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Renormalization-group analysis of the global structure of the period-doubling attractor

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We use a recently developed renormalization-group formalism to study the global properties of the period-doubling attractor. The renormalization scheme can be written in closed form in terms of universal functions. The results of the calculation appear as a smooth spectrum of scaling indices in full agreement with direct numerical calculations. As a special result we obtain the fractal dimension of the attractor to be $D_0 = 0.538\,045\,143\,5$, accurate to one part in 10^{-10}. The only input for the calculation is the Taylor expansion of Feigenbaum's universal function.

As found by Feigenbaum,[1(a)] the attractor obtained at the onset of chaos via period doubling (PD) exhibits two main scales, $\alpha_{\rm PD}$ and $\alpha_{\rm PD}^2$, where $\alpha_{\rm PD} = -2.502\,907\,876\ldots$ is a universal number, and the local scaling on the attractor assumes a value between these two scales as determined by the scaling function σ.[1(b)] Recently, Halsey *et al.*[2] observed that the global structure of the attractor can be described by a smooth universal spectrum of scaling indices, which range over all values between $\alpha_{\rm PD}$ and $\alpha_{\rm PD}^2$. More specifically it was found that the density of points p on the attractor scales locally as $p(l) \sim l^\alpha$ when $l \to 0$, where α is a scaling index. This index is restricted to an interval, $\alpha_{\rm min} < \alpha < \alpha_{\rm max}$ where $\alpha_{\rm max} = \ln 2/\ln|\alpha_{\rm PD}|$ and $\alpha_{\rm min} = \ln 2/\ln \alpha_{\rm PD}^2$. The complete set of scaling indices is obtained via the smooth spectrum $f(\alpha)$, where $f(\alpha)$ is the *global* dimension of the set of points on the attractor where scaling indices of type α are located. Obviously, $f(\alpha) \leq D_0$.

To obtain the spectrum $f(\alpha)$ one calculates the partition function[2]

$$\hat{\Gamma}_n(q,\tau) = \sum_{j=1}^{2^{n-1}} \frac{(1/2^{n-1})^q}{|x_j - x_{j+2^{n-1}}|^\tau} \ , \tag{1}$$

where x_i is the ith iterate of $x=0$ calculated at the onset of chaos. Requiring $\lim_{n\to\infty} \hat{\Gamma}_n(q,\tau) = 1$ determines $\tau(q)$ which can be transformed into the spectrum $f(\alpha)$ by [2,3]

$$\alpha = \frac{d}{dq}\tau(q), \quad f = q\alpha - \tau(q) \ . \tag{2}$$

$\tau(q)$ is related to the generalized dimensions D_q of Hentschel and Procaccia[4] via $\tau = (1-q)D_q$. The spectrum found this way is shown in Fig. 1. Here we shall employ a renormalization-group (RG) theory to determine it. The scheme we use, which follows closely a scheme recently developed for circle maps by Kadanoff,[5] is different from earlier RG treatments of Feigenbaum[1] in the sense that it deals with the global structure of the attractor. Basically, one can write an eigenvalue problem, which determines the behavior of the partition function $\hat{\Gamma}_n(q,\tau)$ as n becomes large. We obtain $\hat{\Gamma}_n(q,\tau) = \lambda^n(\tau) 2^{-q(n-1)}$, where $\lambda(\tau)$ is the maximal eigenvalue. This determines $q(\tau) = \ln\lambda(\tau)/\ln 2$ and Eq. (2) then yields $f(\alpha)$ to very high accuracy.

We now focus on the part of the attractor around $x=0$ and write a generalized partition function in the following way:

$$\Gamma_{n,m,k}(q,\tau) \equiv (\tfrac{1}{2})^{q(n-m-1)} \sum_{j=1}^{2^{n-m-1}} \frac{x_{j2^m}^k}{|x_{j2^m} - x_{j2^m+2^{n-1}}|^\tau} \ . \tag{3}$$

Our original partition function is thus $\hat{\Gamma}_n(q,\tau) = \Gamma_{n,0,0}(q,\tau)$. In our analysis, we shall assume $n \gg m \gg 1$, solve and extrapolate to $m=0$. A central point of the calculation is to split the sum (3) into two parts,[5] one over even j, the other over odd j:

$$\begin{aligned}
2^{-q(n-m-1)} \sum_{j=1}^{2^{n-m-1}} \frac{x_{j2^m}^k}{|x_{j2^m} - x_{j2^m+2^{n-1}}|^\tau} &= 2^{-q(n-m-1)} \sum_{j=1}^{2^{n-m-2}} \frac{x_{2j2^m}^k}{|x_{2j2^m} - x_{2j2^m+2^{n-1}}|^\tau} \\
&\quad + 2^{-q(n-m-1)} \sum_{j=1}^{2^{n-m-2}} \frac{x_{(2j-1)2^m}^k}{|x_{(2j-1)2^m} - x_{(2j-1)2^m+2^{n-1}}|^\tau} \\
&= 2^{-q(n-m-1)} \sum_{j=1}^{2^{n-m-2}} \frac{x_{j2^{m+1}}^k}{|x_{j2^{m+1}} - x_{j2^{m+1}+2^{n-1}}|^\tau} \\
&\quad + 2^{-q(n-m-1)} \sum_{j=1}^{2^{n-m-2}} \frac{[g^{-2^m}(x_{j2^{m+1}})]^k}{|g^{-2^m}(x_{j2^{m+1}}) - g^{-2^m}(x_{j2^{m+1}+2^{n-1}})|^\tau} \ .
\end{aligned} \tag{4}$$

This splitting is shown schematically in Fig. 2 for the case $n=6$, $m=2$.[6] To evaluate Eq. (4) we make use of the following identities that are obtained from Feigenbaum and Čvitanović's functional equation $g(x) = \alpha_{\rm PD}\, g(g(x/\alpha_{\rm PD}))$[1] (in the follow-

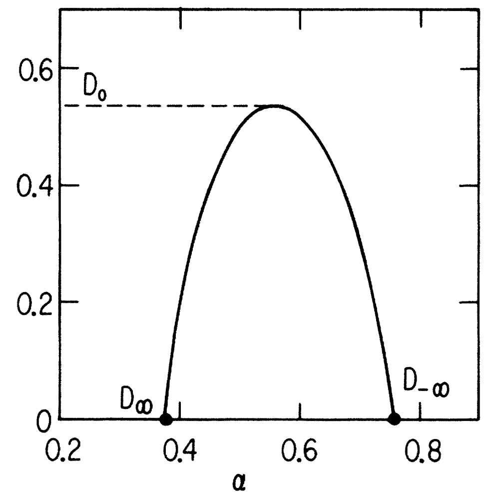

FIG. 1. The spectrum of scaling indices $f(\alpha)$ found for the critical period-doubling attractor. The maximum point is the fractal dimension D_0, and $\alpha_{\rm min}=D_\infty=\ln 2/\ln\alpha_{\rm PD}^2=0.377\,75\ldots$ and $\alpha_{\rm max}=D_{-\infty}=\ln 2/\ln\alpha_{\rm PD}=0.755\,51\ldots$ (Ref. 2).

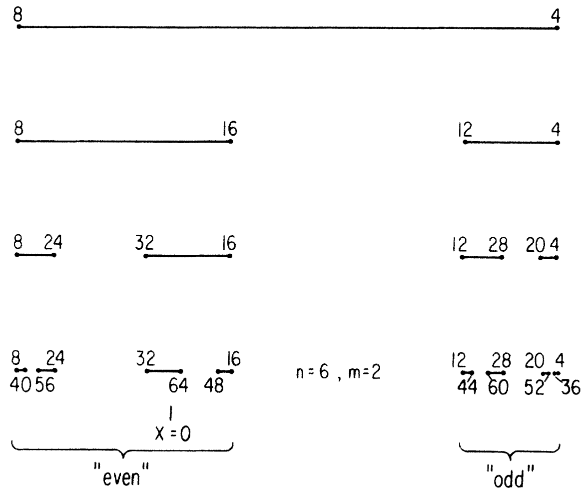

FIG. 2. The construction of a part of the attractor. The numbers refer to iterates of $x=0$. The fourth level in this construction corresponds in our notation to $n=6$, $m=2$.

ing we set $\alpha=\alpha_{\rm PD}$):

$$x_{j2^m}\equiv g^{j2^m}(0)=\alpha^{-m}g^j(0)\ ,$$
$$x_{j2^m+2^{n-1}}=g^{2^{n-1}}(g^{j2^m}(0))=\alpha^{-n+1}g(\alpha^{n-m-1}g^j(0))\ ,\qquad (5)$$
$$g^{-2^m}(x_{j2^m+1})=\alpha^{-m}g^{-1}(\alpha^{-1}g^j(0))\ .$$

The denominator of the second sum in Eq. (4) is expanded to yield

$$g^{-2^m}(x_{j2^m+1})-g^{-2^m}(x_{j2^m+1+2^{n-1}})$$
$$\approx (g^{-2^m}(x))'|_{x=x_{j2^m+1}}(x_{j2^m+1}-x_{j2^m+1+2^{n-1}})$$
$$=(g^{-1}(x))'|_{x=\alpha^{-1}g^j(0)}(x_{j2^m+1}-x_{j2^m+1+2^{n-1}})\ .\qquad (6)$$

Rewriting Eq. (4), with Eqs. (5) and (6) inserted we obtain

$$2^{-q(n-m-1)}\alpha^{-mk}|\alpha|^{m\tau}\gamma_{n-m,k}(\tau)=2^{-q(n-m-1)}\alpha^{-(m+1)k}|\alpha|^{(m+1)\tau}\gamma_{n-m-1,k}(\tau)$$
$$+2^{-q(n-m-1)}\alpha^{-mk}|\alpha|^{(m+1)\tau}\sum_{j=1}^{2^{n-m-2}}\frac{(g^{-1})^k[(g^{-1})']^{-\tau}_{x=\alpha^{-1}g^j(0)}}{|g^j(0)-\alpha^{-(n-m-2)}g(\alpha^{n-m-2}g^j(0))|^\tau}\ ,\qquad (7a)$$

where

$$\gamma_{n-m,k}(\tau)\equiv\sum_{j=1}^{2^{n-m-1}}\frac{(g^j(0))^k}{|g^j(0)-\alpha^{-(n-m-1)}g(\alpha^{n-m-1}g^j(0))|^\tau}\ .\qquad (7b)$$

From Eq. (7a) we observe that the first part of the sum in Eq. (4) (i.e., summing over even j) is just a scaled-down version (by $\alpha^{-k}|\alpha|^\tau$) of the original partition function on the left-hand side. The part obtained from summing over odd j's in Eq. (4) is nearly of the form of $\gamma_{n-m-1,k}$ except for the dependence on $g^{-1}(x)$. To express it in terms of $\gamma_{n-m-1,k}$, we expand the numerator

$$(g^{-1})^k[(g^{-1})']^{-\tau}|_{x=\alpha^{-1}g^j(0)}\ ,$$

i.e., we expand $g^{-1}(x)$ around the point $x_0=g^{-1}(0)$. Since the iterates $g^j(0)$ fall in the interval $[1/\alpha,1]$ the expansion is evaluated on the interval $[1/\alpha,1/\alpha^2]$. This interval is indicated on the graph of $g^{-1}(x)$, Fig. 3. Clearly, the expansion avoids the singularity at $x=1$. We may thus expand in a Taylor series,

$$(g^{-1})^k[(g^{-1})']^{-\tau}_{x=\alpha^{-1}g^j(0)}=\sum_{l=0}^{\infty}A_{kl}(\tau)\alpha^{-l}[g^j(0)]^l\ .\qquad (8)$$

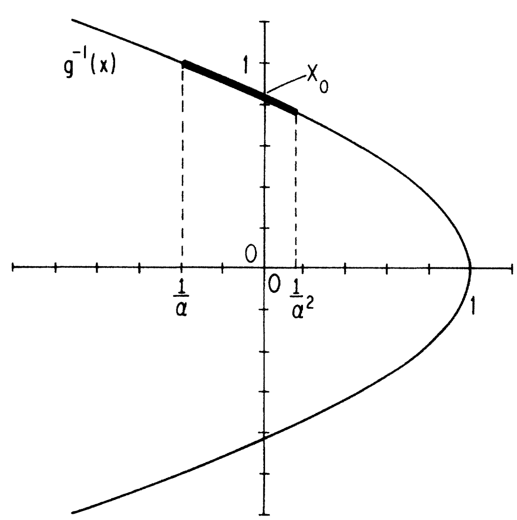

FIG. 3. The inverse $g^{-1}(x)$ of Feigenbaum's universal function. The regime where our expansion is evaluated is indicated by the thick line.

TABLE I. The first ten Taylor coefficients for the inverse of Feigenbaum's universal function, $g^{-1}(x)=x_0+\sum_{l=1} a_l x^l$, with $x_0=0.832\,367\,236\,909$.

a_1	$-0.445\,451\,223\,8$
a_2	$-8.232\,482\,984\,6\times10^{-2}$
a_3	$-5.333\,356\,953\,4\times10^{-2}$
a_4	$-3.045\,383\,410\,6\times10^{-2}$
a_5	$-2.219\,869\,939\,3\times10^{-2}$
a_6	$-1.641\,492\,052\,8\times10^{-2}$
a_7	$-1.299\,018\,648\,4\times10^{-2}$
a_8	$-1.053\,581\,161\,6\times10^{-2}$
a_9	$-8.793\,459\,865\,5\times10^{-3}$
a_{10}	$-7.474\,735\,027\,6\times10^{-3}$

Inserting in Eq. (7), we find the following RG equation for $\gamma_{n,k}(\tau)$:

$$\gamma_{n,k}(\tau)=\alpha^{-k}|\alpha|^\tau\gamma_{n-1,k}(\tau)+|\alpha|^\tau\sum_{l=0}^{\infty}A_{kl}(\tau)\alpha^{-l}\gamma_{n-1,l}(\tau)\ . \tag{9}$$

This is the main equation of the formalism. It is similar to the RG equation obtained for circle maps by Kadanoff.[5] Assuming that for large n the partition function grows as $\gamma_{n,k}=\lambda^n\xi_k$, we find

$$\lambda\xi_k=\alpha^{-k}|\alpha|^\tau\xi_k+|\alpha|^\tau\sum_{l=0}^{\infty}A_{kl}(\tau)\alpha^{-l}\xi_l\ , \tag{10a}$$

which means that λ is the largest eigenvalue of the matrix

$$M_{kl}=|\alpha|^\tau(\alpha^{-k}\delta_{kl}+A_{kl}(\tau)\alpha^{-l})\ . \tag{10b}$$

In the following calculations the size of the matrix is truncated at finite order.

The "input" for the calculation is Feigenbaum's universal function $g(x)$ written as a series,[1] $g(x)=1+\sum_{n=1}^{\infty}g_nx^{2n}$. We can of course move the expansion to x_0, $g(x_0)=0$ (see Fig. 3), and formally write $g(x)=\sum_{l=1}^{\infty}\tilde{a}_l(x-x_0)^l$. The coefficients $\tilde{a}_l$ are easily determined from higher order derivatives,

$$\tilde{a}_l=\frac{1}{i!}\frac{d^lg(x)}{dx^l}\bigg|_{x=x_0}\ .$$

Knowing this series we can analytically invert it to any order (see, for instance, Ref. 7) and obtain $g^{-1}(x)=x_0+\sum_{l=1}^{\infty}a_lx^l$. The first 10 coefficients are displayed in Table I. The matrix (10b) can then be constructed numerically to any order. Alternatively, one can solve all equations on a grid; we have used both methods and the results are indistinguishable.

From the largest eigenvalue λ of M_{kl} we find $\Gamma_{n,0,0}\sim 2^{-q(n-1)}\lambda^n\rightarrow 1$ leading to

TABLE II. Convergence as a function of the number of terms in the expansion of g^{-1} for three specific examples.

	D_0	$q(\tau=10)$	$q(\tau=-10)$
$N=5$	0.53	26.47	−13.235 60
$N=10$	0.5380	26.472 12	−13.235 608
$N=15$	0.538 045 1	26.472 124 1	−13.235 608
$N=25$	0.538 045 14	26.472 124 17	−13.235 608 18
$N=35$	0.538 045 143 5	26.472 124 17	−13.235 608 184

$$q(\tau)=\ln\lambda/\ln2\ . \tag{11}$$

Our results for $q(\tau)$ are excellent and with $N\sim5$ we get convergence at 10^{-4}, whereas with $N\sim35$ we get convergence around 10^{-9}. See Table II for the convergence with N at three different points. The $f(\alpha)$ spectrum calculated this way is indistinguishable from the direct numerical calculation as shown in Fig. 1; though the present calculation is several orders of magnitude more accurate than a direct numerical calculation. As a particular result we obtain the fractal dimension of the attractor: $D_0=0.538\,045\,143\,5(1)$. We believe this is the most accurate estimate to date and agrees down to 10^{-8} with the result found by Grassberger[8] and down to 10^{-6} with the result of Chang and McCown.[9] Actually, a direct numerical estimate of D_0 via Eq. (1) seems to deviate from this estimate even with very many points ($n=4096$). We attribute this to a very slow convergence of the direct calculation which we avoid in our RG formulation.

It is instructive to consider the lowest two approximations in our formulation. To zeroth order it is easy to find

$$q(\tau)=\frac{\tau\ln|\alpha|+\ln(1+|a_1|^{-\tau})}{\ln2}\ . \tag{12}$$

As $\tau=(q-1)D_q\rightarrow-\infty$ we obtain the theoretical result $D_{-\infty}=\ln2/\ln|\alpha|$. As $\tau\rightarrow\infty$ we should expect to find the theoretical value $D_\infty=\ln2/\ln\alpha^2$. This would be the case if in Eq. (12), $|a_1|^{-1}\sim|\alpha|$. As this is true to within 10%, we find that already the lowest approximation is fair. To first order, as $\tau\rightarrow\infty$, we obtain

$$\lambda=\left|\frac{\alpha}{a_1}\exp\left(-\frac{2a_2x_0}{\alpha a_1}\right)\right|^\tau\ .$$

Inserting the numerical values of the a_i's the result is accurate to 1.4%, indicating a very fast convergence.

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