New method for the solution of the logarithmic nonlinear Schrödinger equation via stochastic mechanics

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Via Nelson's stochastic mechanics, a method for the solution of the hydrodynamical version of the logarithmic nonlinear Schrödinger equation (LNLSE), with a time-dependent forced-harmonic-oscillator potential, is presented. Based on a new interpretation of the interplay between dispersion and nonlinearity, a revealing general spreading-wave-packet solution is found. The complete stochastic process associated with the LNLSE is also derived and decomposed into underlying processes of independent nature (classical and quantum). Physical consequences and conditions which allow the existence of solitonlike (nonspreading) solutions are described: Among them, it is found that the zero-point energy is given by $\epsilon_0=\hbar\Omega_0/2$, where $\Omega_0=(\lambda/2)+[(\lambda/2)^2+\omega_0^2]^{1/2}$ with ω_0 and λ being the harmonic frequency and the nonlinearity strength, respectively.

Nonlinear Schrödinger equations (NLSE's) have been an essential and pervasive ingredient of many physical theories.[1–3] As is well known, these equations possess special particlelike solutions, which propagate without change of form, and have some well-defined shape (solitons, kinks).[2] One of the most remarkable NLSE's is that with a logarithmic nonlinear term $(-b\ln|\psi|^2)$. This was proposed by Bialynicki-Birula and Mycielski (BBM) (Ref. 4) and has recently been a center of considerable research.[4–11] While retaining many of the known features of the linear Schrödinger equation, BBM showed that only such a nonlinear term satisfies the condition of separability of noninteracting systems. In all other types of nonlinearities, the existence of even an isolated subsystem would influence the physics of all other subsystems of the Universe.[4] Thus, they concluded that such an equation was the best candidate for a *nonlinear variant of the linear Schrödinger equation.*

The physical reality of the *logarithmic nonlinear Schrödinger equation* (LNLSE) has been tested by neutron interferometer experiments. The results indicate that, as far as atomic physics is concerned, there is no definite basis for such an equation.[5–7]

Very recently, however, Hefter[8] has given robust physical grounds for the use of the LNLSE by applying it to nuclear physics and obtaining qualitative and quantitative positive results. He argues that the only consistent interpretation of such an equation is that the LNLSE holds true for extended objects, e.g., nucleons and α particles, and not for point particles (in the sense of atomic physics) as originally suggested by BBM and subsequently assumed in the experimental tests. Thus, he explains the negative results of the previous tests.

Further, it has been shown that the BBM nonlinear term emerges naturally in the stochastic formulation of quantum mechanics.[10,11] A remarkable fact has been found: The logarithmic nonlinear term originates from an internal stochastic force due to quantum fluctuations (resulting from the action of a hypothetical invariant background field)[12] and its strength is an $\hbar$-dependent parameter ($b=\hbar\lambda/2$). This implies that such a *nonlinearity does not contribute in the classical limit,*[10] as will become transparent with the results of this work.

Another distinct feature of the LNLSE is that it admits special Gaussian-shaped solitonlike solutions (we show below that this is a particular case of a more general solution generated by the crucial interplay between nonlinearity and spreading). This is a central and fundamental point in the experimental tests, which have focused on the investigation of changes in the (longitudinal[5] and lateral[7]) spreading of wave packets.

In this paper, we address this fundamental point from a new perspective. Via Nelson's stochastic mechanics we present a method for the solution of the hydrodynamical version of the LNLSE with a time-dependent forced-harmonic-oscillator potential. Based on a new interpretation of the crucial interplay between dispersion and nonlinearity, we find a general spreading-wave-packet solution and describe the physical consequences and conditions for the very existence of solitonlike solutions. We also derive the complete stochastic process associated with the LNLSE and decompose it, within the same scheme, into underlying processes of independent nature (classical and quantum). Then, we find that the zero-point energy is given by $\epsilon_0=\hbar\Omega_0/2$, where $\Omega_0=(\lambda/2)+[(\lambda/2)^2+\omega_0^2]^{1/2}$ with ω_0 and λ being the harmonic frequency and the nonlinearity strength, respectively. So, we propose that the present theory uses stochastic mechanics only in as much as it allows an alternative representation of the LNLSE, and provides physically intuitive and mathematically more transparent outcomes. Thus, our findings *per se* provide a bridge to a relatively simple alternative way for new investigations on the LNLSE, irrespective of whether stochastic mechanics is fully equivalent to ordinary quantum mechanics, or, at its present status, just significantly overlaps with it.[13]

Let us begin by reformulating the LNLSE through Nelson's stochastic mechanics:[12,14] a stochastic formulation of quantum mechanics in terms of quantum fluctuations of a system, resulting from the action of a stochastic background field. The basic assumption here is that the system under consideration consists of a quantum extended particle in a time-dependent forced-harmonic-oscillator potential [$V=\frac{1}{2}m\omega^2(t)x^2-F(t)x$], whose quantum state corresponds to a Markov process $x(t)$, which is a solution of the stochastic differential equation

$$dx(t)=b(x(t),t)dt+(\hbar/m)^{1/2}dw(t)\ , \tag{1a}$$

where $dw(t)$ is a Wiener process with expectations $\langle dw(t)\rangle=0$, $\langle dw^2(t)\rangle=dt$ and $\hbar/m$ is the diffusion coefficient. The forward drift velocity b is a dynamical variable given by

$$b(x,t)=v(x,t)+(\hbar/2m)[\partial\ln\rho(x,t)/\partial x]\ , \tag{1b}$$

where ρ and v are the probability density and the current velocity of the process determined through

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{2a}$$

and

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{m}\frac{\partial V}{\partial x}=-\frac{1}{m}\frac{\partial}{\partial x}(V_{\rm qu}+V_{\rm BBM})\ , \tag{2b}$$

where V is the time-dependent forced-harmonic-oscillator potential, and $V_{\rm qu}\equiv-(\hbar^2/2m)\rho^{-1/2}(\partial^2\rho^{1/2}/\partial x^2)$ and $V_{\rm BBM}\equiv-(\hbar\lambda/2)\ln\rho$ are the Madelung-Bohm and the Bialynicki-Birula and Mycielski quantum potentials, respectively. In doing so, one can show that the wave function $\psi(x,t)=\rho^{1/2}(x,t)\exp[iS(x,t)]$, where $S(x,t)$ is such that $v(x,t)=(\hbar/m)(\partial S/\partial x)$, satisfies the LNLSE:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\left[\frac{1}{2}m\omega^2(t)x^2-F(t)x-\frac{\hbar\lambda}{2}\ln|\psi|^2\right]\psi\ . \tag{3}$$

We turn now to the central point of our method, i.e., the essential feature emerging from $V_{\rm qu}$ and $V_{\rm BBM}$ (which are *responsible for the dispersion and nonlinearity in the LNLSE,* respectively). This feature is noteworthy in that the force arising from the *total quantum potential* ($V_{\rm qu}+V_{\rm BBM}$) is not like the mechanical force of a wave pushing on a particle with a pressure proportional to the wave intensity. Rather, it acts more like an intrinsic, self-information content intimately attached to the stochastic background field.[14] So, we assume that the expectation (classical) value of the total quantum force vanishes for all times, i.e.,

$$\langle {\rm F}_{\rm qu}\rangle\equiv-\langle\partial(V_{\rm qu}+V_{\rm BBM})/\partial x\rangle=0\ , \tag{4}$$

where $\langle A\rangle\equiv\int A\rho\,dx$ is the expectation value of A taken over an ensemble of equivalent particles. This is a new assumption *as far as treating the LNLSE, on which bears our central interest,* and will later be justified self-consistently. In fact, previous methods[2] for the investigation of solitonlike solutions of the LNLSE (in a similar hydrodynamical form[2]) cannot be employed when a time-dependent external potential of the form as in Eq. (3) is present. As we will show below, no solitonlike solution exists *unless* $\omega(t)$ is a constant (a necessary condition). This answers affirmatively an open question recently formulated by Hasse[2] on "whether there exist solitons in other potentials besides the linear potential." Further, we will verify that, as in Hasse's work,[2] ${\rm F}_{\rm qu}=0$ is a sufficient, but restrictive ($\omega=0$), condition for the existence of a solitonlike solution of the LNLSE. In any case, Eq. (4) will be self-consistently satisfied.

Thus, Eq. (2b) can be reduced to[15]

$$md\langle v\rangle/dt+m\omega^2(t)\langle x\rangle=F(t)\ , \tag{5a}$$

where $\langle v\rangle$ represents the velocity of the coordinate $\langle x\rangle$ [$\langle x\rangle\equiv x_c(t)$] of the center of mass of the system and is given by

$$\langle v\rangle=d\langle x\rangle/dt\equiv\dot{x}_c(t)\ . \tag{5b}$$

Accordingly, the above discussion suggests that Eq. (2b) can be split into (an ansatz)

$$\frac{\partial}{\partial x}\left[\frac{\hbar^2}{2m^2}\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x^2}+\frac{\hbar\lambda}{2m}\ln\rho\right]=k(t)(x-\langle x\rangle) \tag{6}$$

and

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\omega^2(t)x-\frac{F(t)}{m}=k(t)(x-\langle x\rangle)\ , \tag{7}$$

where $k(t)$ is an arbitrary function of time, which will be determined simultaneously with $\langle x\rangle$.

The time development of ρ and v can be uniquely determined from the system of Eqs. (6) and (7) if boundary conditions are imposed on them. The particle is prepared initially in a Gaussian wave packet centered at $x=0$, $\rho(x,0)=[\pi\sigma(0)]^{-1/2}\exp[-x^2/\sigma(0)]$, and with an initial velocity $v(x,0)=v_0(x)$.

Integration of Eq. (6), consistent with the boundary conditions, yields

$$\rho(x,t)=[\pi\sigma(t)]^{-1/2}\exp\{-[x-x_c(t)]^2/\sigma(t)\}\ , \tag{8a}$$

where

$$k(t)=[\hbar^2/m^2\sigma^2(t)-\hbar\lambda/m\sigma(t)]\ , \tag{8b}$$

$$\langle x\rangle\equiv x_c(t)\ . \tag{8c}$$

Next, substituting Eq. (8a) into Eq. (2a) and integrating, one obtains

$$v(x,t)=[\dot\sigma(t)/2\sigma(t)][x-x_c(t)]+\dot{x}_c(t)\ , \tag{9}$$

where the constant of integration must be zero to keep v finite as $\rho\to0$. Notice that Eq. (9) satisfies clearly Eq. (5b).

Inserting Eq. (9) into Eq. (7) it follows that, after simple manipulations,

$$[(\ddot\sigma/2\sigma)-(\dot\sigma^2/4\sigma^2)+\omega^2(t)+(\hbar\lambda/m\sigma)-(\hbar^2/m^2\sigma^2)][x-x_c(t)]+[\ddot{x}_c+\omega^2(t)x_c-F(t)/m]=0\ . \tag{10}$$

Notice also that by averaging Eq. (10), one self-consistently recovers Eq. (5a). So, it may be inferred readily that Eq. (10) is satisfied if

$$\ddot{\alpha}+\omega^2(t)\alpha+\lambda/\alpha-1/\alpha^3=0 \tag{11}$$

and

$$\ddot{x}_c+\omega^2(t)x_c-F(t)/m=0 , \tag{12}$$

where $\sigma=(\hbar/m)\alpha^2$ has been used in Eq. (11).

This emerging picture contains the two significant results which completely determine the solution of the hydrodynamical version of the LNLSE [Eqs. (2a) and (2b)] given by Eqs. (8) and (9): Eq. (11) governs the internal-structure motion of the spreading wave packet while Eq. (12) dictates its external centroid motion. The separability, decomposed as in Eqs. (11) and (12), corroborates the fact that the BBM nonlinearity does not contribute in the classical limit (as we promised to make transparent) and that its nature and effect are purely quantum. This is a *direct consequence of assumption (4),* which must hold true even when $\lambda\rightarrow 0$, since the LNLSE is supposed to be the best candidate for a nonlinear variant of the linear Schrödinger equation[1,8] (see also discussion at the beginning of this work).

To the best of our knowledge, Eq. (11) is a new result. The presence of λ, however, makes it possible for the LNLSE, with or without the quadratic potential, to admit more localized solutions, resembling classical particles, instead of the spreading wave packet of the linear Schrödinger equation. In particular, if $\lambda=F(t)=0$ we reduce the system of Eqs. (11) and (12) to what is called the Ermakov pair of equations,[14] which, by elimination of $\omega^2(t)$ between them, yields a time-dependent invariant of motion.[14] In the sense of Ermakov, we have not been able to find a general time-dependent invariant from Eqs. (11) and (12), unless some restrictive and nontrivial conditions are imposed on $F(t)$. We will report on this remark in a future work.

Next, by following Ruggiero and Zannetti,[16] the stochastic process $x(t)$ can be written as a sum of two independent components $[x(t)=x_c(t)+\xi(t)]$: the classical solution $x_c(t)$ and the pure quantum fluctuations $\xi(t)$. Then, with the help of Eqs. (1), (8), and (9), the stochastic process, associated with the LNLSE, can be written in the five-dimensional representation $[x_c(t),p_c(t),\alpha(t),\beta(t),\xi(t)]$

$$dx_c(t)=[p_c(t)/m]dt , \tag{13a}$$

$$dp_c(t)=-m\omega^2(t)x_c(t)dt+F(t)dt , \tag{13b}$$

$$d\alpha(t)=\beta(t)dt , \tag{14a}$$

$$d\beta(t)=-\omega^2(t)\alpha(t)dt-[\lambda/\alpha(t)]dt+[1/\alpha^3(t)]dt , \tag{14b}$$

$$d\xi(t)=\{[\beta(t)/\alpha(t)]-[1/\alpha^2(t)]\}\xi(t)dt+(\hbar/m)^{1/2}dw(t) . \tag{15}$$

Now, from Eqs. (8) and (11) an evident condition for a solitonlike (nonspreading) solution, i.e., $\alpha(t)=\alpha_0$ (a constant), is given by $\omega(t)=\omega_0$ (also a constant). Then, it follows that the main physical consequences are the following. (1) The current velocity is "incompressible" ($\partial v/\partial x=0$) and is given by $v=\dot{x}_c(t)$. (2) The total quantum force [with the help of Eqs. (6), (8b), and (11)] is given by

$$\mathsf{F}_{\text{qu}}\equiv-\partial(V_{\text{qu}}+V_{\text{BBM}})/\partial x=\omega_0^2[x-x_c(t)] . \tag{16}$$

This is also a new result.

If $\omega_0=0$, we have $\mathsf{F}_{\text{qu}}=0$ (Ref. 17) which recovers the case treated by Hasse,[2] via the "inverse method." Here, we also answer affirmatively his proposed open question on whether there exist solitons in a quadratic potential for the LNLSE. Notice that the presence of the time-dependent linear term in the external potential does not affect the behavior of the internal structure of the wave packet, whatsoever. (3) The stochastic process $x(t)$ reduces to the three-dimensional representation:

$$dx_c=(p_c/m)dt , \tag{17a}$$

$$dp_c=-m\omega_0^2x_c dt+F(t)dt , \tag{17b}$$

$$d\xi=-\Omega_0\xi\, dt+(\hbar/m)^{1/2}dw , \tag{18}$$

where $\Omega_0\equiv(\lambda/2)+[(\lambda/2)^2+\omega_0^2]^{1/2}$.

These equations portray a revealing physical panorama if compared with those obtained by Ruggiero and Zannetti (in the frictionless case):[16] The unique effect of the BBM nonlinearity can be identified only through the "modified frequency" Ω_0 in the stochastic differential equation for the pure quantum fluctuations [Eq. (18)]. Thus, the zero-point energy will be given by

$$\epsilon_0=\hbar\Omega_0/2 . \tag{19a}$$

In the case when $\lambda\ll\omega_0$,

$$\epsilon_0\cong\hbar[\omega_0+(\lambda/2)]/2 , \tag{19b}$$

which agrees with the result one would obtain by other means, e.g., via the perturbative method provided by Roy and Singh.[18] As pointed out by Machado,[19] the latter results [Eqs. (19)] provide a relatively simple alternative way for a numerical investigation of the strength of the BBM nonlinearity: The essential idea is to find the quantum correction on the zero-point energy of our system due to λ, by proceeding along the line of shell-model theories in nuclear physics. (4) The constant solution for Eq. (11), with $\omega_0=0$, is $(\hbar/m)\alpha_0^2=\hbar/m\lambda=\hbar^2/2mb\equiv l^2$, which is the radius of the Gaussian-shaped solitonlike solution originally found by BBM.[4]

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