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Squeezing via two-photon transitions

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The squeezing spectrum for a cavity field mode interacting with an ensemble of three-level "Λ-configuration" atoms by an effective two-photon transition is calculated. The advantage of the three-level Λ system as a squeezing medium, that is, optical nonlinearity without atomic saturation, has recently been pointed out by Reid, Walls, and Dalton. We predict perfect squeezing at the turning points for dispersive optical bistability and good squeezing for a range of other cases. Three-level ladder atoms interacting by an effective two-photon transition are also shown to give perfect squeezing in the dispersive limit.

I. INTRODUCTION

The generation of squeezed states of light is currently the subject of much theoretical and experimental attention. For a general review of squeezed states the reader is referred to Ref. 1 and for recent experiments to Refs. 2–4. These experiments focus on studies of nondegenerate four-wave mixing in resonant[2,3] and nonresonant media.[4] Several theoretical analyses of the effects of spontaneous emission from the medium have been given.[5] Reid and Walls modeled the medium by two-level atoms for resonant media[6–8] and anharmonic oscillators[9] for nonresonant media. They have analyzed both degenerate and nondegenerate[8] four-wave mixing and optical bistability.[7] We note that squeezing has recently been reported to result from the Hanle effect in the V configuration of three-level atoms.[10]

The source of nonlinearity in a one-photon transition in a two-level atom is the saturation, which gives rise to spontaneous emission and hence tends to destroy the squeezing. One is therefore motivated to search for a nonlinearity where atomic saturation and hence an inverted medium is not the principle source of nonlinearity. In this respect Reid, Walls, and Dalton[11] have shown that the three-level Λ medium is particularly promising, since a nonlinearity may be generated via ground-state coherences with negligible upper-state population.

The three-level Λ medium has been the subject of recent theoretical studies[12–16] into optical instabilities, many of which have been demonstrated experimentally.[17–23] In this paper we present a model of a single cavity field mode interacting with an ensemble of three-level Λ atoms. The field mode frequency is assumed to be far from resonance with the upper level, hence an effective Hamiltonian coupling the two lower levels via a two-photon transition is employed. We derive Langevin equations for the atomic and field variables and calculate the squeezing spectrum of the output field from the cavity. In the limit of large two-photon detuning the system is shown to be equivalent to the nonlinear polarizability model of dispersive optical bistability, studied by Drummond and Walls[24] and Collett and Walls,[25] which gives perfect squeezing at the turning points of bistability. We also demonstrate that good squeezing is attainable for intermediate values of the two-photon detuning.

A comparison is made of the squeezing spectra obtained by adiabatically eliminating the atomic variables and those obtained without adiabatic elimination. Limits in which these two results coincide are determined.

We also consider a two-photon transition in a ladder atomic configuration. In the dispersive limit of large two-photon detuning we find that this system also reduces to the nonlinear polarizability model of dispersive optical bistability. Two-photon optical bistability in rubidium vapor with significant detuning has been observed.[26]

II. THE Λ MEDIUM MODEL

We consider N three-level atoms in a Λ configuration. The two lower levels of each atom are coupled by a two-photon interaction with a cavity field mode (Fig. 1). We only consider the limit in which the cavity mode is highly detuned from resonance with the one-photon transitions from a lower to the upper level. This enables the upper level to be adiabatically eliminated and the interaction may be modeled by an effective two-photon process. Our model Hamiltonian is

$$\begin{aligned} H &= H_F + H_E + H_I + H_D \,, \\ H_F &= \hbar\omega_c a^\dagger a + \tfrac{1}{2}\hbar\omega_a \sum_{\mu=1}^{N} \sigma_\mu^z \,, \\ H_E &= i\hbar(E' e^{-i\omega_E t} a^\dagger - E' e^{i\omega_E t} a) \,, \\ H_I &= i\hbar g' a^\dagger a \sum_{\mu=1}^{N} (\sigma_\mu^- - \sigma_\mu^+) \,, \\ H_D &= \sum_\mu (\Gamma_p \sigma_\mu^z + \Gamma_A \sigma_\mu^+ + \Gamma_A^\dagger \sigma_\mu^-) + \Gamma_F a^\dagger + \Gamma_F^\dagger a \,. \end{aligned} \tag{1}$$

H_F is the free Hamiltonian for the cavity field mode of frequency ω_c, having creation and annihilation operators $a^\dagger$ and a, and for the lower levels of the N atoms, separated by energy $\hbar\omega_a$. H_F accounts for the driving of the cavity mode by an external coherent field of amplitude E' and frequency ω_E. H_I models the two-photon interaction of the cavity mode with the lower levels, g' being the

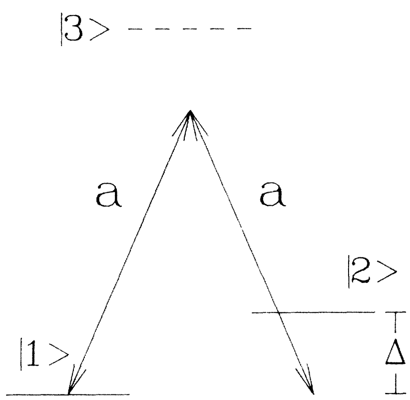

FIG. 1. The three-level Λ atom.

two-photon coupling strength. $\sigma_\mu^z, \sigma_\mu^+, \sigma_\mu^-$ are the Pauli atomic operators for the μth atom. H_D describes the coupling of the atoms to reservoirs $\Gamma_A, \Gamma_A^\dagger$, representing incoherent pumping and damping, and to the reservoir Γ_p, representing phase damping. The final terms couple the field to reservoirs $\Gamma_F, \Gamma_F^\dagger$ describing damping of the cavity mode.

Following the analysis of Haken[27] and Drummond and Walls[28] we may derive a master equation for the density operator ρ from the Hamiltonian (1), including the various damping and incoherent pumping effects. Specifically we have included damping of the cavity mode at rate γ_c, phase damping of coherence between the lower levels at rate γ_p, population decay from level $|2\rangle$ to level $|1\rangle$ at rate ω_{21}, and incoherent population pumping from level $|1\rangle$ to level $|2\rangle$ at rate ω_{12}.

The last two processes allow us to introduce a population inversion between the lower levels, which is necessary for interesting behavior but cannot be generated by the coherent two-photon process. They serve to model mechanisms such as optical pumping via the upper level which would in practice generate population inversion. These mechanisms will be discussed in a future publication.[29] The noise associated with these processes is included in our model.

The master equation for the density operator yields a Fokker-Planck equation for the generalized P-representation function[30] which is valid for a large number of atoms N.[27,28] From this follows the equivalent set of Langevin equations

$$\begin{aligned}
\dot{\alpha} &= E-(\gamma_c - i\phi)\alpha - g\alpha(J^+ - J^-) + \Gamma_\alpha ,\\
\dot{\alpha}^\dagger &= E-(\gamma_c + i\phi)\alpha^\dagger + g\alpha^\dagger(J^+ - J^-) + \Gamma_{\alpha^\dagger} ,\\
\dot{J}^- &= -(\gamma_\perp - i\Delta)J^- + g\alpha\alpha^\dagger D + \Gamma_{J^-} ,\\
\dot{J}^+ &= -(\gamma_\perp + i\Delta)J^- + g\alpha\alpha^\dagger D + \Gamma_{J^+} ,\\
\dot{D} &= -\gamma_{\parallel}(D-D_0) - 2g\alpha\alpha^\dagger(J^- + J^+) + \Gamma_D .
\end{aligned} \tag{2}$$

The stochastic variables $\alpha, \alpha^\dagger$ correspond to the field annihilation and creation operators so that the means $\langle(\alpha^\dagger)^n\alpha^m\rangle$ equal the quantum expectation values $\langle(a^\dagger)^n a^m\rangle$. Similarly J^-, J^+, D correspond to the collective atomic coherence and population inversion operators. E is a scaled driving field amplitude related to the field amplitude in volts per meter incident on the cavity input mirror, E_{expt}, by

$$E_{\text{expt}} = (\hbar\omega_E/2\epsilon_0 V)^{1/2} T_R t^{1/2} N^{1/2} E , \tag{3}$$

where V is the cavity mode volume, t is the input mirror power transmittance, and T_R is the cavity round-trip time. The two-photon coupling parameter g is related to the one-photon coupling $g_1 = e(\omega/2\hbar\epsilon_0 V)^{1/2} D_{13}$, where e is the electron charge and D_{13} the electric dipole matrix element between the upper and lower levels, by $g \approx g_1^2 \Delta_1^{-1} N$, where Δ_1 is the detuning between the driving field and the lower-to-upper-level transition. The following parameters have been introduced in Eqs. (2):

$$\begin{aligned}
&\gamma_{\parallel} = \omega_{12} + \omega_{21}, \quad \gamma_\perp = \tfrac{1}{2}\gamma_{\parallel} + \gamma_p ,\\
&D_0 = \gamma_{\parallel}^{-1}(\omega_{12} - \omega_{21}) .
\end{aligned} \tag{4}$$

ϕ is the cavity to driving field detuning, $\phi = \omega_c - \omega_E$. Δ is the two-photon detuning.

The Gaussian noise terms in the Langevin equations (2) have the following nonzero correlations and their conjugates:

$$\begin{aligned}
&\langle \Gamma_\alpha(t)\Gamma_{J^-}(t')\rangle = g\alpha D\delta(t-t') ,\\
&\langle \Gamma_\alpha(t)\Gamma_D(t')\rangle = -2g\alpha J^+\delta(t-t') ,\\
&\langle \Gamma_{J^-}(t)\Gamma_{J^-}(t')\rangle = 2g\alpha\alpha^\dagger J^-\delta(t-t') ,\\
&\langle \Gamma_{J^-}(t)\Gamma_{J^+}(t')\rangle = [\omega_{12} + \gamma_p(1+D)]\delta(t-t') ,\\
&\langle \Gamma_{J^-}(t)\Gamma_D(t')\rangle = -2\omega_{12}J^-\delta(t-t') ,\\
&\langle \Gamma_D(t)\Gamma_D(t')\rangle = [-4g\alpha\alpha^\dagger(J^- + J^+) + 2\omega_{12}(1-D)\\
&\qquad\qquad + 2\omega_{21}(1+D)]\delta(t-t') .
\end{aligned} \tag{5}$$

The stationary-state means are obtained from Eqs. (2) by setting the left-hand sides to zero and ignoring the noise terms, which have zero mean. We find

$$\begin{aligned}
&D_S = D_0\{1 + X^2/[1+(\Delta')^2]\}^{-1} ,\\
&J_S^- = gD_S|\alpha_S|^2\gamma_\perp^{-1}(1+i\Delta')/[1+(\Delta')^2] ,\\
&\alpha_S = E\{\gamma_c - i\phi - g(\gamma_{\parallel}/\gamma_\perp)^{1/2} D_S iX\Delta'/[1+(\Delta')^2]\}^{-1} ,\\
&J_S^+ = (J_S^-)^*, \quad \alpha_S^\dagger = \alpha_S^* ,\\
&X = |\alpha_S|^2/n_0, \quad n_0 = (\gamma_{\parallel}\gamma_\perp/4g^2)^{1/2} ,\\
&\Delta' = \Delta/\gamma_\perp ,
\end{aligned} \tag{6}$$

where the subscript S indicates a stationary-state mean. The scaled intensity X is found by solving the nonlinear equation

$$E^2 = n_0 X \left[\gamma_c^2 + \left[\phi + g(\gamma_{\parallel}/\gamma_\perp)^{1/2} D_0 \times \frac{\Delta' X}{1+(\Delta')^2 + X^2} \right]^2 \right] . \tag{7}$$

There are two sources of nonlinearity in this equation, each of which may give rise to bistability.[31] As in the one-photon case it may result from atomic saturation. However, there is an additional intensity-dependent refractive index in the two-photon process accounted for by the intensity occurring in the numerator of the last term in Eq. (7). It is the bistability induced by this second effect that will be of interest to us.

III. LINEARIZATION AND SQUEEZING

In order to proceed with finding the stationary states of Eqs. (2) we approximate them by an Ornstein-Uhlenbeck process. This is achieved by linearization and the replacement of the stochastic variables by their stationary-state means. We are thus restricted to the consideration of small fluctuations about the mean values. When large fluctuations occur, such as for perfect squeezing, our approximation breaks down. Nevertheless our method will indicate where good squeezing may be expected. Using a prefix δ to indicate the deviation of a stochastic variable from its mean, we obtain from Eqs. (2) the following linearized vector Langevin equations:

$$\delta\dot{\mathbf{x}} = -\underline{\mathbf{A}}\,\delta\mathbf{x} + \underline{\mathbf{B}}\boldsymbol{\Gamma}\,, \qquad \delta\mathbf{x}^T = (\delta\alpha, \delta\alpha^\dagger, \delta J^-, \delta J^+, \delta D)\,, \tag{8}$$

where $\boldsymbol{\Gamma}$ is a vector of delta-correlated Gaussian random processes. The drift matrix $-\underline{\mathbf{A}}$ is

$$\begin{bmatrix} -\gamma_c + i\phi + g(J_S^- - J_S^+) & 0 & g\alpha_S & -g\alpha_S & 0 \\ 0 & -\gamma_c - i\phi - g(J_S^- - J_S^+) & -g\alpha_S^* & g\alpha_S^* & 0 \\ gD_S\alpha_S^* & gD_S\alpha_S & -\gamma_\perp + i\Delta & 0 & g\,|\alpha_S|^2 \\ gD_S\alpha_S^* & gD_S\alpha_S & 0 & -\gamma_\perp - i\Delta & g\,|\alpha_S|^2 \\ -2g(J_S^- + J_S^+)\alpha_S^* & -2g(J_S^- + J_S^+)\alpha_S & -2g\,|\alpha_S|^2 & -2g\,|\alpha_S|^2 & -\gamma_\parallel \end{bmatrix}. \tag{9}$$

The diffusion matrix $\underline{\mathbf{D}} = \underline{\mathbf{B}}\,\underline{\mathbf{B}}^T$ is

$$\begin{bmatrix} 0 & 0 & g\alpha_S D_S & 0 & -2g\alpha_S J_S^+ \\ 0 & 0 & 0 & g\alpha_S^* D_S & -2g\alpha_S^* J_S^- \\ g\alpha_S D_S & 0 & 2g\,|\alpha_S|^2 J_S^- & \omega_{12} + \gamma_p(1+D_S) & -2\omega_{12} J_S^- \\ 0 & g\alpha_S^* D_S & \omega_{12} + \gamma_p(1+D_S) & 2g\,|\alpha_S|^2 J_S^+ & -2\omega_{12} J_S^+ \\ -2g\alpha_S J_S^+ & -2g\alpha_S^* J_S^- & -2\omega_{12} J_S^- & -2\omega_{12} J_S^+ & D_{DD} \end{bmatrix}, \tag{10}$$

$$D_{DD} = -4g\,|\alpha_S|^2 (J_S^+ + J_S^-) + 2\omega_{12}(1 - D_S) + 2\omega_{21}(1 + D_S)\,.$$

The stationary-state solution to the Ornstein-Uhlenbeck process [Eq. (8)] is known. In particular, the covariance spectrum in the stationary state is[30]

$$\underline{\mathbf{S}}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \langle \delta\mathbf{x}(t), \delta\mathbf{x}^T(0) \rangle dt = (\underline{\mathbf{A}} + i\omega\underline{\mathbf{I}})^{-1} \mathbf{D} (\underline{\mathbf{A}}^T - i\omega\underline{\mathbf{I}})^{-1}\,. \tag{11}$$

The quadrature phase X_θ of the field in which we seek squeezing is defined by

$$X_\theta = a e^{-i\theta} + a^\dagger e^{i\theta}\,, \tag{12}$$

where the angle θ specifies the particular quadrature of interest. We now consider the squeezing in the continuum of output modes from a cavity with a single input-output mirror, all other mirrors being perfectly reflecting. Such a cavity prevents unwanted vacuum fluctuations from entering and thus gives the best squeezing.[25] We characterize the squeezing by $V(X_\theta)$, the variance of X_θ. A coherent state has $V(X_\theta) = 1$, squeezing corresponds to $V(X_\theta) < 1$ for some θ and $V(X_\theta) = 0$ represents perfect squeezing. These results are also true for the component of squeezing at frequency ω, $V(X_\theta, \omega)$, which is obtained by replacing the covariances in $V(X_\theta)$ by their components at frequency ω,

$$V(X_\theta, \omega) = 1 + 2\gamma_c \{ S_{\alpha\alpha^\dagger}(\omega) + S_{\alpha^\dagger\alpha}(\omega) + [e^{-2i\theta} S_{\alpha\alpha}(\omega) + e^{2i\theta} S_{\alpha^\dagger\alpha^\dagger}(\omega)] \}\,. \tag{13}$$

Choosing θ to maximize the squeezing for given ω we find

$$V_{\max}(X_\theta, \omega) = 1 + 2\gamma_c [S_{\alpha\alpha^\dagger}(\omega) + S_{\alpha^\dagger\alpha}(\omega) - 2\,|S_{\alpha\alpha}(\omega)|\,]\,. \tag{14}$$

It is a straightforward numerical task to calculate the squeezing using these equations. First find the stationary-state means from Eqs. (6) and (7) and evaluate the matrix elements of $\underline{\mathbf{A}}$ and $\underline{\mathbf{D}}$, Eqs. (9) and (10). If $\underline{\mathbf{A}}$ is positive definite the stationary state is stable and we may proceed to use Eq. (11) to find the spectral matrix and hence the squeezing.

IV. ELIMINATION OF THE ATOMIC MEDIUM

The variables in our Langevin equations (8) fall into two sets; the atomic variables J^+, J^-, D and the field variables $\alpha, \alpha^\dagger$. In order to simplify such equations the assumption that one or the other set of variables follows the other and hence may be adiabatically eliminated is commonly made. This approximation may be justified by the larger damping of the eliminated variables. When the atomic variables are damped much more than the field we refer to a high-Q situation and to a low-Q situation when the field is more strongly damped than the atoms.[28,32]

In the Appendix we show that for calculation of the zero-frequency covariance spectral component of an Ornstein-Uhlenbeck process such justification for the elimination of variables is unnecessary. We also examine the conditions under which the spectrum is well approximated after elimination of variables.

Since our interest lies in the squeezing spectrum of the field variables we eliminate the atomic variables. The first two rows of Eqs. (8) are the linearized field equations

$$\begin{aligned}\delta\dot{\alpha} &= -[(\gamma_c - i\phi) + g(J_S^+ - J_S^-)]\delta\alpha \\ &\quad - g\alpha_S(\delta J^+ - \delta J^-) + \Gamma_\alpha , \\ \delta\dot{\alpha}^\dagger &= -[(\gamma_c + i\phi) - g(J_S^+ - J_S^-)]\delta\alpha^\dagger \\ &\quad + g\alpha_S^*(\delta J^+ - \delta J^-) + \Gamma_{\alpha^\dagger} .\end{aligned} \tag{15}$$

Setting the left-hand sides of the remaining equations (8) to zero and solving we find

$$\begin{aligned}\delta J^+ - \delta J^- = {} & [\gamma_\parallel(\gamma_\perp^2 + \Delta^2) + 4\gamma_\perp g^2 |\alpha_S|^4]^{-1}\{2ig\Delta[2(J_S^- + J_S^+)g|\alpha_S|^2 - D_S\gamma_\parallel](\alpha_S\delta\alpha^\dagger + \alpha_S^*\delta\alpha) \\ & - i\Delta\gamma_\parallel(\Gamma_{J^-} + \Gamma_{J^+}) - (\gamma_\perp\gamma_\parallel + 4g^2|\alpha_S|^4)(\Gamma_{J^-} - \Gamma_{J^+}) - 2i\Delta g|\alpha_S|^2\Gamma_D\} .\end{aligned} \tag{16}$$

Substitution of this result into Eqs. (15) yields field equations from which the zero-frequency squeezing may be found using Eq. (13) with $\omega = 0$. However, we restrict our attention to two simple limits, the case of zero two-photon detuning, $\Delta = 0$, and the dispersive case of large two-photon detuning.

In the case $\Delta = 0$ the field equations (15) reduce to

$$\begin{aligned}\delta\dot{\alpha} &= -(\gamma_c - i\phi)\delta\alpha + \Gamma'_\alpha , \\ \delta\dot{\alpha}^\dagger &= -(\gamma_c + i\phi)\delta\alpha^\dagger + \Gamma'_{\alpha^\dagger} , \\ \Gamma'_\alpha &= g\alpha_S\gamma_\perp^{-1}(\Gamma_{J^-} - \Gamma_{J^+}) + \Gamma_\alpha , \\ \Gamma'_{\alpha^\dagger} &= -g\alpha_S^*\gamma_\perp^{-1}(\Gamma_{J^-} - \Gamma_{J^+}) + \Gamma_{\alpha^\dagger} .\end{aligned} \tag{17}$$

Using Eqs. (10) we find the correlation matrix elements for these equations to be

$$\begin{aligned}D_{\alpha\alpha} &= -z\alpha_S^2, \quad D_{\alpha\alpha^\dagger} = z|\alpha_S|^2 , \\ z &= -2g^2\gamma_\perp^{-1}\{D_S + \gamma_\perp^{-1}[g|\alpha_S|^2(J_S^- + J_S^+) \\ &\quad - \omega_{12} - \gamma_p(1 + D_S)]\} .\end{aligned} \tag{18}$$

The maximum zero-frequency squeezing for this system is found using Eq. (14) to be zero. It may be shown that in the limit of eliminated atomic variables no squeezing is found for any frequency when $\Delta = 0$.

Next we consider the case of large two-photon detuning. Specifically we assume that the two-photon detuning is large compared to $\gamma_\perp$ and that the atom is unsaturated,

$$|\Delta/\gamma_\perp| \gg 1, \quad 1 \gg X/|\Delta'| = |\alpha_S|^2/(n_0|\Delta'|) . \tag{19}$$

The stationary-state means found using Eqs. (6) and (7) are

$$\begin{aligned}& D_S \approx D_0, \quad J_S^- \approx i\Delta^{-1}D_S g|\alpha_S|^2 , \\ & \alpha_S \approx E[\gamma_c - i\phi - ig(\gamma_\parallel/\gamma_\perp)^{1/2}D_S X/\Delta']^{-1} .\end{aligned} \tag{20}$$

In the dispersive limit, Eqs. (19), we find for the linearized field equations

$$\begin{aligned}\delta\dot{\alpha} &= -(\gamma_c - i\phi + 2w|\alpha_S|^2)\delta\alpha - w\alpha_S^2\delta\alpha^\dagger + \Gamma'_\alpha , \\ \delta\dot{\alpha}^\dagger &= -(\gamma_c + i\phi - 2w|\alpha_S|^2)\delta\alpha^\dagger + w\alpha_S^{*2}\delta\alpha + \Gamma'_{\alpha^\dagger} , \\ \Gamma'_\alpha &= i\Delta^{-1}g\alpha_S(\Gamma_{J^-} + \Gamma_{J^+}) \\ &\quad + 2i\gamma_\parallel^{-1}\Delta^{-1}g^2\alpha_S|\alpha_S|^2\Gamma_D + \Gamma_\alpha , \\ \Gamma'_{\alpha^\dagger} &= -i\Delta^{-1}g\alpha_S^*(\Gamma_{J^-} + \Gamma_{J^+}) \\ &\quad - 2i\gamma_\parallel^{-1}\Delta^{-1}g^2\alpha_S^*|\alpha_S|^2\Gamma_D + \Gamma_{\alpha^\dagger} , \\ w &= -2ig^2\Delta^{-1}D_S .\end{aligned} \tag{21}$$

Using Eqs. (10) the correlation matrix elements for these equations are

$$D_{\alpha\alpha} = -w\alpha_S^2, \quad D_{\alpha\alpha^\dagger} = 0 . \tag{22}$$

Thus the dispersive-limit Langevin equations (21) are precisely those for the nonlinear polarizability model of dispersive optical bistability.[24] Collett and Walls[25] have presented the squeezing spectrum in the quadrature phase chosen for optimal squeezing at $\omega = 0$,

$$\begin{aligned}V(X_\theta, \omega) &= 1 + \frac{4\gamma_c\epsilon(2\gamma_c\epsilon - f)}{(\gamma_c^2 + \omega^2 - \lambda)^2 + 4\gamma_c^2\lambda} , \\ f &= \frac{(\gamma_c^2 + \omega^2 - \lambda)(\gamma_c^2 - \lambda) + 4\gamma_c^2(\lambda + \epsilon^2)}{[(\gamma_c^2 - \lambda)^2 + 4\gamma_c^2(\lambda + \epsilon^2)]^{1/2}} , \\ \lambda &= (2\epsilon + \phi)^2 - \epsilon^2, \quad \epsilon = -iw|\alpha_S|^2 .\end{aligned} \tag{23}$$

For $\omega=0$ this result applies to our system in the dispersive limit. According to the Appendix the elimination of the equations for $\delta J^-,\delta J^+$ would leave the field spectrum valid for $\omega/|\Delta| \ll 1$. Due to the dispersive-limit condition $X \ll |\Delta'|$ and since the field equations do not directly couple to the inversion, the elimination of the δD equation from the system (8) also leaves the low-frequency spectrum unchanged. Compare Figs. 3(a) and 3(b), which are further discussed below.

Perfect squeezing in one quadrature phase X_θ is associated by the uncertainty principle with infinite uncertainty in the conjugate quadrature phase $X_{\theta+\pi/2}$. From Eq. (11) for the covariance spectrum we see that the infinite fluctuations at frequency ω will occur when the matrix $\underline{\mathbf{A}}+i\omega\underline{\mathbf{I}}$ is singular. Such situations are candidates for the occurrence of good squeezing. In particular, perfect zero-frequency squeezing may occur when the drift matrix $\underline{\mathbf{A}}$ is singular. Such critical points are found at the turning points for optical bistability. The critical points of Eqs. (21) occur for $\gamma_c^2+\lambda=0$ and then the squeezing spectrum equation (23) reduces to[25]

$$V(X_\theta,\omega)=1-\frac{4\gamma_c^2}{4\gamma_c^2+\omega^2}\ , \tag{24}$$

which gives perfect zero-frequency squeezing $V(X_\theta,0)=0$.

We can use the full linearized model without elimination of the atomic variables, Eqs. (8), to verify our previous statements concerning the validity of the squeezing spectrum in the dispersive limit after elimination of the atomic variables. Figures 2(a) and 2(b) are, respectively, the cavity amplitude $|\alpha_S|$ and the zero-frequency component of the squeezing plotted against the driving field amplitude E for a low-Q situation, $\gamma_\perp/\gamma_c=0.1$. Figure 3(a) is the squeezing spectrum numerically calculated using the full model while Fig. 3(b) is a plot of Eq. (23) which was obtained after elimination of the atomic variables. The parameter values correspond to the dispersive limit, Eqs. (19), and the quadrature phase has been chosen for optimal squeezing at $\omega=0$. It is apparent that they agree well for $\omega/\gamma_c \ll \Delta/\gamma_c$.

The full model has a feature at $\omega=\Delta$. The size of this feature decreases as $\gamma_\perp$ increases and it can be understood as a resonant effect of the fluctuations with the atomic oscillations.

We have also used the full model to investigate how squeezing behaves as we move from the dispersive limit into the absorptive regime. The sequence of Figs. 4, 5, and 6 is for a high-Q situation, $\gamma_\perp/\gamma_c=10$, and the ratio $\Delta/\gamma_\perp$ is, respectively, 100, 10, and 1. A marked decrease in squeezing at the turning point of the lower branch occurs as $\Delta/\gamma_\perp$ decreases. However, even for $\Delta/\gamma_\perp=1$ the squeezing near the turning point of the upper branch is good. Note that as $\Delta/\gamma_\perp$ decreases part of the upper branch becomes unstable and in fact becomes inaccessible for $\Delta/\gamma_\perp$ too low.

To summarize the results of this section we can expect to find good zero-frequency squeezing at the turning points of dispersive optical bistability. This is true regardless of the relative time scales of relaxation in the field and in the atomic variables, that is, for all cases from low Q, Fig. 2, through to high Q, Fig. 4.

V. THE LADDER MEDIUM

In this section we apply the preceding analysis to an effective two-photon interaction of a cavity field mode with N three-level atoms in a ladder configuration, Fig. 7. The Hamiltonian for this system is

$$H=H_F+H_E+H_I'+H_D\ , \tag{25}$$

where H_F,H_E,H_D are defined in Eq. (1) and the new interaction Hamiltonian is[33]

$$H_I'=i\hbar g'\sum_{\mu=1}^{N}(a^{\dagger 2}\sigma_\mu^- e^{-i\mathbf{k}\cdot\mathbf{r}_\mu}-a^2\sigma_\mu^+ e^{i\mathbf{k}\cdot\mathbf{r}_\mu})\ , \tag{26}$$

where $\mathbf{k}$ is the field wave vector and $\mathbf{r}_\mu$ the position vector of the μth atom. From this Hamiltonian the resulting Langevin equations are[33]

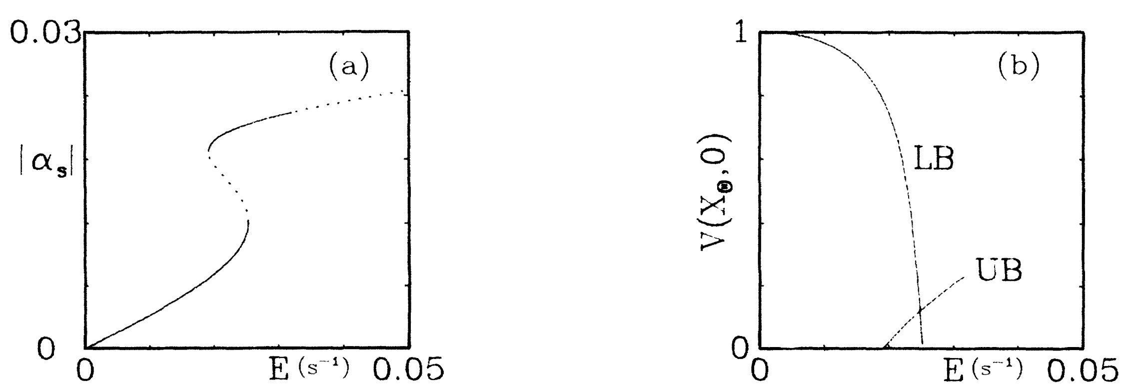

FIG. 2. (a) State equation (7) for the Λ system in the dispersive limit. Dashed line indicates instability. Parameter values: $\Delta/\gamma_\perp=-100$, $\phi/\gamma_c=3$, $\gamma_\perp/\gamma_c=\gamma_\parallel/\gamma_c=0.1$, $g=200$, $D_0=1$. (b) Squeezing $V(X_\theta,0)$ vs driving amplitude E for upper branch (UB) and lower branch (LB) of state equation of (a).

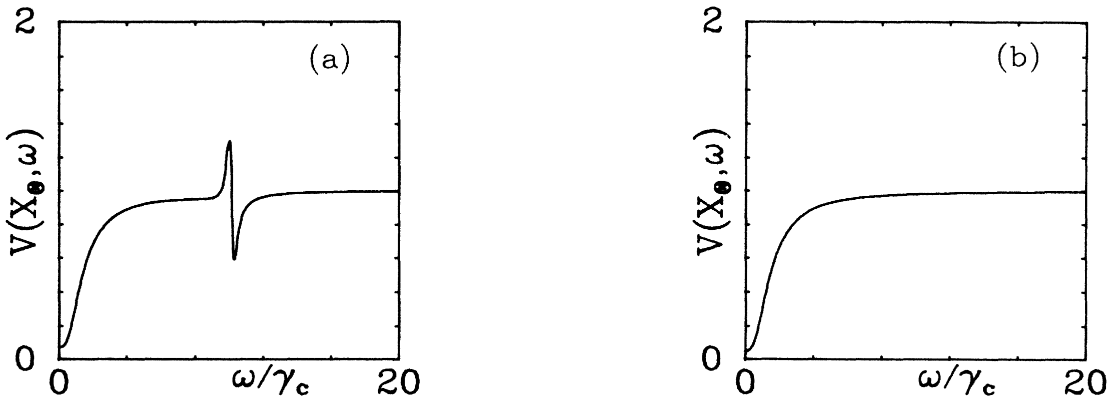

FIG. 3. (a) Squeezing spectrum $V(X_\theta,\omega)$ for the Λ system calculated numerically without adiabatic elimination of the atoms. Parameters as for Fig. 2(a) with $E=0.025$ on the lower branch, ω in units of γ_c, note $\Delta/\gamma_c=10$. (b) $V(X_\theta,\omega)$ calculated from Eq. (23). Parameters as for (a).

$$\begin{aligned}
\dot{\alpha}&=E-(\gamma_c-i\phi)\alpha+2g\alpha^\dagger J^-+\Gamma_\alpha ,\\
\dot{\alpha}^\dagger&=E-(\gamma_c+i\phi)\alpha^\dagger+2g\alpha J^+ +\Gamma_{\alpha^\dagger} ,\\
\dot{J}^-&=-(\gamma_\perp-i\Delta)J^-+g\alpha^2D+\Gamma_{J^-} ,\\
\dot{J}^+&=-(\gamma_\perp+i\Delta)J^-+g(\alpha^\dagger)^2D+\Gamma_{J^+} ,\\
\dot{D}&=-\gamma_{\parallel}(D+1)-2g[J^-(\alpha^\dagger)^2+J^+\alpha^2]+\Gamma_D .
\end{aligned} \tag{27}$$

We have set the pumping rate $\omega_{12}=0$ so that $\gamma_{\parallel}=\omega_{21}$, and the two-photon detuning $\Delta=2\omega_E-\omega_a$. The other parameters are as defined in Sec. II. Note that the population difference between levels 1 and 2 is simply maintained by spontaneous emission, so we are dealing with the case of two-photon optical bistability.[26,31,34,35] Comparing the equations for $\alpha,\alpha^\dagger$ with those for the Λ system, Eqs. (2), and momentarily ignoring their stochastic nature we see that in the Λ case the field is coupled to the imaginary quantity J^+-J^- while in the present case the field conjugate couples to J^-. Thus the ladder case includes the possibility of atomic loss which can be expected to degrade the squeezing.

The nonzero noise correlations are

$$\begin{aligned}
\langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle&=2gJ^-\delta(t-t') ,\\
\langle\Gamma_{J^-}(t)\Gamma_{J^-}(t')\rangle&=2gJ^-\alpha^2\delta(t-t') ,\\
\langle\Gamma_{J^-}(t)\Gamma_{J^+}(t')\rangle&=\gamma_p(1+D)\delta(t-t') ,\\
\langle\Gamma_D(t)\Gamma_D(t')\rangle&=\{\gamma_{\parallel}(1+D)\\
&\quad-2g[J^-(\alpha^\dagger)^2+J^+\alpha^2]\}\delta(t-t') ,
\end{aligned} \tag{28}$$

and their conjugates. From Eqs. (27) we find the stationary-state means to be

$$\begin{aligned}
D_S&=-\{1+X^2/[1+(\Delta')^2]\}^{-1} ,\\
J_S^-&=gD_S\alpha^2\gamma_\perp^{-1}(1+i\Delta')/[1+(\Delta')^2] ,\\
\alpha_S&=E\left[\gamma_c-i\phi+\tfrac{1}{2}\gamma_{\parallel}(1+i\Delta')/[1+(\Delta')^2]\right.\\
&\qquad\left.\times\frac{X}{1+X^2/[1+(\Delta')^2]}\right]^{-1} .
\end{aligned} \tag{29}$$

Since we only expect good squeezing when loss effects are small compared to dispersive effects we confine our attention to the dispersive case of large two-photon detuning.

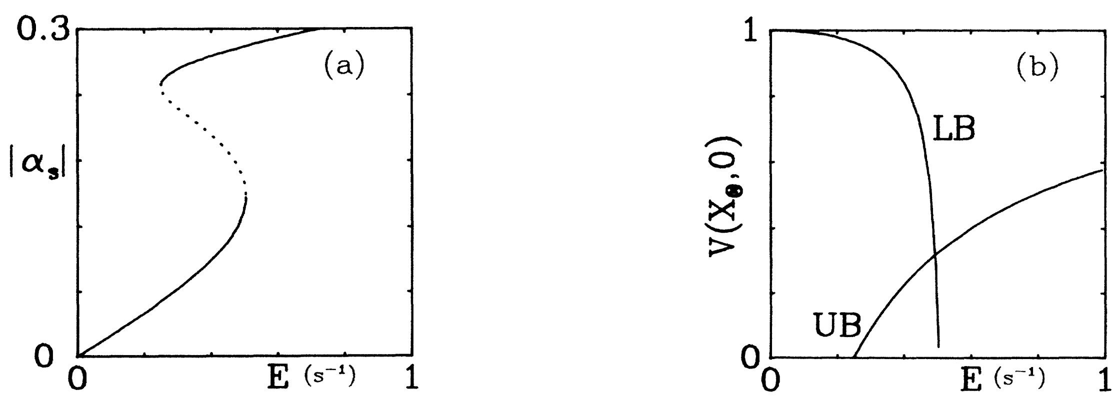

FIG. 4. (a) State equation (7) for the Λ system. Parameter values: $\Delta/\gamma_\perp=-100$, $\phi/\gamma_c=5$, $\gamma_\perp/\gamma_c=\gamma_{\parallel}/\gamma_c=10$, $g=200$, $D_0=1$. (b) $V(X_\theta,0)$ vs E for parameters of (a).

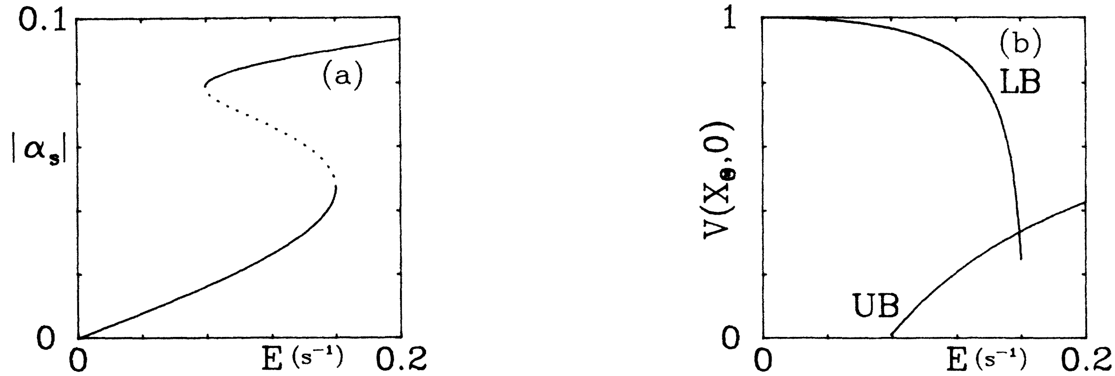

FIG. 5. (a) State equation (7) for the Λ system. $\Delta/\gamma_\perp = 10$, other parameters as for Fig. 4(a). (b) $V(X_\theta,0)$ vs E for parameters of (a).

When the dispersive conditions [Eqs. (19)] hold we find the stationary-state means to be

$$D_S \approx -1,\quad J_S^- \approx -i\Delta^{-1} g\alpha^2 ,$$
$$\alpha_S \approx E[\gamma_c - i\phi + ig(\gamma_\parallel/\gamma_\perp)^{1/2} X/\Delta']^{-1} . \tag{30}$$

Comparing with the Λ-system dispersive-limit result [Eqs. (20)] shows that the field equation is identical in this limit.

After linearization of Eqs. (27) and elimination of the atomic variables we obtain in the dispersive limit the Langevin equations

$$\delta\dot{\alpha} = -(\gamma_c - i\phi + 2w\,|\alpha_S|^2)\delta\alpha - w\alpha_S^2\delta\alpha^\dagger + \Gamma''_\alpha ,$$
$$\delta\dot{\alpha}^\dagger = -(\gamma_c + i\phi - 2w\,|\alpha_S|^2)\delta\alpha^\dagger + w\alpha_S^{*2}\delta\alpha + \Gamma''_{\alpha^\dagger} ,$$
$$\Gamma''_\alpha = 2i\Delta^{-1} g\alpha_S^*(\Gamma_{J_-} + 2i\gamma_\parallel^{-1}\Delta^{-1} g^2\alpha_S^4\Gamma_{J_+}) + 2i\gamma_\parallel^{-1}\Delta^{-1} g^2\alpha_S\,|\alpha_S|^2\Gamma_D + \Gamma_\alpha , \tag{31}$$
$$\Gamma''_{\alpha^\dagger} = -2i\Delta^{-1} g\alpha_S(\Gamma_{J_+} - 2i\gamma_\parallel^{-1}\Delta^{-1} g^2\alpha_S^{*4}\Gamma_{J_-}) - 2i\gamma_\parallel^{-1}\Delta^{-1} g^2\alpha_S^*\,|\alpha_S|^2\Gamma_D + \Gamma_{\alpha^\dagger} ,$$
$$w = 2i\Delta^{-1} g^2 .$$

Using Eqs. (28) the correlation matrix elements for these equations are

$$D_{\alpha\alpha} = -w\alpha_S^2,\quad D_{\alpha\alpha^\dagger} = 0 . \tag{32}$$

Comparison with Eqs. (21) and (22) for the Λ case, taking $D_S \approx -1$, shows the two systems to be identical in this limit.

Thus in the dispersive limit of large two-photon detunings the ladder medium will also display perfect squeezing at the turning points for optical bistability. Figures 8(a) and 8(b) are plots of the state equation and squeezing from the ladder medium versus the driving field E for parameters corresponding to the Λ case of Fig. 4. While the graphs of the state equations are essentially identical the squeezing on the upper branch for the ladder case is significantly less than that in the Λ case, even for $\Delta/\gamma_\perp = 100$. As the detuning is decreased and the absorptive regime approached (Fig. 9) the squeezing disappears much more rapidly than for the Λ medium (Fig. 5). This is due to the atomic loss becoming significant. Clearly the squeezing on the upper branch has been more seriously degraded than on the lower branch, the loss effects being more significant for larger $|\alpha_S|$.

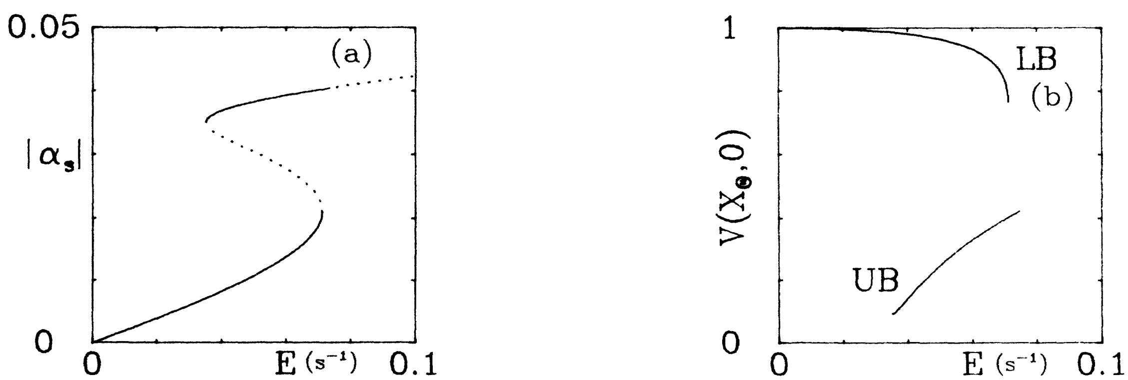

FIG. 6. (a) State equation (7) for the Λ system. $\Delta/\gamma_\perp = 1$, other parameters as for Fig. 4(a). (b) $V(X_\theta,0)$ vs E for parameters of (a).

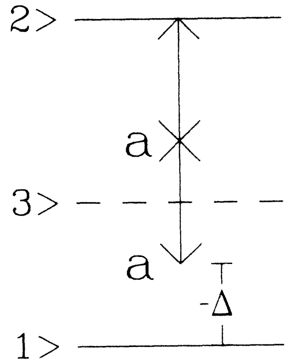

FIG. 7. The three-level ladder atom.

VI. DISCUSSION

We now summarize the results of our work and compare them with the work of Reid and Walls[7] who used a two-level atomic medium and a one-photon transition to generate optical bistability. In the dispersive regime of large two-photon detuning our linearized models predict perfect squeezing at the turning points of optical bistability. The parameter values required to achieve dispersive optical bistability via this transition appear to be achievable. For example, a system with upper-level lifetime $\approx 10^{-7}$ s, one-photon detuning $\Delta_1 \approx 10^{11}$ s^{-1}, mode volume ≈ 1 cm^3, and $N \approx 10^{14}$ implies our $g \approx 200$ s^{-1}. For the parameters of Fig. 2(a) the turning points occur for $E \approx 0.02$, so using Eq. (3) for a cavity with round-trip time 10^{-9} s we find $E_{\rm expt} \approx 10$ W cm^{-2}.

To understand the physical origin of the different squeezing behaviors of the one-photon[7] and two-photon transition models of dispersive optical bistability we first examine the one-photon Bloch equations[28]

$$\begin{aligned} \dot{\alpha} &= E - (\gamma_c - i\phi)\alpha + gJ^- , \\ \dot{J}^- &= -(\gamma_\perp - i\Delta)J^- + g\alpha D , \\ \dot{D} &= -\gamma_{\parallel}(D - D_0) - 2g(\alpha J^+ + \alpha^* J^-) , \end{aligned} \tag{33}$$

and their conjugates. The steady-state solutions of these equations are

$$\begin{aligned} D_S &= D_0 \left[1 + \frac{X}{1+(\Delta')^2} \right]^{-1} , \\ J_S^- &= g D_S \alpha_S \gamma_\perp^{-1} \frac{1+i\Delta'}{1+(\Delta')^2} , \\ \alpha_S &= E \left\{ \gamma_c - i\phi - g^2 \gamma_\perp^{-1} D_0 [1+(\Delta')^2]^{-1} \right. \\ &\quad \left. \times \frac{1+i\Delta'}{1+X/[1+(\Delta')^2]} \right\}^{-1} . \end{aligned} \tag{34}$$

Comparing these steady states with those for the two-photon ladder Eq. (29) we note that the only source of nonlinearity in the one-photon case is through the generation of inversion D_S. Thus bistability cannot be attained in the dispersive limit for which the inversion is negligible and the nonlinear polarizability limit hence holds.[7,28] However, in the two-photon model there is an extra source of nonlinearity which does not require the generation of significant inversion. Thus bistability can be obtained while satisfying the conditions for the nonlinear polarizability model to hold.

In conclusion the linearized models we have used predict perfect squeezing at the turning points of dispersive optical bistability generated by two-photon transitions in either the ladder or Λ atomic systems. Hence the experimenter need only find dispersive bistability to have established a system capable of good squeezing. The particular advantage of the Λ system is that the atomic loss is absent and hence it is much less dependent on large two-photon detunings for the generation of good squeezing.

ACKNOWLEDGMENTS

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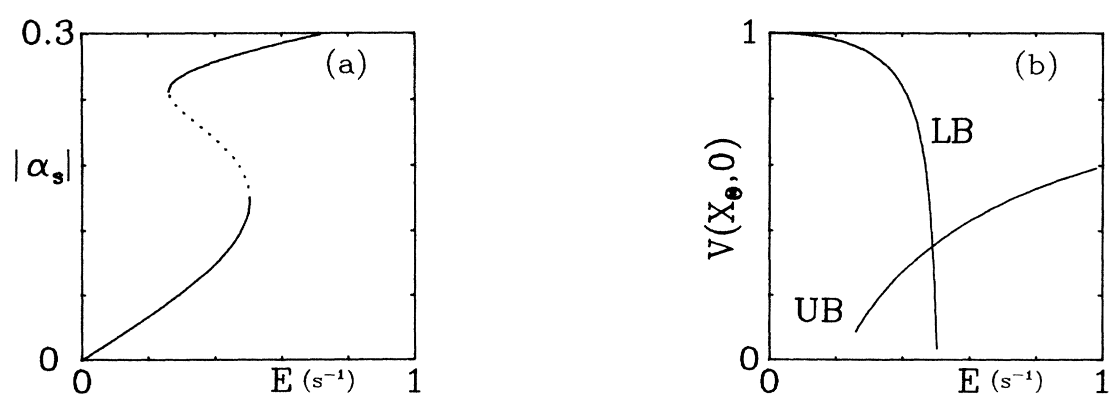
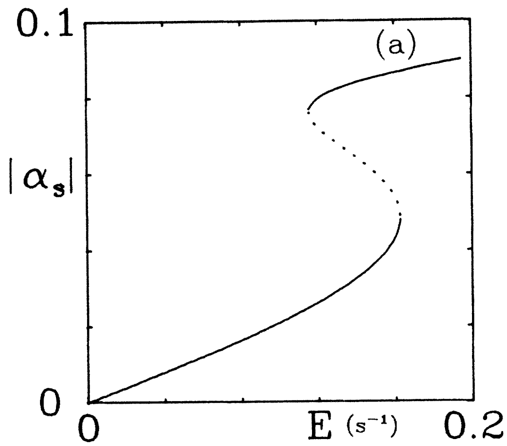


FIG. 8. (a) State equation for the ladder system. Parameters as for Fig. 4(a), except $\Delta/\gamma_\perp = 100$, $D_0 = -1$. (b) $V(X_\theta,0)$ vs E for the ladder system. Parameters as for (a).



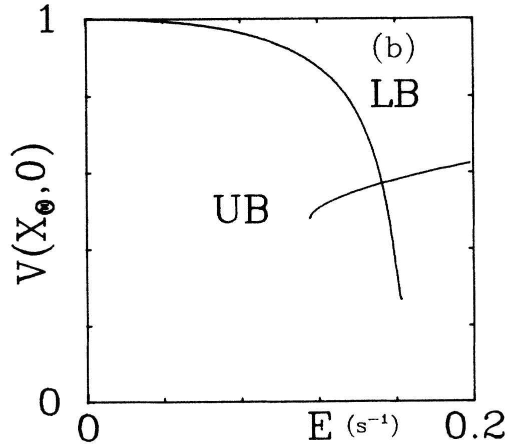


FIG. 9. (a) State equation for the ladder system. Parameters as for Fig. 5(a), except $\Delta/\gamma_\perp=10$, $D_0=-1$. (b) $V(X_\theta,0)$ vs E for the ladder system. Parameters as for (a).

APPENDIX: STATIONARY-STATE VARIABLE ELIMINATION

Consider the Langevin equations for a multivariate Ornstein-Uhlenbeck process

$$\dot{\mathbf{x}}=-\underline{\mathbf{A}}\mathbf{x}+\underline{\mathbf{B}}\mathbf{\Gamma}\ , \tag{A1}$$

where $\mathbf{\Gamma}$ is a vector of independent delta-correlated Gaussian random processes. The spectral matrix of the stationary-state covariances of the variables $\mathbf{x}$ is

$$\underline{\mathbf{S}}(\omega)=(\underline{\mathbf{A}}+i\omega\underline{\mathbf{I}})^{-1}\underline{\mathbf{B}}\underline{\mathbf{B}}^T(\underline{\mathbf{A}}^T-i\omega\underline{\mathbf{I}})^{-1}\ . \tag{A2}$$

This spectrum may be obtained by naively solving the equations

$$i\omega\tilde{\mathbf{x}}_1=-\underline{\mathbf{A}}\tilde{\mathbf{x}}_1+\underline{\mathbf{B}}\tilde{\mathbf{\Gamma}},\quad -i\omega\tilde{\mathbf{x}}_2=-\underline{\mathbf{A}}\tilde{\mathbf{x}}_2+\underline{\mathbf{B}}\tilde{\mathbf{\Gamma}}\ , \tag{A3}$$

the tildes indicating Fourier transforms. Now

$$\tilde{\mathbf{x}}_1=(\underline{\mathbf{A}}+i\omega\underline{\mathbf{I}})^{-1}\underline{\mathbf{B}}\tilde{\mathbf{\Gamma}},\quad \tilde{\mathbf{x}}_2=(\underline{\mathbf{A}}-i\omega\underline{\mathbf{I}})^{-1}\underline{\mathbf{B}}\tilde{\mathbf{\Gamma}}\ . \tag{A4}$$

Taking the expectation value of the product $\tilde{\mathbf{x}}_1\tilde{\mathbf{x}}_2^T$ we find

$$\langle\tilde{\mathbf{x}}_1\tilde{\mathbf{x}}_2^T\rangle=(\underline{\mathbf{A}}+i\omega\underline{\mathbf{I}})^{-1}\underline{\mathbf{B}}\underline{\mathbf{B}}^T(\underline{\mathbf{A}}^T-i\omega\underline{\mathbf{I}})^{-1}\ , \tag{A5}$$

which is just the spectrum Eq. (A2).

The naive procedure for eliminating variables from Eq. (A1), usually referred to as adiabatic elimination, sets the left-hand sides of the relevant rows of Eq. (A1) to zero. These rows are then naively solved for the relevant set of variables, treating the noise terms as inhomogeneities, which are substituted into the uneliminated rows.

The implication of this procedure for the Eqs. (A3) is that the left-hand sides of the rows corresponding to the eliminated variables are set to zero. For the case $\omega=0$ this is of no consequence as the left-hand sides are already zero. Thus the zero-frequency component of the stationary-state spectrum is unchanged by the elimination of variables whose spectrum is not of interest.

When the left-hand sides of the eliminated rows of Eqs. (A3) are small compared to the right-hand sides the elimination procedure will also yield a good approximation to the spectrum $\mathbf{S}(\omega)$. Fixing our attention on a particular variable to be eliminated, it may be that on the left-hand side of its row it is multiplied by a complex quantity γ. Roughly speaking we can expect the spectrum to be well approximated after the elimination of this variable for frequencies satisfying $|\omega|\ll|\gamma|$.

As an example we apply these results to the elimination of the atomic coherence variables from the linearized system Eq. (8). The spectrum can be expected to be valid after the elimination for frequencies satisfying $|\omega|\ll|\gamma_\perp+i\Delta|$ or equivalently

$$|\omega/\gamma_c|\ll\frac{\gamma_\perp}{\gamma_c}|1+i\Delta/\gamma_\perp|\ . \tag{A6}$$

ω/γ_c is the frequency in units of the cavity linewidth and $\Delta/\gamma_\perp$ is the two-photon detuning in units of the atomic linewidth. For small detuning $|\Delta/\gamma_\perp|<1$ this result reduces to $|\omega/\gamma_c|\ll\gamma_\perp/\gamma_c$, while for large detunings it becomes $|\omega/\gamma_c|\ll\gamma_\perp/\gamma_c|\Delta/\gamma_\perp|$.

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