where

$$L_0[\delta] = g_0(\lambda)\omega\delta + \frac{1-\lambda}{\lambda}\frac{\partial}{\partial\psi}\left(\frac{\sin(\pi\psi)\delta}{1+\cos(\pi\psi)}\right) ,$$

$$\kappa^1[\delta] = -g_0^{-1/2}\delta'' - \frac{\pi(1-\lambda)^2\sin(\pi\psi)}{[1+\cos(\pi\psi)]^2}g_0^{-3/2}\delta' + \left(\frac{\pi^2(1-\lambda)^2\cos(\pi\psi)}{[1+\cos(\pi\psi)]^2 g_0^{3/2}} - \frac{3\pi^2\lambda^2(1-\lambda)^2}{[1+\cos(\pi\psi)]^2 g_0^{5/2}}\right)\delta \ ,$$

and

$$g_0(\lambda) = \lambda^2 + \frac{(1-\lambda)^2\sin^2(\pi\psi)}{[1+\cos(\pi\psi)]^2} \ .$$

To proceed further, we note that δ can be decomposed into $\delta_1\cos(\frac{1}{2}k\psi) + \delta_2\sin(\frac{1}{2}k\psi)$, and that similarly $\delta - \gamma\kappa^1[\delta]$ can be expressed as $F_1\cos(\frac{1}{2}k\psi) + F_2\sin(\frac{1}{2}k\psi)$. These definitions assure that $\delta_{1\,(2)}$ and $F_{1\,(2)}$ are periodic under $\psi \to \psi \pm 2$ and are even (odd) under $\psi \to -\psi$. Similarly, L_0 can be decomposed into even and odd pieces. Finally, we can eliminate the a_n from Eq. (2) via the principal-value integral

$$\frac{\partial}{\partial\psi}\frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{d\psi'}{\psi'-\psi}(\delta - \gamma\kappa^1[\delta]) = L_0[\delta] \ . \tag{3}$$

Making use of the periodicity and parities, and assuming k positive, we find the coupled stability equations

$$L_0[\delta_1] = \frac{1}{2}k\frac{\sin(\pi\psi)}{1+\cos(\pi\psi)}\delta_2 - \int_{-1}^{1} d\psi'\left[\cot\left(\frac{\pi}{2}(\psi-\psi')\right)(F_1' + \tfrac{1}{2}kF_2) - \tfrac{1}{2}kF_1\right] ,$$
$$L_0[\delta_2] = -\frac{1}{2}k\frac{\sin(\pi\psi)}{1+\cos(\pi\psi)}\delta_1 - \int_{-1}^{1} d\psi'\left[\cot\left(\frac{\pi}{2}(\psi-\psi')\right)(F_2' - \tfrac{1}{2}kF_1)\right] . \quad (4)$$

For finite fixed γ, steady-state solutions only exist at discrete λ.[3,9] Furthermore, the $k=0$ stability problem has an exact translation mode at $\omega=0$. We, therefore, wish to understand the change in this eigenvalue as we increase k. We will see that this mode moves to *positive* Reω, signaling an instability.

The simplest way to see that this will occur is to focus on the structure of Eq. (4) for very small k. We can think of these equations as forming the 2×2 supermatrix

$$\begin{pmatrix} \omega - A(k) & C(k) \\ D(k) & \omega - B(k) \end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = 0 .$$

One can easily verify that C and D are both $O(k)$ and that $A(k=0)$ is just the symmetric mode stability operator studied in Ref. 3. Therefore, up to terms $O(k^2)$, the zero mode will be sensitive only to terms in A linear in k. The only such term is the piece of the first principal value integral which is *not* multiplied by the cotangent kernel. This can be verified by noting that all the terms in F_i proportional to δ_j, $j\neq i$ are also proportional to k, and that there are no such linear terms in F_i proportional to δ_i. In fact, this term is not analytic around $k=0$; the presence of this type of term can be traced back to the singularity in the principal value integral in Eq. (3). For small γ, $F_1 \sim \delta_1$ and hence this piece is *positive*. This suggests that $(\partial \text{Re}\omega/\partial k)|_{k=0} \geq 0$.

To check this, we have discretized these operators and studied the spectrum numerically. The algorithms are similar to those used in Ref. 3 at $k=0$ and need not be explained in detail. The only point worth noting is that the mode in question is well behaved near $\psi=1$ and therefore, converges quadratically in the number of points we use for the discretization.

In Fig. 1 we plot Reω vs k at $\gamma = 8.3\times 10^{-3}$, $\lambda = 0.539$. This corresponds to an allowed width up to corrections of order γ^2, which we verify by noting that $\omega(0)=0$. We verify that there is indeed an instability for finite k and that the instability grows faster as k increases. For our channel of L fingers, this suggests a maximum for the mode at $k=\pi$, corresponding to an alternation of enhanced and suppressed growth for each and every pair of neighboring fingers. The actual magnitude of the instability changes very slowly as γ (and the appropriate λ) is changed in the range $1\times 10^{-3} \leq \gamma \leq 1\times 10^{-2}$. This can be understood by recognizing that the above mentioned term in $A(k)$ linear in k is *independent* of γ, and that this term dominates for small γ.

The best experimental effort to date to measure this finger amalgamation was performed by Maher.[7] Our analysis assumes zero viscosity for the invading fluid, a case he attempted to realize by invading water into paraffin oil. He discovered that fingers tended to coalesce in pairs via suppression of any finger momentarily left lagging behind its neighbor. Furthermore, the relevant dimensionless time scale for the process to occur was 5–10 units at $\gamma \sim 0.001$. These results are consistent with the instability derived here. Inasmuch as the initial state in his experiment does not precisely correspond to an array of parallel fingers, a more detailed comparison is not possible. A truer experimental realization might be performed in a Hele-Shaw cell with a divider which equilibrates the system to two parallel fingers which then ends at some finite distance down the channel. Also, one can study this instability by numerically simulating the time-dependent equations of motion starting from the initial configuration of a slightly perturbed multifinger array.

In recent experiments on diffusion-controlled growth,[10,11] a clear distinction has emerged between globally stable patterns (called homogeneous in Ref. 11) and globally unstable, fractal-like ones. Based on our results here, we can speculate that the difference between these two modes of growth can be traced to the type of global instability discovered here. Specifically, we can argue that there are, in general, four possibilities governing diffusively controlled patterns corresponding to global stability or instability and local stability or instability. We have shown here that Saffman-Taylor fingers, although *locally* stable, are globally unstable to finger competition. Directional cells,[12] however, are most probably globally stable and remain globally stable even after the onset of the *local* sidebranching instability as the velocity is increased. Certainly, true snowflakes exhibit a global stability that remains to be understood in detail. Similarly, the compact tip-splitting patterns seen in radial Hele-Shaw,[13] the aforementioned electrochemical deposition,[10,11] and microsterias growth[14] all exhibit global stability

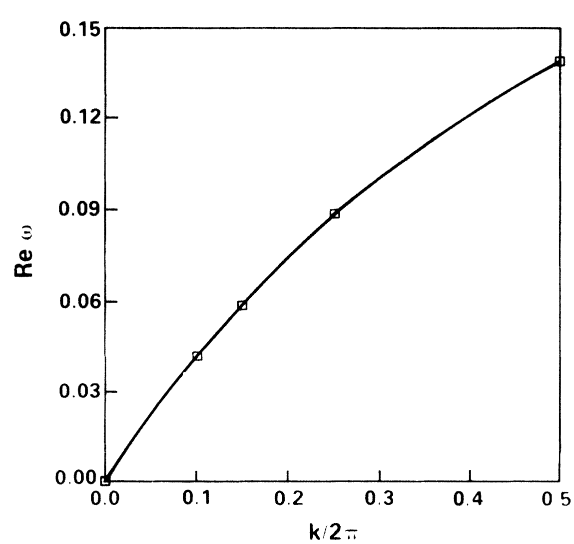

FIG. 1. Growth rate vs modulation wave vector.

in spite of the *local* tip-splitting mode. Finally, patterns produced by diffusion-limited aggregation such as those seen with low surface tension and with non-Newtonian fluids,[15] or in porous media,[16] exhibit both instabilities. Clearly, much work is needed to turn the above hypotheses into facts.

In summary, we have shown that the tendency of Saffman-Taylor fingers to coalesce and approach a single-finger final state can be understood by means of a new modulatory instability. This instability may provide an important mechanism for the evolution of a wide spectrum of differing patterns via diffusively controlled growth.

Note added in proof. After completion of this work, we were made aware of a paper by G. Trygvasson and H. Aref, J. Fluid Mech. **154**, 287 (1985), which numerically studied finger competition; the results given there are in qualitative agreement with the ideas discussed here.

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