Autocorrelations in the center manifold of dissipative systems

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(Received 4 October 1985)

The phase-space contraction for the systems under consideration corresponds to an asymptotic-center manifold functional dependence of the fast-relaxing variables, $X_{f,j}$'s, on the slowly-relaxing ones, the $X_{s,i}$'s. The propagation of perturbations along the center manifold is obtained from the Green-function sensitivity matrix. Scaling and self-similarity relations among the matrix elements are found. This fact allows for a simplification in the computation of the $X_{f,j}$-$X_{f,j}$ and the $X_{s,i}$-$X_{s,i}$ autocorrelations. The validity of the results is confirmed in two different contexts: (a) An analytical derivation of power spectra at the onset of periodic instabilities is performed. We demonstrate that in the infinite relaxation time limit for slow variables, the power spectra for all the $X_{f,j}$'s converge to the same distribution. This result is in accord with previous computations. (b) The power spectrum for a randomly driven anharmonic damped oscillator is computed at the asymptotic-center manifold regime and tested *vis-á-vis* previous plots exhibiting a very good agreement.

I. INTRODUCTION

We shall consider dissipative dynamical systems with a separation of the relaxation time scales which enables us to apply a center manifold (CM) reduction.[1-5] The formulation presented in Sec. II is general and extends previous results on the statistical enslaving of fast-relaxing variables[6-9] to an infinite-dimensional phase space. The approach is inspired in the finite-dimensional reduction to Poincaré-Jordan normal form put forward by Hassard and co-workers.[10] The CM reductions have been used to study the onset of symmetry-breaking instabilities in open reactive systems operating far from the thermodynamic equilibrium regime.[4-8] The dissipative structures are identified with the CM's. In this work we shall concentrate on the role of the fast-relaxing variables at the onset of instabilities in different contexts as described below.

The functional dependence of the fast modes ($X_{f,j}$'s) on the slow modes (the $X_{s,i}$'s) is given by the CM analytical expansions. *This dependence constitutes an asymptotic description of the phase-space contraction which occurs in dissipative systems before the onset of bifurcating instabilities if there exists a separation of relaxation-time scales.*

The CM expansions will be used explicitly to obtain the fundamental sensitivity propagators[11-13] which determine the response of the subordinated variables to instantaneous perturbations in the $X_{f,j}$'s and in the X_s along the CM. As we shall show in Secs. III and IV, considerable simplification in the calculation of the autocorrelation for the fast variables takes place at critical regimes. *This simplification is due to the scaling and self-similarity relations among the Green-function matrix sensitivity elements.*[11-13] *This fact will be demonstrated in this work. Such relations will be shown to hold in general in the framework of statistical subordination at critical regimes.* The general formulas for sensitivity propagators are derived in Sec. II. Sections III and IV are devoted to the specialization of the results in order to obtain the power spectra for the enslaved variables. The fluctuations of the $X_{f,j}$'s about the CM need to be calculated consistently. This means that the second moments are evaluated by averaging using the time-independent factor $Q(\mathbf{X}_f \mid \mathbf{X}_s)$ of the probability density functional $P(\mathbf{X}_f,\mathbf{X}_s,t)$. This factor is a Gaussian peaked at the CM and it is parametrically dependent on the $X_{s,i}$'s since it represents a conditional probability.[4] This allows for a continuous flow of probability about the CM.

The role of noise precursors on periodic instabilities[14-16] leading to bifurcations is analyzed with these techniques. When there exists a separation in the magnitude of the Floquet exponents which determine the stability of the cycle, a CM reduction becomes possible. In the case of transcritical bifurcations we prove that when taking the limit of infinite relaxation time for the enslaving modes, the power spectra for all the $X_{f,j}$'s converge to one single distribution. These results confirm the scaling and self-similarity relations as demonstrated in Sec. III.

The same reduction scheme is applied in Sec. IV to a damped randomly driven anharmonic oscillator. A detailed derivation is carried out by first reducing the system to Poincaré normal form in the spirit of the derivation of Sec. II. The results are in accord with those reported in previous works (cf. Ref. 17).

A close examination of the enslaving modes for this oscillator is fully justified since the dynamics representing many realistic physical systems[18,19] are reproduced even for parameter values which depart considerably from the criticality. Examples of systems equivalent to our study case in a neighborhood of the codimension-2 bifurcation are

(a) The oscillatory convection driven by the Soret-Dufour effect.[18] The slowly-relaxing enslaving modes describe the evolution of the amplitudes of the dominant velocity mode when the frequency of oscillations is small, that is, in the regime corresponding to the onset of the center manifold.

(b) The ordinary differential equation for a single-mode laser with a saturable absorber.[19] It can be shown (cf. Ref.

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19) that the system is entirely equivalent to the Soret-Dufour-driven convection problem by means of a linear transformation of variables. Again one can state that the dominant mode lies in the center manifold.

II. SENSITIVITY PROPAGATORS AT THE CENTER MANIFOLD

Consider a dissipative system whose evolution is determined by a semiflow in a Hilbert space H such that the countable set of eigenmodes ψ_j's forms a complete set. Such an assumption has been confirmed in hydrodynamic problems[20] and in reaction-diffusion systems.[21] We shall assume that the eigenmodes have been labeled so that the first K modes have as their corresponding coordinates the slowly-relaxing degrees of freedom. The CM is tangent at the stationary state to the space spanned by the first K eigenmodes at criticality.

We shall introduce the following notation:

$$\{\psi_i\}i=1,\ldots,K=\text{set of eigenmodes whose coordinates are the slowly-relaxing degrees of freedom ,} \tag{1}$$

$$\{\psi_j\}j=K+1,\ldots=\text{set of eigenmodes whose coordinates are the fast-relaxing degrees of freedom ,} \tag{2}$$

$$\{\phi_m\}=\text{set of eigenmodes for the adjoint problem .} \tag{3}$$

The following biorthogonality relations hold:

$$(\phi_m,\psi_h)=\delta_{mh} , \tag{4}$$

The coordinates for the point p belonging to H are defined by

$$\begin{aligned} &X_{s,k}=(\phi_k,p)\ (\text{slow coordinates},k=1,\ldots,K) ,\\ &X_{f,j}=(\phi_j,p)\ (\text{fast coordinates},j=K+1,\ldots) . \end{aligned} \tag{5}$$

$$\begin{aligned} N(n)=&\ \text{set of } K\text{-component vectors with}\\ &\text{non-negative integer coordinates}\\ &\text{and norm bigger than } n . \end{aligned} \tag{6}$$

The norm of $\mathbf{v}$ is defined by

$$|\mathbf{v}|=\sum_{i=1,\ldots,K}\mathbf{v}_i . \tag{7}$$

We obtain

$$N(1)\supset N(2)\supset N(3)\supset\cdots\supset N(n-1)\supset N(n)\supset\cdots \tag{8}$$

$$c_i(\mathbf{v})=\begin{cases}1 \text{ if } v_i\neq 0 ,\\ 0 \text{ otherwise ,}\end{cases}\qquad c_i(\mathbf{e}_j)=\delta_{ij} , \tag{9}$$

$$\begin{aligned} &\mathbf{X}=(X_{s,1},X_{s,2},\ldots,X_{s,K}) ,\\ &\mathbf{X}^{\mathbf{v}}=X_{s,1}^{v_1}X_{s,2}^{v_2},\ldots,X_{s,K}^{v_K} . \end{aligned} \tag{10}$$

A representation commonly encountered in hydrodynamics for a point p belonging to the locally attractive portion of the phase space is given by a series of the form

$$p=\sum_{i=1}^{K}X_{s,i}\psi_i+\sum_{\mathbf{v}\in N(2)}\mathbf{X}^{\mathbf{v}}p_{\mathbf{v}} . \tag{11}$$

Such a representation *is not* an eigenfunction expansion. The elements $p_{\mathbf{v}}$ will be called the h functions; they are orthogonal to the set defined by relation (1) and they obey inhomogeneous equations worked out in detail in Ref. 20. Our approach will be different: *we shall directly obtain the coordinates $X_{f,j}$'s as functions of the $X_{s,i}$'s and represent the point p with an eigenfunction expansion. The dependence accounts for the CM enslaving of the fast-relaxing variables.* This method extends that devised by Hassard and co-workers for finite-dimensional phase spaces.[10]

Accordingly, we write

$$p=\sum_{i=1}^{K}X_{s,i}\psi_i+\sum_{j=K+1}^{\infty}\tilde{X}_{f,j}(\mathbf{X})\psi_j . \tag{12}$$

The CM expansion is an analytic expansion,[4] therefore

$$X_{f,j}=\tilde{X}_{f,j}=\sum_{\mathbf{w}\in N(2)}u_{\mathbf{w},j}\mathbf{X}^{\mathbf{w}} . \tag{13}$$

In order to provide recurrent relations to evaluate the $u_{\mathbf{w},j}$'s, we expand the h functions in terms of the eigenfunctions:

$$p_{\mathbf{v}}=\sum_{j=K+1}^{\infty}h_{\mathbf{v},j}\psi_j . \tag{14}$$

Combining equations (12) with (13) and (11) with (14) we get

$$h_{\mathbf{v},j}=u_{\mathbf{v},j} . \tag{15}$$

In what follows we shall assume that there is more than one enslaving variable (K bigger than 1). The case of a single enslaving mode has been treated extensively (see, for example, Ref. 5). We shall come back to this situation in the next two sections. The kth eigenvalue λ_k is the damping constant for the degree of freedom associated to the kth eigenmode. The equations of motion for orbits in the CM have the form

$$\dot{X}_{s,i}=\sum_{\mathbf{r}\in N(1)}a_{i,\mathbf{r}}\mathbf{X}^{\mathbf{r}} . \tag{16a}$$

Moreover,

$$a_{i,\mathbf{e}_j}=\lambda_i\delta_{ij} . \tag{16b}$$

After a relaxation time of the order of

$$T_{\mathrm{CM}}=\sup_{j=K+1,\ldots}|\lambda_j|^{-1} \tag{17}$$

we obtain from Eqs. (12)—(16a), to first approximation:

$$\sum_{k=1}^{K}\sum_{\mathbf{p}\in N(1)} a_{k,\mathbf{p}}\mathbf{X}^{\mathbf{p}}\psi_k+\sum_{j=K+1}^{\infty}\sum_{\mathbf{q}'\in N(2)}\sum_{i=1}^{K}\sum_{\mathbf{r}\in N(1)} a_{i,\mathbf{r}}\mathbf{X}^{\mathbf{r}}q_i'\mathbf{X}^{\mathbf{q}'-c_i(\mathbf{q}')\mathbf{e}_i}u_{\mathbf{q}',j}\psi_j=\sum_{k=1}^{K}X_{s,k}\lambda_k\psi_k+\sum_{j=K+1}^{\infty}\sum_{\mathbf{q}\in N(2)}\lambda_j\mathbf{X}^{\mathbf{q}}u_{\mathbf{q},j}\psi_j \ . \quad (18)$$

Hence, we have $a_{k,\mathbf{e}_m}=\delta_{km}\lambda_k$ which proves relation (16b). From Eq. (18) we obtain

$$\lambda_j u_{\mathbf{q}}=\sum_{i=1}^{K}\widetilde{\sum_{\mathbf{q}'\in N(2)}}\widetilde{\sum_{\mathbf{r}\in N(1)}} a_{i,\mathbf{r}}q_i'u_{\mathbf{q}',j} \ . \quad (19)$$

The symbol $\widetilde{\sum}_{\mathbf{q}'\in N(2)}\widetilde{\sum}_{\mathbf{r}\in N(1)}$ indicates that the sums are extended over all $r,\mathbf{q}'$s such that

$$\mathbf{q}'+\mathbf{r}-c_i(\mathbf{q}')\mathbf{e}_i=\mathbf{q} \quad \text{for all } i=1,\ldots,K \ . \quad (20)$$

The recursive relations given in Eq. (19) have a starting point given by the adiabatic elimination method:[1-4] To lowest order in the $X_{s,i}$'s and within the situation described in Eq. (17), the $u_{\mathbf{q},j}$'s are determined from the approximation $\dot{X}_{f,j}\approx 0$ which leads to

$$u_{\mathbf{e}_i,j}=\delta_{ij} \ . \quad (21)$$

The reader is referred to Secs. III and IV for specific calculations.

In order to determine the propagation of a perturbation along the center manifold, we need to evaluate the Green-function sensitivity elements given by the derivatives

$$G_{(h,i),(u,j)}(t,t')=\frac{\partial X_{h,i}(t)}{\partial X_{u,j}(t')} \quad h,u=s \text{ or } f, \quad i,j=1,\ldots \quad (22)$$

subject to the following restriction,

$$t>t'\geq T_{\mathrm{CM}} \ . \quad (23)$$

For convenience we define

$$M_{j,i}(t)=\frac{\partial \widetilde{X}_{f,j}}{\partial X_{s,i}}(t) \ . \quad (24)$$

Making use of Eq. (24), we obtain the following relations:

$$G_{(f,j),(s,i)}(t,t')=M_{j,i}(t)H(t-t')e^{\lambda_i(t-t')} \ . \quad (25)$$

Here H is the Heaviside step function:

$$G_{(s,i),(f,j)}(t,t')=[M_{j,i}(t')]^{-1}e^{\lambda_i(t-t')}H(t-t') \ . \quad (26)$$

And also

$$G_{(f,j),(f,n)}(t,t')=M_{j,i}(t)[M_{n,i}(t')]^{-1}H(t-t')e^{\lambda_i(t-t')} \ . \quad (27)$$

Equations (25)—(27) exhibit the scaling and self-similarity relations. The multiplicative laws among the Green-function coefficients are satisfied. Indeed, one can verify that

$$G_{(h,i),(u,j)}(t,t')G_{(u,j),(m,n)}(t',t'')=G_{(h,i),(m,n)}(t,t'') \ , \quad (28)$$

where $t>t'>t''$ and $h,u,m=s$ or f and $i,j,n=1,\ldots$.

The general group property for the multiplication of the Green-function matrix, given by

$$\underline{G}(t,t')\underline{G}(t',t'')=\underline{G}(t,t'') \quad (28')$$

is verified when the system is confined to the CM, that is, for a time scale given by the inequality $t''\geq T_{\mathrm{CM}}$. Indeed, when the system is restricted to its locally attractive portion of the phase space, the equations (25)—(27) hold and therefore relation (28) is valid as well.

The propagators are related to each other according to the following commutative diagram

$$\begin{array}{ccc} & M_{j,i}(t) & \\ (s,i),(s,i) & \longrightarrow & (f,j),(s,i) \\ \Big\downarrow M_{n,i}(t')^{-1} & & \Big\downarrow M_{n,i}(t')^{-1} \\ (s,i),(f,n) & \longrightarrow & (f,j),(f,n) \\ & M_{j,i}(t) & \end{array} \quad (29)$$

The symbol $(h,i),(m,j)$ denotes $G_{(h,i),(m,j)}(t,t')$. The arrows can be reversed and such an operation corresponds to multiplication by the inverse of the quantity indicated. The word "commutative" in the previous paragraph means that the following paths are equivalent:

$$\begin{aligned} &(s,i),(s,i)\rightarrow(f,j),(s,i)\rightarrow(f,j),(f,n) \ , \\ &(s,i),(s,i)\rightarrow(s,i),(f,n)\rightarrow(f,j),(f,n) \ . \end{aligned} \quad (30)$$

III. CENTER MANIFOLD FOR A PERIODIC INSTABILITY

The Floquet exponents associated with a periodicity in dissipative systems determine the stability of the orbit in phase space. The mean relaxation time for the modes of the transient response problem are thus given by the reciprocal of the absolute value of the Floquet exponents. We shall assume that all the Floquet exponents are in the left-hand side of the complex plane and examine the system right before the onset of the bifurcating instability. The results hold when there exists a separation of time scales. The coordinates associated with the slowly-relaxing eigenmodes are given by

$$X_{s,i}=e^{\lambda_i t}F(t) \quad i=1,\ldots,K \ . \quad (31)$$

We assume that, after a time rescaling, $F(t)$ is periodic with period 2π. To fix the ideas, we shall concentrate on the case of a transcritical bifurcation. That means $K=1$ [we therefore drop the subindex i in Eq. (31)] and also the following relation holds:

$$\lim_{\mathrm{Re}\,\lambda_1\rightarrow 0}\mathrm{Im}\lambda_1=0 \ . \quad (32)$$

The CM expansion given in Eq. (13) takes the simple form

$$\tilde{X}_{f,j}=\sum_{n=2}^{\infty} u_{n,j}X_s^n,\quad X_{f,j}\sim O(X_s^n j),\quad n_j\geq 2\ . \tag{33}$$

The second moments for the fluctuations about the center manifold are given by

$$\overline{f_j^2}=\overline{(X_{f,j}-\bar{X}_{f,j})^2}=\overline{(X_{f,j}-\tilde{X}_{f,j})^2}\ . \tag{34}$$

The averages are taken over an ensemble of realizations of δ-correlated Gaussian white noise which is taken as the precursor for the unfolding of the bifurcation (cf. Ref. 14).

The calculation of the quantity defined in Eq. (34) is carried out by factorizing the probability density functional into time-independent factors for the fast-relaxing variables and a time-dependent factor for X_s. The validity of this procedure has been established.[4,5] It holds under the restriction imposed by Eq. (17).

The time-independent factor corresponding to $X_{f,j}$ is given by[5]

$$Q(X_{f,j}\mid X_s)=[L_j(X_s)]^{1/2}\pi^{-1/2}\exp\{-L_j(X_s)[X_j-\tilde{X}_j(X_s)]^2\}\ , \tag{35}$$

$$L_j(X_s)=(\mathrm{Re}\lambda_j)/N_j+O(X_s^2)\ , \tag{36}$$

where N_j is the intensity of the jth component of the noise. The distribution is a Gaussian peaked at the CM with a width which is dependent on the position in the CM.

Then we obtain

$$\overline{f_j^2}=\int dX_{f,j}(X_{f,j}-\tilde{X}_{f,j})^2Q(X_{f,j}\mid X_s)\ . \tag{37}$$

This formula will be used to obtain Fig. 1.

A plot of the second moment as a function of $\overline{X_{f,j}}^2$ is given for the case presented in Sec. IV. Note the growth of the fluctuations at criticality. This plot can be compared *vis-á-vis* the work given in Ref. 22 dealing with fluctuations at criticality.

The autocorrelation function is given by

$$C_j(\tau)=\lim_{T\to\infty}\left[\int_0^T M_j(t+\tau)M_j(t)^{-1}e^{\lambda_1\tau}\overline{f_j^2}dt\right]\frac{1}{T}\ . \tag{38}$$

Since we have only one slow variable, we have also dropped the subindex i in Eqs. (24)–(30).

From relation (33), at the adiabatic approximation limit, we get

$$C_j(\tau)=(\overline{f_j^2})_{\mathrm{eq}}\,e^{n_j\lambda_1\tau}G(\tau)\ , \tag{39}$$

where $G(\tau)$ is given by

$$G(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\frac{F(t+\tau)}{F(t)}\right]^{n_j-1}dt\ . \tag{40}$$

Therefore, we get

$$G(\tau)=\sum_{n=-\infty}^{+\infty} g_n e^{in\tau}\ . \tag{41}$$

The power spectrum can now be obtained by Fourier transforming the autocorrelation function. It is given by

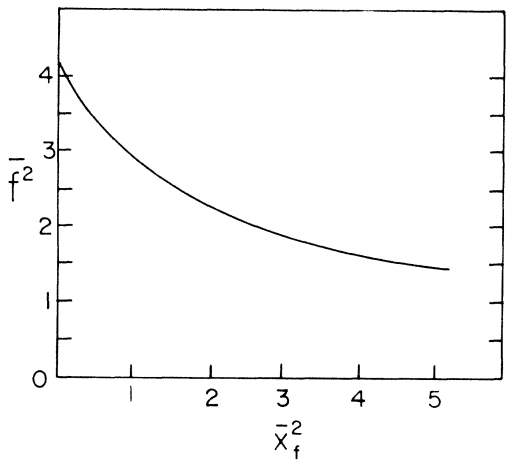

FIG. 1. Plot of the average second moment $\overline{f^2}$ versus the c.m. coordinate $\overline{X_f}^2$. The choice of the parameters is as in Fig. 2. The fast variable fluctuates about the c.m.

$$S_j(\omega)=(\overline{f_j^2})_{\mathrm{eq}}\,2n_j\mid\lambda_1\mid\times\sum_{n=-\infty}^{+\infty} g_n\{[n_j^2\mid\lambda_1\mid^2+(\omega-n)^2]^2\}^{-1}\ . \tag{42}$$

As a consequence of the scaling relation (27) we have *for all* j's

$$\lim_{\lambda_1\to 0}S_j(\omega)=\sum_{n=-\infty}^{+\infty}\delta(\omega-n)\ . \tag{43}$$

The results obtained in Ref. 14 show that the power spectrum for the autocorrelation of *any fast variable* is a sum of the Lorentzians peaked at the same frequencies as the ones given in relation (42). We have thus arrived at the same power spectra applying the CM reduction and demonstrated that the fact that they are given by sums of Lorentzians peaked at the same frequencies *for any fast variable* is a consequence of the simplified nature of the Green function as given by relation (27) which accounts for the contraction in phase space. It follows that in the infinite relaxation-time limit, all the spectra will converge to one single line spectrum as given by relation (43). This is so since the exponential decay in the autocorrelation [as given by Eq. (39)] becomes unity.

IV. AUTOCORRELATION IN THE CENTER MANIFOLD FOR A RANDOMLY DRIVEN ANHARMONIC DAMPED OSCILLATOR

In this example we only have one enslaving variable given by

$$X_s=X-\frac{1}{\beta}\dot{X}\ , \tag{44}$$

X = position coordinate ,

β = damping constant ,

and one enslaved (fast-relaxing) variable:

$$X_f=\frac{1}{\beta}\dot{X}\ , \tag{45}$$

The transformation T defined by Eqs. (44) and (45) corresponds to the shift from the position-momentum representation to the $X_s - X_f$ representation which is in the Poincaré normal form (for a general derivation of such transformations see Ref. 10).

The deterministic equation is

$$\ddot{X}_1 = \alpha X + \beta \dot{X} + AX^3 + BX^2\dot{X} ,$$
$$\xi(t) = T^{-1}\begin{bmatrix} 0 \\ \xi_f(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \xi_f(t) , \tag{46}$$
$$A,B,\beta < 0; \quad |\alpha| \ll |\beta| .$$

With this particular choice of the noise $\xi(t)$ we make sure that the time scale for the fluctuations is comparable to that of X_f in the $X_s - X_f$ representation (cf. Ref. 5).

The theory developed in Sec. II applies to the system in Poincaré normal form:

$$\begin{bmatrix} \dot{X}_s \\ \dot{X}_f \end{bmatrix} = \begin{bmatrix} -\alpha/\beta & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} X_s \\ X_f \end{bmatrix} + \begin{bmatrix} \phi(X_s,X_f) \\ -\phi(X_s,X_f) \end{bmatrix} + \begin{bmatrix} 0 \\ \xi_f(t) \end{bmatrix} , \tag{47}$$
$$\phi(X_s,X_f) = -(A/\beta)X_s^3 - [B + 3A/\beta]X_s^2 X_f - [2B + 3A/\beta]X_s X_f^2 - [B + A/\beta]X_f^3 .$$

From the adiabatic approximation, it follows that the propagator defined by Eq. (27) is

$$G_{ff}(t,t') = \left[-\frac{3A}{\beta^2}[X_s(t)]^2 + O[X_s^4(t)] \right] \times \left[-\frac{3A}{\beta^2}[X_s(t')]^2 + O[X_s^4(t')] \right]^{-1} \times \exp\left[\frac{2|\alpha|}{\beta}(t-t') \right] \tag{48}$$

assuming $t > t'$. Since we intend to do an explicit calculation, the quantity L as defined in Eqs. (35) and (36) needs to be calculated. We get

$$L(X_s) = -\frac{\beta}{N} + \frac{-B - (3A/\beta)}{N(1/2)}(X_s^2) . \tag{49}$$

Since the results will be tested *vis-à-vis* those obtained from an analog computer simulation given in Ref. 17, we shall introduce the following dimensionless parameters (cf. Ref. 17):

$$\Delta = \frac{AN/2}{\beta\alpha^2}, \quad \eta = -\frac{\beta}{2\sqrt{|\alpha|}}, \quad \rho = \frac{\sqrt{|\alpha|}}{(1/2)N} . \tag{50}$$

The first one can be regarded as a nonlinearity parameter and the second one, as a damping parameter. Equation (13) for this particular case reads

$$\tilde{X}_f \cong (\tfrac{1}{8}N^2\Delta\rho^3\eta^{-1})X_s^3 + O(X_s^5) . \tag{51}$$

The plot displayed in Fig. 1 was obtained using the working equation (37), together with Eqs. (49)—(51) and using a distribution of the form given by Eq. (35). In accord with Ref. (17), we chose the resonant frequency

$$\sqrt{|\alpha|} = 660 \text{ rps}, \quad B = 0, \quad N = 7000 \text{ s}^{-1} ,$$
$$\eta = 1.5, \quad \Delta = 0.33 .$$

The behavior of the fluctuations about the CM in the region near criticality can be correlated very well with fluctuations in the order parameters in the sense of Haken (the reader can compare Fig. 1 with a plot for the order parameter given in Ref. 22.)

The autocorrelation function is given by

$$C(\tau) = \lim_{T\to\infty} \int_0^T \frac{1}{T} \left[-\frac{3A}{\beta^2}[X_s(t+\tau)]^2 \right] \times \left[-\frac{3A}{\beta^2}[X_s(t)]^2 \right]^{-1} \times e^{(2|\alpha|/\beta)\tau} \overline{f^2(t)} dt . \tag{52}$$

The corresponding spectral density is obtained by the standard Fourier transform of $C(\tau)$. The location of the peak and the shape of the plot as exhibited in Fig. 2 are remarkably close to that obtained with the analog computer simulation as described below.

The choice of the parameters given in the caption for Fig. 2 was made so that the data for the ratio of the fundamental frequency over the effective linear frequency is available from Ref. 17. In our case, the effective linear frequency as obtained from a self-consistent statistical linearization is approximately 690 Hz. We predict that the peak in the spectral density occurs at 710 Hz in very good agreement with the computer simulation experiment and with the statistical linearization prediction. The value of the spectral density estimated from our calculation is -21 dB. The computer experiment yields -19 dB and the statistical linearization, in contrast, predicts a sharper peak at -16 dB. In the low-frequency regime our predictions are approximately 1—2 dB above those given by the statistical linearization. This is due to the fact that our calculations do not involve an effective linearized poten-

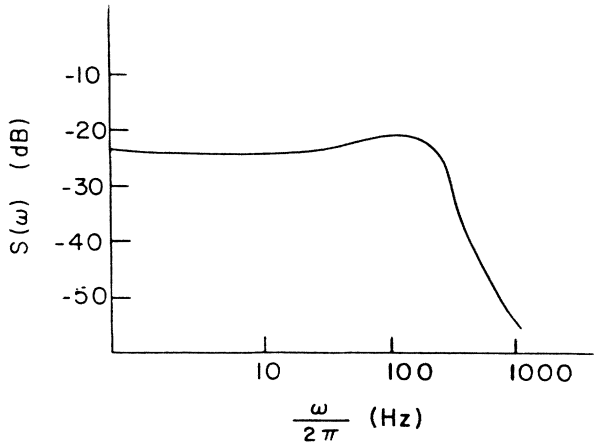

FIG. 2. Spectral density plot obtained by means of the fast-Fourier-transform algorithm and the working equation (52). The parameter values are resonant frequency $=660$ rps; $B=0$; $N=7000\ \text{s}^{-1}$; $\eta=0.5$; $\Delta=1$.

tial, which is smaller at low frequencies than the rigorous potential (the regime under consideration involves frequencies below the effective value obtained by statistical linearization.) At frequencies higher than the effective value the three methods show very good agreement since the power spectrum differs in less than 1 dB from one method to another.

ACKNOWLEDGMENTS

We acknowledge support from the U.S. Office of Naval Research and the U.S. Air Force Office of Scientific Research.

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