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New method for determining the largest Liapunov exponent of simple nonlinear systems

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A new method is introduced which allows a direct measurement of the largest positive or negative Liapunov exponent λ_m of simple dynamical systems. It is shown that λ_m can be determined experimentally by varying the coupling between two equal systems and observing the mutual correlation of their motions. An electronically simulated driven pendulum is studied as an example. The measured values of λ_m agree well with results obtained by other methods. The method can only be applied to systems for which one knows the variables in phase space and for which the coupling can be realized experimentally.

Chaotic dynamical systems can be characterized by the fact that initially nearby trajectories in phase space separate exponentially in time. The rates of separation in different directions are measured by the Liapunov exponents.[1,2] Recently, algorithms have been developed[3,4] and used to extract information about the Liapunov exponent from an experimentally measured time series.[5–7]

We introduce in this paper a new and independent method by which the largest Liapunov exponent of simple systems can be directly measured in an experiment by turning a knob.

The basic idea is to couple two identical versions of the dynamical system. We show that the critical coupling at which both systems first run out of phase, as the coupling is decreased, is apart from a factor of $\frac{1}{2}$, equal to the largest Liapunov exponent λ_m. Then, we demonstrate the applicability of this method by using it to measure both positive and negative λ_m for a driven pendulum.[8–10] The results are compared with the lower bound K_2 for the Kolmogorov entropy, which is obtained independently via the Grassberger-Procaccia algorithm,[4] and to numerical computations of λ_m.

We describe the dynamical system by a set of autonomous differential equations

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}) \ , \tag{1}$$

where $\mathbf{X}$ is a d-dimensional vector, the dot denotes the time derivative, and $\mathbf{f}(\mathbf{X})$ is a nonlinear function of the components of $\mathbf{X}$. By coupling two replicas of the system linearly in all components with the same coupling parameter c (which can be positive or negative), we obtain

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{f}(\mathbf{X}) + c(\mathbf{Y}-\mathbf{X}) \ , \\ \dot{\mathbf{Y}} &= \mathbf{f}(\mathbf{Y}) + c(\mathbf{X}-\mathbf{Y}). \end{aligned} \tag{2}$$

Linear stability analysis yields for the "phase difference" $\xi = \mathbf{X}-\mathbf{Y}$,

$$\dot{\boldsymbol{\xi}} = \dot{\mathbf{X}} - \dot{\mathbf{Y}} = \mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y}) - 2c\boldsymbol{\xi} = [M(t) - 2c\mathbf{1}]\boldsymbol{\xi} + O(\xi^2) \ , \tag{3}$$

where $M(t) = \partial f_i(\mathbf{X}(t))/\partial x_j$ is the Jacobian matrix of the system [Eq.(1)] and $\mathbf{X}(t) = \mathbf{X}(t, \mathbf{X}(0))$ is the solution of Eq. (1). Equation (3) can be integrated:

$$\boldsymbol{\xi}(t) = \left[e^{-2ct} T \exp\left(\int_0^t d\tau M(\tau)\right)\right]\boldsymbol{\xi}(0) \ , \tag{4}$$

where T is the time ordering operator. Since the eigenvalues ϵ_i of $\lim_{t\to\infty} [T\exp(\int_0^t d\tau M(\tau))]^{1/t}$ are related to the Liapunov exponents λ_i of our system via[11] $\lambda_i = \ln|\epsilon_i|$, we have for $t \to \infty$

$$\langle |\boldsymbol{\xi}(t)| \rangle \sim \exp[(\lambda_m - 2c)t], \tag{5}$$

where the average $\langle\ \rangle$ is taken over all initial conditions $\mathbf{X}(0)$ and all directions of $\boldsymbol{\xi}(0)$, and λ_m is the largest Liapunov exponent which automatically dominates all averages. Equation (5) shows that for $2c > \lambda_m$ both systems stay in phase and therefore have the λ_m of the uncoupled system described by Eq. (1). They become out of phase at a critical value of the coupling parameter $c^* = \lambda_m/2$ from which we obtain the largest Liapunov exponent λ_m.

In the experiment we simulate electronically two driven pendulums and measure the largest Liapunov exponent via the critical coupling. The circuit simulating the driven pendulum is a sample-and-hold circuit introduced and discussed by Henry and Prober.[12] The normalized pendulum equation[13]

$$\ddot{\theta} + \frac{1}{Q}\dot{\theta} + \sin\theta = A\cos(\omega t) \ , \tag{6}$$

where θ is the pendulum angle, Q is the quality factor, A and ω are the amplitude and frequency of the driving torque, is equivalent to the autonomous system

$$\dot{x} = y, \quad \dot{y} = -\frac{1}{Q}y - \sin x + A\cos z, \quad \dot{z} = \omega \ , \tag{7}$$

where $x = \theta$, $y = \dot{\theta}$, and $z = \omega t$. By considering two replicas of Eq. (7) and coupling them as in Eq. (2) we can interpret the different components of $\dot{\mathbf{X}} = -c(\mathbf{X}-\mathbf{Y}) \ldots$ as "elastic" coupling $c(\theta_1 - \theta_2)$ and "frictional" coupling $c(\dot{\theta}_1 - \dot{\theta}_2)$. Transforming back, the equations for the coupled pendulums become

$$\begin{aligned} &\ddot{\theta}_1 + \frac{1}{Q}\dot{\theta}_1 + \sin\theta_1 + 2c(\dot{\theta}_1 - \dot{\theta}_2) \\ &\qquad + (c/Q + 2c^2)(\theta_1 - \theta_2) = A\cos(\omega t) \ , \\ &\ddot{\theta}_2 + \frac{1}{Q}\dot{\theta}_2 + \sin\theta_2 + 2c(\dot{\theta}_2 - \dot{\theta}_1) \\ &\qquad + (c/Q + 2c^2)(\theta_2 - \theta_1) = A\cos(\omega t) \ . \end{aligned} \tag{8}$$

In our case, the third variable z is nonchaotic, as it is proportional to the time, so that the components z_1, z_2 need not be coupled.

It turns out experimentally that the influence of the coupling $2c(\dot{\theta}_1-\dot{\theta}_2)$ on the phase correlation is more pronounced than that of the coupling $(c/Q+2c^2)(\theta_1-\theta_2)$. Therefore, we choose to vary only the $\dot{\theta}$ coupling.[14] We measure the value of the critical coupling c^* by observing the change of the mutual correlation of the oscillators on an oscilloscope as the coupling is decreased.

In Fig. 1 the measured averaged phase fluctuations

$$\frac{1}{T^*}\int_0^{T^*} dt(\dot{\theta}_1(t)-\dot{\theta}_2(t))^2, \quad T^*=100\frac{2\pi}{\omega}$$

are plotted as a function of the coupling parameter c. The positive Liapunov exponent is measured [Fig. 1(a)] by decreasing c and observing a substantial increase of the phase fluctuations at $c^*=\lambda_m/2>0$. The negative Liapunov exponent is measured [Fig. 1(b)] by increasing the absolute magnitude of c (here $c<0$) and observing the increase of the phase fluctuations at $c^*=\lambda_m/2<0$.

In Fig. 2 we compare the largest positive Liapunov exponent λ_m, measured via the critical coupling, with the entropy parameter K_2 and with numerically obtained values of λ_m. K_2 is independently determined from a measured time series[4] of $\dot{\theta}_1(t)$. It yields a lower bound to the Kolmogorov entropy K.[15] The numerical results for λ_m in Fig. 2(c) are obtained by integrating the equation of a forced pendulum, using the method of Shimada and Nagashima.[16]

Figure 2 shows that the normalized values of λ_m obtained by the three different methods agree reasonably well with each other. We observe three chaotic bands with $\lambda_m>0$ whose positions, widths, and heights are up to a variation of the order of 10%–20% the same for all methods. It is remarkable that even K_2 which yields only a lower bound to λ_m displays a similar structure to the calculated values of λ_m, although the absolute magnitude of K_2 is smaller by a factor of 2. The remaining differences between Figs. 2(a) and 2(c) are due to the fact that our two oscillators are not completely identical. It can be shown that in this case one measures the averaged Liapunov exponent of the coupled system.[17]

For externally driven systems we can even measure the largest negative Liapunov exponent in the nonchaotic

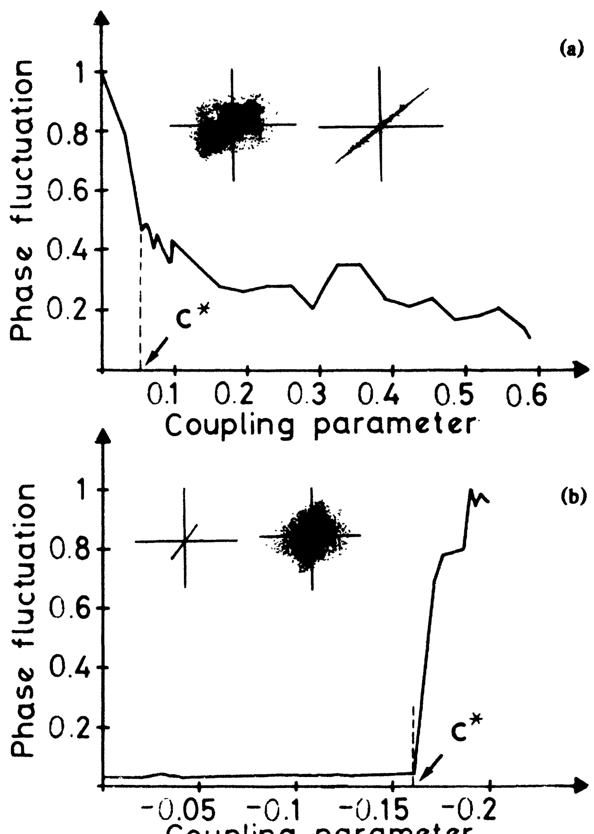

FIG. 1. The time-averaged phase fluctuations $(1/T^*)\int_0^{T^*} dt(\dot{\theta}_1(t)-\dot{\theta}_2(t))^2$ [(normalized to their individual maximum values) are plotted as a function of the coupling parameter. The insets show digitized oscilloscope traces of $\dot{\theta}_1(t)$ vs $\dot{\theta}_2(t)$]. (a) Measurement of positive λ_m: The phase fluctuations are large for weak coupling, as both oscillators are out of phase (left section of inset). For strong coupling the phase fluctuations decrease, as both oscillators are forced to move in phase with each other (right section of inset). (b) Measurement of negative λ_m: The phase fluctuations are small for weak coupling, as both oscillators are in phase with each other (left section of inset). At a strong (negative) coupling the phase fluctuations increase sharply, as both oscillators move out of phase.

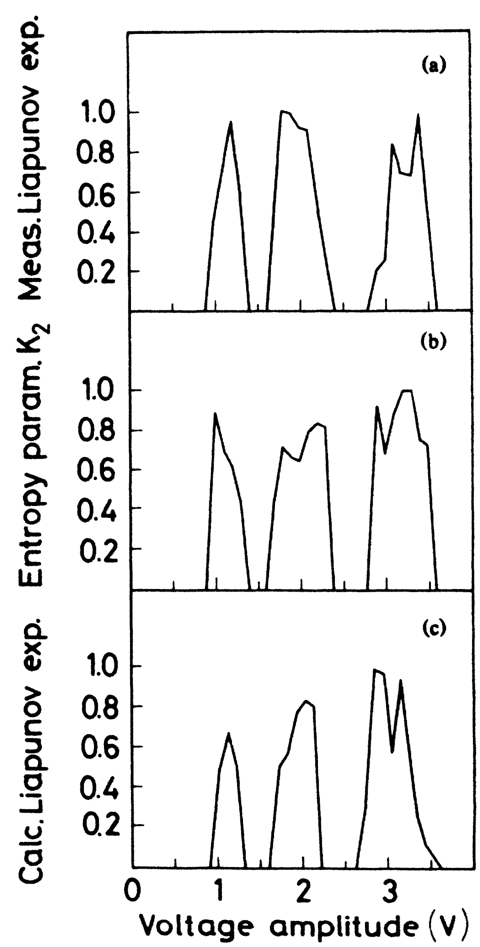

FIG. 2. Comparison of (a) the measured positive Liapunov exponent λ_m with (b) the entropy parameter K_2, and with (c) the numerically calculated values of λ_m as a function of the driving voltage amplitude. The lines are drawn by linear interpolation between subsequent points, which are determined with a resolution of 0.1 V along the voltage axis. In all three sections of the figure the variable on the ordinate is normalized to its largest value in the voltage range of 0–4 V. The absolute magnitudes of λ_m in (a) and (c) are 0.15 and 0.17, respectively, and that of K_2 in (b) is 0.073.

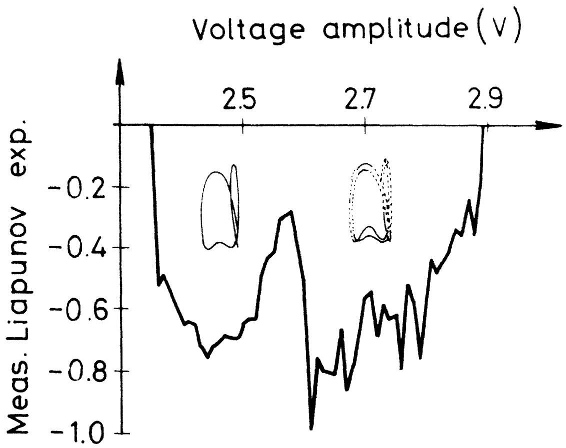

FIG. 3. The measured negative Liapunov exponent λ_m is shown as a function of the driving voltage amplitude. An increase of the Liapunov exponent at about 2.6 V is seen, where the oscillators exhibit a period doubling bifurcation. The inset shows phase portraits (θ vs $\sin\theta$) of one of the oscillators before and after the bifurcation.

state.[11] In Fig. 3 the negative Liapunov exponent λ_m, measured via the critical coupling, is shown as a function of the driving voltage amplitude. At about 2.6 V λ_m shows a significant increase which is associated with a period-doubling bifurcation of the oscillators at this voltage (see inset Fig. 3).

In conclusion, we have shown that it is possible to measure directly the largest positive or negative Liapunov exponent of a driven pendulum. This is performed by observing the mutual correlation of two equivalent coupled systems. The method can only be used for those systems for which one knows the variables in the phase space (this determines the nature of the coupling) and for which the coupling can be realized experimentally. Systems to which the method can be applied are, for instance, magnetoelastic beams,[18] p-n junctions,[19,20] and Josephson junctions.[21-25]

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