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Collapse and revival phenomena in the Jaynes-Cummings model with cavity damping

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The density-matrix equation for a single atom interacting with a single mode of a cavity with finite Q is solved for the time dependence of various physical quantities using the dressed-atom approximation. The effect of the cavity damping on the collapse and revival phenomena and on the photon statistics is studied in detail.

The Jaynes-Cummings model[1-4] of a two-level atom interacting with the electromagnetic field in a lossless cavity is one of the few exactly soluble models in quantum optics. It enables one to calculate all the quantum-mechanical properties of a system. It predicts many interesting effects such as vacuum-field Rabi oscillations, revival and collapse of Rabi oscillations in presence of a coherent field,[2,3] etc. It is now becoming possible to test experimentally[4,5] many of the predictions of this model. In realistic situations, one does experiments in cavities with finite Q and hence, one should know how the predictions of this model are affected by the relaxation of photons in the cavity. This is currently very extensively studied,[4,6-8] and in this paper we adopt the dressed-atom approximation to obtain analytic results for the time depencence of various physical quantities, such as inversion, photon number distribution, and field amplitude and fluctuations. Numerical computation of the analytic results for the time dependence of excitation probabilities, photon correlation, and squeezing properties of the field for different values of the cavity damping parameters are also given.

The Jaynes-Cummings model considers the interaction of a single two-level atom characterized by spin-$\frac{1}{2}$ angular momentum operators $S^{\pm}, S^z$ with a single mode of the electromagnetic field characterized by annihilation and creation operators a and $a^\dagger$, respectively. For simplicity we will assume that the field is at resonance with the atomic frequency ω. For the cavity problem we must account for the leakage of photons from the cavity at the rate 2κ. The master equation for the density matrix of the combined system is

$$\frac{\partial\rho}{\partial t} = -i/\hbar[H,\rho] - \kappa(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a) \ , \tag{1}$$

$$H = \hbar\omega(S^z + a^\dagger a) + \hbar g(S^+ a + S^- a) \ , \tag{2}$$

with g denoting the coupling between the atom and field. On introducing the eigenstates of H,

$$|\psi_{\pm n}\rangle = (|n,\tfrac{1}{2}\rangle \pm |n+1, -\tfrac{1}{2}\rangle)/\sqrt{2} \ , \tag{3}$$

with eigenvalues,

$$\lambda_{\pm n} = \omega(n+\tfrac{1}{2}) \pm g(n+1)^{1/2} \ , \tag{4}$$

and on defining

$$W(t) = e^{iHt}\rho(t)e^{-iHt} \ , \tag{5}$$

and using the dressed-atom (secular) approximation, which as shown in Ref. 4 holds for $2\kappa n^2 << g\sqrt{n+1}$, the equations for the diagonal elements of W are found to be

$$\begin{aligned}\langle\Psi_{+n}|\dot W(t)|\Psi_{+n}\rangle = 2\kappa[&\Gamma_{+(n+1)}\langle\Psi_{+(n+1)}|W(t)|\Psi_{+(n+1)}\rangle + \Gamma_{-(n+1)}\langle\Psi_{-(n+1)}|W(t)|\Psi_{-(n+1)}\rangle \\ &-(\Gamma_{+n}+\Gamma_{-n})\langle\Psi_{+n}|W(t)|\Psi_{+n}\rangle] \ ,\end{aligned} \tag{6}$$

$$\begin{aligned}\langle\Psi_{-n}|\dot W(t)|\Psi_{-n}\rangle = 2\kappa[&\Gamma_{+(n+1)}\langle\Psi_{-(n+1)}|W(t)|\Psi_{-(n+1)}\rangle + \Gamma_{-(n+1)}\langle\Psi_{+(n+1)}|W(t)|\Psi_{+(n+1)}\rangle \\ &-(\Gamma_{+n}+\Gamma_{-n})\langle\Psi_{-n}|W(t)|\Psi_{-n}\rangle] \ ,\end{aligned} \tag{7}$$

$$\langle\Psi_{\pm n}|\dot W(t)|\Psi_{\mp n}\rangle = -2\kappa(n+\tfrac{1}{2})\langle\Psi_{\pm n}|W(t)|\Psi_{\mp n}\rangle \ , \tag{8}$$

where Γ's are given by

$$\Gamma_{\pm n} = (\sqrt{n+1} \pm \sqrt{n})^2/4 \ . \tag{9}$$

The solution of the above coupled differential equations will yield the time dependence of the elements of ρ. To solve Eqs. (6) and (7), we add these to get a simpler equation:

$$\begin{aligned}\dot F_n &\equiv \langle\Psi_{+n}|\dot W(t)|\Psi_{+n}\rangle + \langle\Psi_{-n}|\dot W(t)|\Psi_{-n}\rangle \\ &= 2\kappa[-(n+\tfrac{1}{2})F_n + (n+\tfrac{3}{2})F_{n+1}] \ .\end{aligned} \tag{10}$$

Let us consider the solution of the dynamical problem assuming that initially the field has photon number distribution p_n of photons, and the atom is in the excited state. Let us write

$$\rho(0) = W(0) = \sum_{n=0}^{N} p_n|n,\tfrac{1}{2}\rangle\langle n,\tfrac{1}{2}| + \sum_{m\neq n=0}^{N} p_{mn}|m,\tfrac{1}{2}\rangle\langle n,\tfrac{1}{2}| \ . \tag{11}$$

The limit $N\to\infty$ will be taken at the end of the calculations. $W(0)$ can be written in terms of the dressed states as

$$W(0)=\frac{1}{2}\sum_{n=0}^{N} p_n(|\Psi_{+n}\rangle\langle\Psi_{+n}|+|\Psi_{-n}\rangle\langle\Psi_{-n}|+|\Psi_{+n}\rangle\langle\Psi_{-n}|+|\Psi_{-n}\rangle\langle\Psi_{+n}|)$$
$$+\frac{1}{2}\sum_{m\neq n=0}^{N} p_{mn}(|\Psi_{+m}\rangle\langle\Psi_{+n}|+|\Psi_{-m}\rangle\langle\Psi_{-n}|+|\Psi_{+m}\rangle\langle\Psi_{-n}|+|\Psi_{-m}\rangle\langle\Psi_{+n}|) \quad . \tag{12}$$

The solution of Eq. (8) is straightforward and is given by

$$\langle\Psi_{\pm n}|W(t)|\Psi_{\mp n}\rangle=\frac{1}{2}\exp[-2\kappa(n+\tfrac{1}{2})t]p_n \quad . \tag{13}$$

Now, to solve (10), we write it as

$$F_n(t)=\exp[-2\kappa(n+\tfrac{1}{2})t]F_n(0)+2\kappa(n+\tfrac{3}{2})\int_0^t \exp[-2\kappa(n+\tfrac{1}{2})(t-\tau)]F_{n+1}(\tau)d\tau \quad . \tag{14}$$

We start with $n=N$ and iterate Eq. (14) for successive smaller values of n. This procedure leads to

$$F_n(t)=\exp[-2\kappa(n+\tfrac{1}{2})t]\sum_{j=n}^{N}\frac{(j+\frac{1}{2})![1-\exp(-2\kappa t)]^{j-n}}{(j-n)!(n+\frac{1}{2})!}p_j \quad . \tag{15}$$

Using now Eqs. (3), (5), (13), and (15), we can evaluate the time evolution of various physical quantities. For example, the probability $T_1(t)$ of finding the atom in the excited state irrespective of the field state is found to be[9]

$$T_1(t)=\frac{1}{2}\sum_{n=0}^{N}\exp[-2\kappa(n+\tfrac{1}{2})t]\left[p_n\cos[2g\sqrt{(n+1)}t]+\sum_{j=n}^{N}\frac{(j+\frac{1}{2})![1-\exp(-2\kappa t)]^{j-n}}{(j-n)!(n+\frac{1}{2})!}p_j\right] \quad . \tag{16}$$

The probability of finding the atom in the ground state will be $1-T_1(t)$. This is one of our key results and can be used to study the phenomena of collapse and revivals in the Jaynes-Cummings model for the finite value of cavity Q. For this purpose we take the field to be in a coherent state so that

$$p_{mn}=z^m z^{*n}\exp(-|z|^2)/\sqrt{m!}\sqrt{n!} \quad , \tag{17}$$

with $p_n=p_{nn}$, and let $N\rightarrow\infty$. The numerical results so obtained are displayed in Fig. 1 for two values of the parameter κ consistent with the secular approximation $2\kappa\times|z|^{3/2}<<g$. For comparison, the curve for $\kappa=0$ is also shown.

The dressed-state solution (15) also enables us to find the photon statistics. Calculations show that the probability $p_n(t)$ of finding n photons in the field at time t is given by

$$p_n(t)=\exp[-2\kappa(n+\tfrac{1}{2})t]p_n\cos^2[g\sqrt{(n+1)}t]+\exp[-2\kappa(n-\tfrac{1}{2})t]p_{n-1}\sin^2(g\sqrt{n}t)$$
$$+\frac{1}{2}\left[\exp[-2\kappa(n+\tfrac{1}{2})t]\sum_{j=n+1}^{\infty}\frac{(j+\frac{1}{2})![1-\exp(-2\kappa t)]^{j-n}}{(j-n)!(n+\frac{1}{2})!}p_j+(1-\delta_{n0})(\text{terms with } n\rightarrow n-1)\right]+\kappa\delta_{n0}\int_0^t F_0(\tau)d\tau. \tag{18}$$

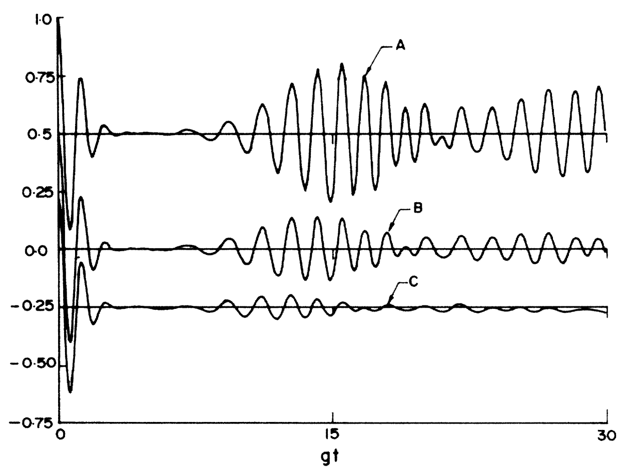

FIG. 1. The probability of finding the atom in the excited state as a function of time when the field is initially in a coherent state with $|z|^2=5$. The curve A is $T_1(t)$ for cavity relaxation parameter $\kappa=0.0$; the curves B and C represent, respectively, excitation probabilities for $\kappa=0.005$ $[T_1(t)-\frac{1}{2}]$ and $\kappa=0.015$ $[T_1(t)-\frac{3}{4}]$.

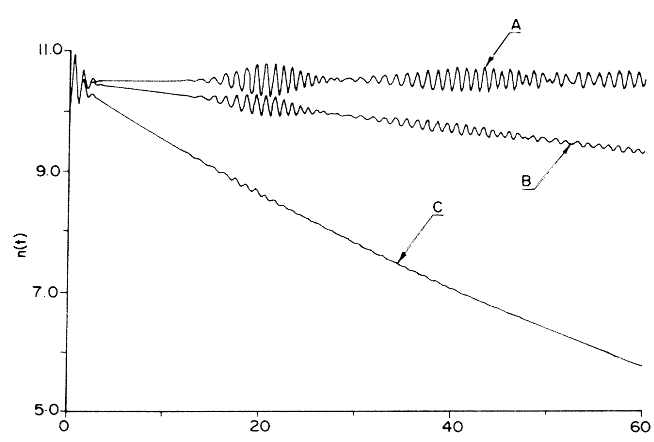

FIG. 2. The mean photon number $n(t)=\langle a^{\dagger}(t)a(t)\rangle$. Curves A, B, and C are for $\kappa/g=0$, 0.001, and 0.005, respectively, and for $|z|^2=10$.

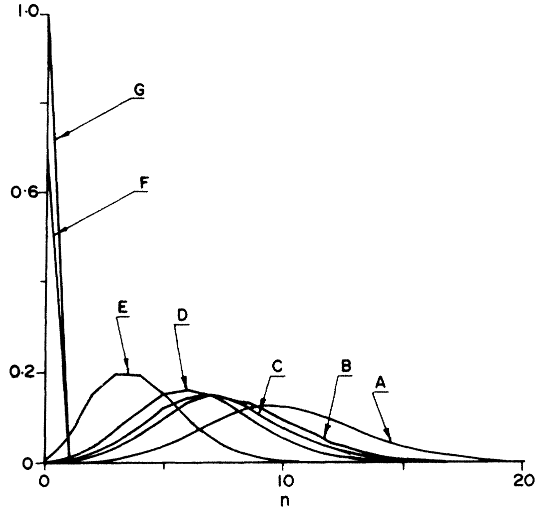

FIG. 3. The photon number distribution $p_n(t)$ for $\kappa/g=0.005$ and $|z|^2=10$. Curves A–G correspond to $gt=0.0$, 30.0, 38.0, 50.0, 100.0, 500.0, and 1500.0, respectively.

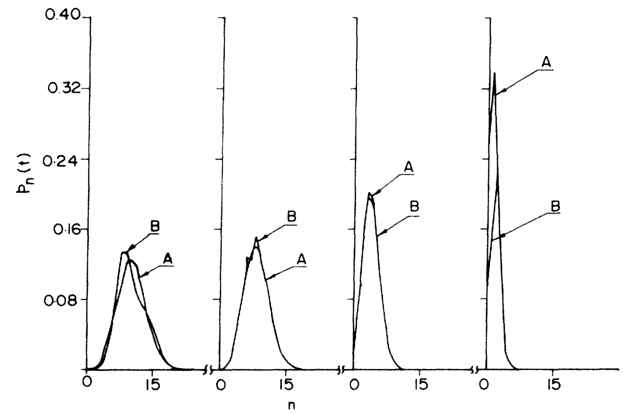

FIG. 4. Comparison of $p_n(t)$ with the Poisson distribution $\exp[-n(t)]n(t)^n/n!$ (curves A), for $gt=2.0$, 25.0, 100.0, and 200.0, with the time increasing from left to right.

The first two terms in Eq. (18) are the usual contributions to the photon statistics, but these are now damped at the rate $2\kappa(n+\frac{1}{2})$. The remaining terms arise from the leakage of photons, i.e., from the dissipative effects in the cavity. For times such that $\kappa t \ll 1$, the effects of the dissipation are not important. In Fig. 2, we present the time developement of the mean photon number $\langle a^\dagger(t) \times a(t)\rangle = n(t)$, assuming that the field is initially in state (17). A comparison of Figs. 1 and 2 shows that the dissipation effects change the photon numbers more significantly than the atomic inversion. Figure 3 shows the photon number distribution $p_n(t)$ for various times.

In Fig. 4 we give a comparison of $p_n(t)$ with the Poisson distribution (17), with $|z|^2 \rightarrow \langle a^\dagger(t)a(t)\rangle$. It is interesting to observe that the deviations from the Poisson distribution, though noticeable, are not very significant.

Perhaps a better appreciation of the statistics can be had by examining.

$$g^{(2)}(t) = \frac{\langle a^{\dagger 2}(t)a^2(t)\rangle - \langle a^\dagger(t)a(t)\rangle^2}{\langle a^\dagger(t)a(t)\rangle^2} \tag{19}$$

as a function of time t. For a strictly coherent field $g^{(2)}=0$, whereas negative values of $g^{(2)}$ lead to the antibunching of the field. Figure 5 shows the revival and oscillation in the bunching and antibunching characteristics of the field.

Finally, we study the squeezing properties of the field operator $(a^\dagger + a)$ by evaluating

$$S(t) = \langle :(a^\dagger + a)^2: \rangle - \langle a^\dagger + a\rangle^2 \ , \tag{20}$$

where : : denotes the normal ordering. Negative values of $S(t)$ imply squeezing. To evaluate $S(t)$ we also need to know the off-diagonal elements of $W(t)$ in the dressed states. We find that

$$\langle \Psi_{\pm(n+m)}|\dot{W}(t)|\Psi_{\pm n}\rangle = -2\kappa[n+(m+1)/2]\langle \Psi_{\pm(n+m)}|W(t)|\Psi_{\pm n}\rangle, \quad (m=\pm 1, \pm 2) \ . \tag{21}$$

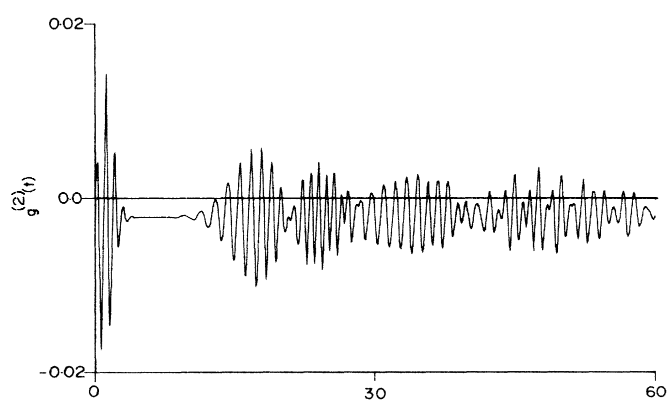

FIG. 5. The normalized intensity correlation function $g^{(2)}(t)$ as a function of time for the initial coherent state with $|z|^2=10$ and $\kappa/g=0.001$.

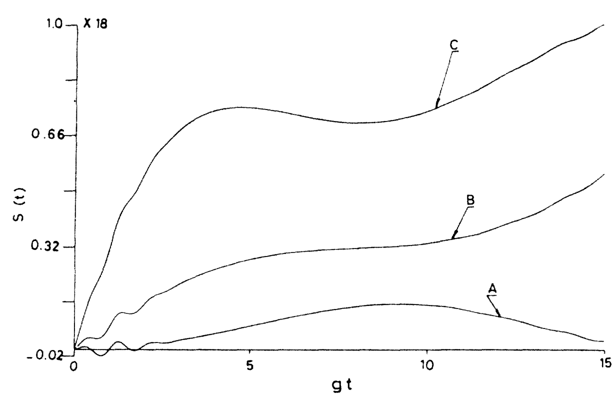

FIG. 6. The squeezing $S(t)$ in $(a^\dagger + a)$ as a function of time when the field is initially in coherent state with $|z|^2=10$. Curves A–C are for $\kappa/g=0.0$, 0.001, and 0.005, respectively.

Using Eq. (21), and for the initial state given by Eq. (17), we obtain

$$\langle a^m + a^{+m} \rangle = z^m \sum_{n=0}^{\infty} p_n \exp\{-2\kappa[n+(m+1)/2]t\} [\sqrt{n+1} \cos(gt\sqrt{n+m+1}) \cos(gt\sqrt{n+1}) + \sqrt{n+m+1} \sin(gt\sqrt{n+m+1}) \sin(gt\sqrt{n+1})]/\sqrt{n+1} + \text{c.c.} \quad (22)$$

$S(t)$ can now be easily evaluated. In Fig. 6 we have plotted $S(t)$ for three values of κ. For $\kappa=0$ there is a noticeable amount of squeezing.[10] The cavity damping is seen to have an appreciable effect on the squeezing properties.

We conclude by mentioning that our analytical solution for the density matrix can be used to study several fundamental questions in the context of quantum Brownian motion, such as the decay of the atomic coherences due to the atom's interaction with the cavity. Our solution can also be used to investigate the dynamics for very general input states of field, such as a mixture of coherent and incoherent fields. The *diagonalization,*[11] i.e., the transformation of an initial coherent superposition of field coherent states to a state which is an incoherent superposition of coherent states can also be studied using the results of this paper.

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