Quantum fluctuations in absorptive bistability without adiabatic elimination

H. J. Carmichael
Department of Physics, University of Arkansas, Fayetteville, Arkansas 72701
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A linearized theory of fluctuations for absorptive bistability based on the positive P representation is developed without adiabatic elimination of the atoms or the field. An analytic expression for the steady-state covariance matrix is derived from which the size of quantum-statistical effects can be estimated without restriction to the good- or the bad-cavity limit. When the atom and field relaxation rates are similar the intensity correlation function of the transmitted light exhibits an oscillatory relaxation associated with vacuum Rabi splitting.

I. INTRODUCTION

The extensive literature on fluctuations in absorptive bistability is restricted almost exclusively to treatments in the good-cavity and bad-cavity limits.[1] These limits simplify the analysis by allowing for the adiabatic elimination of the atoms or the field, thus reducing the dimensions of the mathematical description. Experiments may not find these limits so convenient, however. For example, recent experiments on absorptive bistability using optically prepumped sodium atomic beams have an atomic decay rate just two or three times faster than the cavity decay rate.[2] These experiments achieve good quantitative agreement with the theory for homogeneously broadened two-level atoms, and it now seems feasible to move to measurements of quantum-statistical effects.[1,3-8] Such measurements will require adherence to very restrictive experimental design. Consider photon antibunching as an example.[5-7] Atomic lifetimes are short in the optical regime and the most manageable time scales are then found in the good-cavity limit. However, the predicted effect is very small in a large system. This calls for a small-cavity design where a decay rate only slightly slower than the atomic decay rate is all that can reasonably be achieved. Since design for the smallness of the effect is so critical, factors of two or three in estimating its size are important. It is not sufficient to merely observe that the effect varies inversely as the saturation photon number n_s or the number of interacting atoms N. These are related by $n_s = N/4C\mu$, where C is the bistability parameter and μ is the ratio of cavity and atomic linewidths. They can differ by orders of magnitude and it is necessary to have all of the factors of C and μ in place for an accurate estimate. Existing theories cannot provide this precision for $\mu \sim 1$. These considerations have motivated the present work in which I develop a linearized quantum-statistical theory of absorptive bistability without adiabatically eliminating the atoms or the field.

Aside from providing quantitative precision between the good-cavity and bad-cavity limits, my general treatment reveals one notable new feature which is missed in both of these limits. When the atomic and cavity decay rates are similar the relaxation of fluctuations can be oscillatory for arbitrarily small intensities, and exact resonance of the driving field, cavity, and atoms. These oscillations are displayed in the intensity correlation function of the transmitted light and will give rise to a doublet in the incoherent component of the transmitted spectrum. They arise from the so-called vacuum Rabi splitting,[9,10] where the degenerate first excited state of the composite system of atoms and cavity mode is split by the atom-field interaction. There are no Rabi oscillations in the population inversion, but the normal modes of the coupled field and atomic polarization are moved from resonance with the driving field; hence the oscillations in field characteristics. I give a novel treatment of this effect in terms of a coupled harmonic oscillator model derived using the Schwinger representation. This treatment demonstrates the role played by atomic and cavity decay in vacuum Rabi splitting for the first time.[11]

In the following section I briefly review the model for absorptive bistability and the methods of the positive P representation which are used to obtain a quantum-statistical formulation in terms of a linearized Fokker-Planck equation. In Secs. III and IV I solve for the steady-state covariance matrix and find expressions for the ratio of incoherent and coherent intensities, the second-order correlation function, and the variance of fluctuations in the field quadratures for the transmitted light. Section V discusses the oscillation associated with vacuum Rabi splitting and the coupled oscillator model for this effect. Section VI provides a summary and conclusions.

II. MODEL AND LINEARIZED THEORY OF FLUCTUATIONS

I consider a collection of N homogeneously broadened two-level atoms interacting on resonance with a single quantized ring-cavity mode

$$\mathbf{E}(z,t) = i\hat{\mathbf{e}}(\hbar\omega_0/2\epsilon_0 V_Q)^{1/2}[a(t)e^{ik_0 z} - a^{\dagger}(t)e^{-ik_0 z}] , \tag{2.1}$$

where $a^{\dagger}$ and a are creation and annihilation operators for cavity photons, ω_0 is the resonant frequency,

$k_0=\omega_0/c$, $\hat{\mathbf{e}}$ is a polarization vector, V_Q is the quantization volume, and ϵ_0 is the vacuum permittivity (SI units are used); the atoms are described by pseudospin operators $\sigma_\pm^j$, σ_z^j, $j=1,\ldots,N$, for each atom, and collective atomic operators

$$J_\pm=\sum_{j=1}^{N} e^{\pm ik_0 z_j}\sigma_\pm^j , \tag{2.2a}$$

$$J_z=\sum_{j=1}^{N}\sigma_z^j , \tag{2.2b}$$

with commutation relations $[\sigma_+^j,\sigma_-^k]=2\sigma_z^j\delta_{jk}$ and $[\sigma_\pm^j,\sigma_z^k]=\mp\sigma_\pm^j\delta_{jk}$. The cavity mode is resonantly excited by the incident field

$$\mathbf{E}_i(z,t)=\hat{\mathbf{e}}\mathscr{E}_i e^{-i(\omega_0 t-k_0 z)}+\text{c.c.} , \tag{2.3}$$

where $\mathscr{E}_i$ is a complex amplitude. Each atom is radiatively damped by spontaneous emission to modes other than the privileged cavity mode and the cavity field is damped by losses at partially reflecting mirrors. Then the system of atoms plus cavity mode is described by the master equation

$$\begin{aligned}\frac{d\rho}{dt}&=\mathscr{E}[a^\dagger-a,\rho]+g[a^\dagger J_--aJ_+,\rho]\\&\quad+\tfrac{1}{2}\gamma\sum_{j=1}^{N}(2\sigma_-^j\rho\sigma_+^j-\sigma_+^j\sigma_-^j\rho-\rho\sigma_+^j\sigma_-^j)\\&\quad+\kappa(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a) ,\end{aligned} \tag{2.4}$$

where ρ is the density operator in a frame rotating at the frequency ω_0, γ is the atomic decay rate, κ is the decay rate for the cavity field,

$$g=(\omega_0\mu^2/2\hbar\epsilon_0 V_Q)^{1/2} \tag{2.5}$$

is the atom-field coupling constant, with $\mu=\hat{\mathbf{e}}\cdot\boldsymbol{\mu}$ the atomic dipole moment, and

$$\mathscr{E}=-i(2\epsilon_0 V_Q/\hbar\omega_0)^{1/2}e^{i\phi_T}(\sqrt{T}\mathscr{F}/\pi)\mathscr{E}_i \tag{2.6}$$

is a real driving field amplitude, where $\mathscr{F}$ is the cavity finesse, T is the input mirror transmission coefficient, and ϕ_T is the phase change on transmission at the input mirror.

I use the positive P representation,[12] in which Eq. (2.4) is converted to a Fokker-Planck equation, and then to equivalent Ito stochastic differential equations:[6,13]

$$d\alpha=\tfrac{1}{2}\mu(-\alpha+2Cv+Y)d\tau , \tag{2.7a}$$

$$d\alpha_*=\tfrac{1}{2}\mu(-\alpha_*+2Cv_*+Y)d\tau , \tag{2.7b}$$

$$dv=\tfrac{1}{2}(-v+\alpha m)d\tau+N^{-1/2}(\alpha v)^{1/2}dW_1 , \tag{2.7c}$$

$$dv_*=\tfrac{1}{2}(-v_*+\alpha_* m)d\tau+N^{-1/2}(\alpha_* v_*)^{1/2}dW_2 , \tag{2.7d}$$

$$\begin{aligned}dm&=(-m-1-\tfrac{1}{2}\alpha v_*-\tfrac{1}{2}\alpha_* v)d\tau\\&\quad+(N/2)^{-1/2}(m+1-\tfrac{1}{2}\alpha v_*-\tfrac{1}{2}\alpha_* v)^{1/2}dW_3 ,\end{aligned} \tag{2.7e}$$

where dW_1, dW_2, and dW_3 are independent Wiener processes;

$$\tau=\gamma t,\ \ \mu=2\kappa/\gamma,\ \ C=Ng^2/\kappa\gamma , \tag{2.8a}$$

$$Y=n_s^{-1/2}(\mathscr{E}/\kappa),\ \text{with}\ \ n_s=\gamma^2/8g^2 ; \tag{2.8b}$$

and complex variables $(\alpha,\alpha_*,v,v_*,m)$ lie in one-to-one correspondence with system operators, such that normally ordered averages in the rotating frame are given by

$$\begin{aligned}&\langle a^{\dagger n}a^m J_+^p J_z^r J_-^q\rangle\\&\quad=n_s^{(n+m)/2}(N/\sqrt{2})^{p+q}(N/2)^r(\alpha_*^n\alpha^m v_*^p m^r v^q)_{\text{av}} ,\end{aligned} \tag{2.9}$$

where $(\ )_{\text{av}}$ denotes an ensemble average over the ten-dimensional stochastic process defined by Eqs. (2.7). The more familiar methods which derive such a quantum-stochastic formulation on the basis of the Glauber-Sudarshan-Haken representation[14] obtain Eqs. (2.7) with $\alpha_*=\alpha^*$, $v_*=v^*$, and m real. However, this symmetry is not preserved. First, since dW_1 and dW_2 are independent the noise terms in Eqs. (2.7c) and (2.7d) are not complex conjugate. Also, $m+1-\frac{1}{2}\alpha v^*-\frac{1}{2}\alpha^* v$ can be negative, and therefore the noise term in Eq. (2.7e) can become complex. The difficulty arises from non-positive-definite diffusion in the associated Fokker-Planck equation. As written, the ten-dimensional system defined by Eqs. (2.7) has positive-definite diffusion. It is rigorously derived by generalizing the usual nonanalytic characteristic function to one analytic in five independent complex variables.[6,12,13]

In the absence of noise ($N\to\infty$) steady-state solutions to Eqs. (2.7) have

$$\bar{v}=\bar{\alpha}\bar{m},\ \ \bar{v}_*=\bar{\alpha}_*\bar{m},\ \ \bar{m}=-1/(1+\bar{\alpha}_*\bar{\alpha}) , \tag{2.10}$$

where $\bar{\alpha}=\bar{\alpha}_*$ and $\bar{\alpha}_*\bar{\alpha}$ satisfies the cubic equation

$$\bar{\alpha}_*\bar{\alpha}[1+2C/(1+\bar{\alpha}_*\bar{\alpha})]^2=Y^2 . \tag{2.11}$$

Here the overbar denotes the steady state. As $\bar{\alpha}_*\bar{\alpha}$ can be complex in this formalism there are actually three steady-state solutions in the ten-dimensional space for *all* values of C and Y. However, physical realizations of Eqs. (2.7) have $(\alpha_*)_{\text{av}}=(\alpha)_{\text{av}}^*$, $(v_*)_{\text{av}}=(v)_{\text{av}}^*$, and $(m)_{\text{av}}=(m)_{\text{av}}^*$. Then for physical steady states,

$$\bar{\alpha}=\bar{\alpha}_*=X,\ \ \bar{v}=\bar{v}_*=X\bar{m},\ \ \bar{m}=-1/(1+X^2) , \tag{2.12}$$

$$X[1+2C/(1+X^2)]=Y , \tag{2.13}$$

where X is a real field amplitude. For small noise ($N\gg1$) I linearize Eqs. (2.7) around the physical steady states, writing

$$\eta=\bar{\eta}+\xi , \tag{2.14}$$

with

$$\eta=(\alpha,\alpha_*,v,v_*,m)^T , \tag{2.15a}$$

$$\bar{\eta}=(X,X,-X/(1+X^2),-X/(1+X^2),-1/(1+X^2))^T , \tag{2.15b}$$

$$\xi=(\Delta\alpha,\Delta\alpha_*,\Delta v,\Delta v_*,\Delta m)^T . \tag{2.15c}$$

Here T denotes the transposition. Then ξ obeys the linear equations

$$d\xi=\underline{A}\xi\,d\tau+\underline{B}\,d\mathbf{W} , \tag{2.16}$$

where

$$d\mathbf{W}=(0,0,dW_1,dW_2,dW_3)^T , \tag{2.17}$$

$$\underline{A}=\frac{1}{2}\begin{bmatrix} -\mu & 0 & 2\mu C & 0 & 0 \\ 0 & -\mu & 0 & 2\mu C & 0 \\ -1/(1+X^2) & 0 & -1 & 0 & X \\ 0 & -1/(1+X^2) & 0 & -1 & X \\ X/(1+X^2) & X/(1+X^2) & -X & -X & -2 \end{bmatrix} , \tag{2.18}$$

and

$$\underline{B}=N^{-1/2}X(1+X^2)^{-1/2}\text{diag}(0,0,i,i,2) . \tag{2.19}$$

Equations (2.16) are defined in a ten-dimensional real space. However, we are only interested in the moments appearing in Eq. (2.9), hence in moments of the complex variables $\xi_1,\xi_2,\ldots,\xi_5$. We are not interested, for example, in $(\xi_1^x\xi_1^y)_{\text{av}}$—although this average contributes to $(\xi_1^2)_{\text{av}}=[(\xi_1^x+i\xi_1^y)^2]_{\text{av}}$. It is then readily shown that in the stationary state the physical correlation matrix

$$\underline{G}(\tau)=\{\boldsymbol{\xi}(\tau)[\boldsymbol{\xi}(0)]^T\}_{\text{av}} \tag{2.20}$$

can be calculated in a five-dimensional space:

$$\frac{d}{d\tau}\underline{G}(\tau)=\underline{A}\underline{G}(\tau) , \tag{2.21}$$

where the steady-state covariance matrix $\underline{G}\equiv\underline{G}(0)$ satisfies[15]

$$\underline{A}\underline{G}+\underline{G}\underline{A}^T=-\underline{B}\,\underline{B}^T . \tag{2.22}$$

The spectrum of fluctuations is given by

$$\begin{aligned}\tilde{\underline{G}}(\omega)&=(2\pi)^{-1}\int_{-\infty}^{\infty}d\tau\, e^{-i\omega\tau}\underline{G}(\tau)\\ &=(2\pi)^{-1}(\underline{A}-i\omega\underline{I})^{-1}\underline{B}\,\underline{B}^T(\underline{A}^T+i\omega\underline{I})^{-1} .\end{aligned} \tag{2.23}$$

These are the usual formulas which would be obtained from the Glauber-Sudarshan-Haken representation. The fact that $\underline{D}=\underline{B}\,\underline{B}^T$ represents a non-positive-definite diffusion can simply be overlooked.

III. STEADY-STATE CORRELATIONS

Equation (2.22) defines a set of 15 linear equations for the elements of the covariance matrix $\underline{G}$. Since the fluctuations $\boldsymbol{\xi}$ are Gaussian distributed, moments of all orders can be calculated once these equations have been solved. This problem is tractable analytically, at least in this absorptive case. Since $\underline{A}$ and $\underline{D}=\underline{B}\,\underline{B}^T$ are real, the 15 equations decouple into a set of nine equations for the real part of $\underline{G}$ and a set of six equations for its imaginary part:

$$\underline{A}\,\underline{G}_x+\underline{G}_x\underline{A}^T=-\underline{B}\,\underline{B}^T , \tag{3.1a}$$

$$\underline{A}\,\underline{G}_y+\underline{G}_y\underline{A}^T=\underline{0} , \tag{3.1b}$$

where

$$\underline{G}=\underline{G}_x+i\underline{G}_y . \tag{3.2}$$

The homogeneous equation (3.1b) must have the trivial solution $\underline{G}_y=\underline{0}$, which leaves only the nine equations of Eq. (3.1a) to be solved. The calculations are tedious so I go directly to the results: (1) Field-field correlations

$$n_s^{-1}\langle\Delta a^\dagger\Delta a\rangle=X^2\lambda , \tag{3.3a}$$

$$n_s^{-1}\langle\Delta a\,\Delta a\rangle=X^2(\lambda-P) , \tag{3.3b}$$

where

$$P=N^{-1}2C\frac{\mu}{\mu+1}\left[1-\frac{X}{Y}\right] , \tag{3.4}$$

$$\lambda=P\frac{X^2}{1+X^2}\left[\frac{dY}{dX}\right]^{-1}\left\{1+\frac{1}{2}\frac{Y}{X}\frac{(\mu+3)(2-X^2)+\mu(1+X^2)\frac{dY}{dX}}{(\mu+3)\left[1+X^2+\mu\left[1+\frac{1}{2}\frac{Y}{X}\right]\right]-\mu(1+X^2)\frac{dY}{dX}}\right\} . \tag{3.5}$$

(2) Atom-field correlations

$$n_s^{-1/2}(N/\sqrt{2})^{-1}\langle\Delta a\,\Delta J_+\rangle=(2C)^{-1}X^2\lambda , \tag{3.6a}$$

$$n_s^{-1/2}(N/\sqrt{2})^{-1}\langle\Delta a\,\Delta J_-\rangle=(2C)^{-1}X^2(\lambda-P) , \tag{3.6b}$$

$$n_s^{-1/2}(N/2)^{-1}\langle\Delta a\,\Delta J_z\rangle=(2C)^{-1}X\left[\frac{Y}{X}\lambda-Q\right] , \tag{3.6c}$$

where

$$Q=(\mu+3)^{-1}\left[2(1+X^2)\frac{dY}{dX}\lambda+X^2\left[\frac{Y}{X}-2\right]P\right] . \tag{3.7}$$

(3) Atom-atom correlations

$$(N/\sqrt{2})^{-2}\langle\Delta J_+\,\Delta J_-\rangle=(2C)^{-2}X^2(\lambda+\mu^{-1}Q) , \tag{3.8a}$$

$$(N/\sqrt{2})^{-2}\langle\Delta J_-\,\Delta J_-\rangle=(2C)^{-2}X^2\left[\lambda-P+\mu^{-1}\left[Q-\frac{Y}{X}P\right]\right] , \tag{3.8b}$$

$$(N/\sqrt{2})^{-1}(N/2)^{-1}\langle\Delta J_z\,\Delta J_-\rangle=(2C)^{-2}X\left[\frac{Y}{X}\lambda+\mu^{-1}Q\right] , \tag{3.8c}$$

$$(N/2)^{-2}\langle \Delta J_z\,\Delta J_z\rangle = (2C)^{-2}X^2\left[\frac{Y}{X}\left[\frac{Y}{X}-2\right]\lambda-\left[\frac{Y}{X}-1\right]Q+2\frac{Y}{X}P-\mu^{-1}\left[Q-2\frac{Y}{X}P\right]\right]. \quad (3.8d)$$

In these expressions Y/X and dY/dX are calculated from Eq. (2.13).

This analytic solution is algebraically complicated and there is little to be gained from a detailed dissection of these results. A number of general observations can usefully be made, however. First, note that λ is proportional to $(dY/dX)^{-1}$. It appears in each of the moments, guaranteeing that all diverge at instability points $dY/dX=0$. Then the good- and bad-cavity limits can be taken formally in these expressions. By considering these limits we gain some idea of the structure of the general solution and the way in which quantum noise scales with system size. First the good-cavity limit. The results obtained with adiabatic elimination of the atoms[3,5,6] are recovered from Eqs. (3.3) for $\mu \ll 1$, $C\mu \ll 3$. The requirement $C\mu \ll 3$ is imposed by the terms $\mu Y/X$ and $\mu dY/dX$ in Eq. (3.5). It reflects the increased decay rate $(1+2C)\kappa$ which governs field dynamics at weak intensities where the unsaturated absorber spoils the cavity. In the good-cavity limit λ, P, and Q are all inversely proportional to $n_s = N/4C\mu$, which then governs the scaling with system size. Since $n_s \gg N$ if the requirement $C\mu \ll 3$ is to be met, the fluctuations are smaller than indicated by a scaling with N^{-1}. Note, however, that the scaling with n_s^{-1} applies only to Eqs. (3.3) and (3.6). The dominant terms in Eqs. (3.8) are those in $\mu^{-1}Q$ and $\mu^{-1}P$. Atom-atom correlations then scale as N^{-1}. This can be appreciated by considering the adiabatic elimination of the atoms from Eqs. (2.7). Observe that the noise sources drive the atomic variables both directly and indirectly via coupling to the field. Then the two terms in each of Eqs. (3.8)—proportional to μ^{-1} and otherwise—have distinct origins. The dominant terms, in $\mu^{-1}P$ and $\mu^{-1}Q$, which scale as N^{-1}, come directly from the atomic noise sources. The terms in λ, P, and Q, which scale as n_s^{-1}, come from the fluctuating field to which the atoms are slaved, i.e., from fluctuations passed from the atomic noise sources through the field and back to the atoms. The significance of these two terms is illuminated further by dividing each correlation function into contributions from like-atom and unlike-atom correlations:

$$\langle \Delta J_\alpha\,\Delta J_\beta\rangle = N\langle \Delta\sigma_\alpha^j\,\Delta\sigma_\beta^j\rangle + N(N-1)\langle \Delta\sigma_\alpha^j\,\Delta\sigma_\beta^{k\neq j}\rangle\,, \quad (3.9)$$

where α and β are each either $+$, $-$, or z. Like-atom correlations can be calculated directly from the steady-state averages

$$\langle \sigma_\pm^j\rangle = -(1/2\sqrt{2})X(1+X^2)^{-1}$$

and

$$\langle \sigma_z^j\rangle = -\tfrac{1}{2}(1+X^2)^{-1}\,,$$

whereby it can be shown that the dominant terms in Eqs. (3.8), scaling as N^{-1}, come from like-atom correlations, and the terms scaling as n_s^{-1} come from unlike-atom correlations. This is not surprising since communication between different atoms must be mediated by the field. From this observation it follows that atom-atom correlations are negligible in the good-cavity limit ($n_s \gg N$).[16]

Finally, the bad-cavity limit. Results obtained with adiabatic elimination of the field[3,4,7] are recovered from Eqs. (3.8) for $\mu \gg 3$, $\mu \gg 2C$ (to guarantee $\mu^{-1}Y/X \ll 1$ for weak fields), and $\mu \gg X$ (to guarantee $\mu^{-1}Q \ll \lambda$ for strong fields). The requirements $\mu \gg 2C$ and $\mu \gg X$, respectively, reflect the need for the field to adiabatically follow the collective atomic decay rate $(1+2C)\gamma/2$ for weak fields and the Rabi frequency $\gamma X/\sqrt{2}$ for strong fields. Adiabatic elimination of the field sets $n_s^{-1/2}\Delta a = 2C(N/\sqrt{2})^{-1}\Delta J_-$. Thus correlations involving field and polarization operators should be related by multiples of $2C$. This relationship is correctly reflected by Eqs. (3.3a), (3.3b), (3.6a), (3.6b), (3.8a), and (3.8b). In the bad-cavity limit λ, P, and Q are all proportional to $2C/N$ and the fluctuations scale as N^{-1}.

IV. INCOHERENT INTENSITY, PHOTON ANTIBUNCHING, AND SQUEEZING

Moments of the cavity field fluctuations are calculated from Eqs. (3.3)—(3.5). A number of analytic expressions can be obtained which generalize earlier results for the incoherent intensity,[3,4,6,7] photon antibunching,[5–7] and squeezing.[8] I summarize these here.

The incoherent spectrum for the transmitted light is given by $\tilde{G}(\omega)_{21}$, which can be computed from Eq. (2.23). Spectra for arbitrary values of μ can readily be obtained numerically. The integrated spectrum is just the incoherent intensity, given by Eq. (3.3a). The ratio of incoherent to coherent intensity is a useful measure for assessing the possibility of measuring interesting spectral features against the background of strong coherent transmission. From Eqs. (3.3a), (3.4), and (3.5),

$$I_{\rm inc}/I_{\rm coh} = n_s^{-2}\langle \Delta a^\dagger\,\Delta a\rangle/X^2 = N^{-1}2C\frac{\mu}{\mu+1}\frac{X^2}{1+X^2}\left[1-\frac{X}{Y}\right]\left[\frac{dY}{dX}\right]^{-1}\left[1+\frac{1}{2}\frac{Y}{X}\frac{(\mu+3)(2-X^2)+\mu(1+X^2)\frac{dY}{dX}}{(\mu+3)\left[1+X^2+\mu\left[1+\frac{1}{2}\frac{Y}{X}\right]\right]-\mu(1+X^2)\frac{dY}{dX}}\right]. \quad (4.1)$$

Of course, with the divergence at $dY/dX=0$ an arbitrarily large incoherent intensity is indicated for a close enough approach to instability points. However, this is not to be trusted as the divergence evidences failure of the linearized theory. A fully nonlinear theory based on the numerical simulation of Eqs. (2.7) will be discussed elsewhere.[17]

Photon antibunching and squeezing in absorptive bistability are intimately related.[8,18] Squeezing occurs in the field quadrature in phase with the driving field. The self-homodyning of squeezed amplitude fluctuations produces photon antibunching after the fashion used in the detection of squeezed light.[19] The normalized second-order correlation function for the transmitted light is given by

$$g^{(2)}(0)=\langle(a^\dagger)^2a^2\rangle/\langle a^\dagger a\rangle^2$$
$$=1+(n_sX^2+\langle\Delta a^\dagger\Delta a\rangle)^{-2}\{n_sX^24\langle:(\Delta A_1)^2:\rangle+2n_s^{1/2}X[\langle(\Delta a^\dagger)^2\Delta a\rangle+\text{c.c.}]+\langle(\Delta a^\dagger)^2\Delta a^2\rangle-\langle\Delta a^\dagger\Delta a\rangle^2\}\ , \tag{4.2}$$

where : : denotes normal ordering and $A_1=\frac{1}{2}(a^\dagger+a)$. In this linearized theory the third-order moments are zero since fluctuations are Gaussian; furthermore, all higher-order terms must be dropped for a consistent treatment of fluctuations to lowest order. Then

$$g^{(2)}(0)=1+4n_s^{-1}\langle:(\Delta A_1)^2:\rangle/X^2\ , \tag{4.3}$$

where, from Eqs. (3.3) and (3.4),

$$4n_s^{-1}\langle:(\Delta A_1)^2:\rangle/X^2=4I_{\text{inc}}/I_{\text{coh}}-N^{-1}4C\frac{\mu}{\mu+1}\left[1-\frac{X}{Y}\right]. \tag{4.4}$$

As $I_{\text{inc}}/I_{\text{coh}}$ must be positive, and vanishes for $X^2\to0$, while

$$1-X/Y=2C/(1+X^2+2C)$$

is maximized for $X^2=0$, the maximum antibunching effect is given by

$$\lim_{X^2\to0}g^{(2)}(0)=1-N^{-1}4C\frac{\mu}{\mu+1}\frac{2C}{1+2C}\ . \tag{4.5}$$

Squeezing is indicated by a negative value for $\langle:(\Delta A_1)^2:\rangle$ and hence its obvious correlation with photon antibunching in this system. Unlike photon antibunching, however, it is not maximized for $X^2\to0$. Squeezing depends on absolute photon number. It is impossible to have large squeezing and small mean photon number because reduced quantum fluctuations in one field quadrature are required by Heisenberg uncertainty to be accompanied by increased fluctuations in the other quadrature, thus,

$$\langle(\Delta A_1)^2\rangle\langle(\Delta A_2)^2\rangle\geq\tfrac{1}{16}$$

with

$$\langle(\Delta A_1)^2\rangle\ll\tfrac{1}{4}\Longrightarrow\langle(\Delta A_2)^2\rangle\gg\tfrac{1}{4}\Longrightarrow\langle\Delta a^\dagger\Delta a\rangle\gg1\ .$$

From Eqs. (4.3) and (4.4)

$$4\langle:(\Delta A_1)^2:\rangle=n_sX^2[g^{(2)}(0)-1]$$
$$=X^2(\mu+1)^{-1}\left[\left[\frac{4C}{N}\frac{\mu}{\mu+1}\right]^{-1}4I_{\text{inc}}/I_{\text{coh}}-\left[1-\frac{X}{Y}\right]\right], \tag{4.6}$$

which clearly vanishes for $X^2=0$. Equation (4.6) serves to illustrate a further point arising from this dependence of squeezing on photon number. The term in square brackets is independent of overall scalings with μ and N, the only important scale factor being the $(\mu+1)^{-1}$ outside the bracket. This suggests that squeezing is necessarily a much smaller effect in a bad cavity ($\mu+1\to\mu\gg1$) than in a good cavity ($\mu+1\to1$).[8] However, this observation is misleading. The scaling with μ^{-1} in a bad cavity simply reflects the reduced photon number *inside* a cavity which permits ready photon escape. Squeezing will be detected at a photodetector *outside* the cavity where it is the product of photon flux at the detector and photon counting time that is important. The quantity Q which characterizes the subpoissonian statistics in a homodyne detection scheme outside the cavity[19] is given by

$$Q=\eta(2\kappa_0)T4\langle:(\Delta A_1)^2:\rangle$$
$$=\eta\gamma T\kappa_0(\kappa_0+\kappa_i)^{-1}4\mu\langle:(\Delta A_1)^2:\rangle\ , \tag{4.7}$$

where η is a collection efficiency, T is the counting time, and $2\kappa_0$ and $2\kappa_i$ are introduced as photon escape rates from the cavity through the output and input mirrors, respectively, with $\kappa=\kappa_0+\kappa_i$. This result is valid for short counting times and can be generalized to arbitrary T in an expression which involves integration over time-dependent field correlation functions.[20–22] Notice that from Eqs. (4.6) and (4.7), if we take $\kappa_i\ll\kappa_0\simeq\kappa$, the only change in the overall scaling of effects going from a good to a bad cavity is the change $2\kappa T\to\gamma T$. In the bad cavity we must count for a time set by the atomic time scale, not the much faster cavity time scale, to obtain a comparable number of photon counts; correlated photons are available over this longer time. The meaningful comparison between the the good- and bad-cavity limits is that between

$Q/\eta 2\kappa T$ and $Q/\eta\gamma T$. From Eqs. (4.6) and (4.7), in the good-cavity limit,

$$Q/\eta 2\kappa T = -2CX^2(1-2X^2)[(1+X^2)^3+2C(1-X^4)]^{-1} , \tag{4.8}$$

while in the bad-cavity limit,

$$Q/\eta\gamma T = -2CX^2(1-X^2)[(1+X^2)^2+2C(1-X^2)]^{-1} . \tag{4.9}$$

Both of these expressions give their maximum squeezing for large C. From Eq. (4.8) the maximum effect occurs for $X^2=2-\sqrt{3}$ with

$$Q/\eta 2\kappa T = -\tfrac{1}{2}(2-\sqrt{3}) \simeq -0.134 ,$$

which agrees with the minimum $\langle(\Delta A_1)^2\rangle \simeq 0.22$ reported by Lugiato and Strini.[8] From Eq. (4.9) a maximum effect $Q/\eta\gamma T=-1$ occurs just below the turning point on the lower branch, $X^2 \lesssim 1$. This is a much *larger* effect. In fact, it is the result for a cavity with photon escape rate γ containing a field which is *perfectly squeezed.* It should be compared with the maximum squeezing $Q/\eta\gamma T=-\frac{1}{8}$ in resonance fluorescence.[23] Of course, Q calculated from Eqs. (4.8) and (4.9) is necessarily small since the short counting time result, Eq. (4.7), requires $2\kappa T \ll 1$ and $\gamma T \ll 1$. However, this result for the bad cavity suggests that large squeezing will be attainable for longer photon counting times. A detailed investigation of this possibility requires the solutions for time-dependent correlation functions [Eq. (2.21)]. In the next section I obtain such solutions for weak fields, but not in the general case. Further study of squeezing is postponed to a later publication.

V. PHOTON ANTIBUNCHING AND VACUUM RABI SPLITTING

Photon antibunching is maximum in the weak-field limit where it is straightforward to calculate the full time-dependent correlation function

$$\begin{aligned} g^{(2)}(\tau) &= 1+n_s^{-1}\{\langle \Delta a^\dagger(\tau)\Delta a(0)\rangle + \langle \Delta a^\dagger(0)\Delta a(\tau)\rangle \\ &\quad +2\mathrm{Re}[\langle \Delta a(\tau)\Delta a(0)\rangle]\}/X^2 \\ &= 1+\{G_{21}(\tau)+G_{12}(\tau)+2\mathrm{Re}[G_{11}(\tau)]\}/X^2 . \end{aligned} \tag{5.1}$$

In this section I calculate $g^{(2)}(\tau)$ for the limit $X^2 \to 0$ and demonstrate that an oscillation which arises for $\mu \sim 1$ has its origin in the so-called vacuum Rabi splitting.[9,10]

The correlation functions $G_{21}(\tau)$, $G_{12}(\tau)$, and $G_{11}(\tau)$ are calculated from Eq. (2.21). Writing

$$G_{ij}^0 = \lim_{X^2\to 0} G_{ij}/X^2 , \tag{5.2}$$

we find $G_{21}^0(\tau)=G_{12}^0(\tau)=0$, while G_{11}^0 satisfies

$$\frac{d}{d\tau}\begin{bmatrix} G_{11}^0 \\ G_{31}^0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\mu & 2\mu C \\ -1 & -1 \end{bmatrix}\begin{bmatrix} G_{11}^0 \\ G_{31}^0 \end{bmatrix} , \tag{5.3}$$

where Eqs. (3.3b) and (3.6b) give, respectively,

$$G_{11}^0(0) = -N^{-1}2C\frac{\mu}{\mu+1}\frac{2C}{1+2C} , \tag{5.4a}$$

$$G_{31}^0(0) = (2C)^{-1}G_{11}^0 . \tag{5.4b}$$

The solution for $g^{(2)}(\tau)$ is then

$$\lim_{X^2\to 0} g^{(2)}(\tau) = 1-N^{-1}4C\frac{\mu}{\mu+1}\frac{2C}{1+2C} \times e^{-(1/4)(\mu+1)\tau}\left[\cosh\Omega\tau+\frac{\mu+1}{4\Omega}\sinh\Omega\tau\right] , \tag{5.5}$$

with

$$\Omega = [\tfrac{1}{16}(\mu-1)^2-\tfrac{1}{2}\mu C]^{1/2} . \tag{5.6}$$

The decay of this correlation function is oscillatory for $\mu C > (\mu-1)^2/8$ as illustrated by Fig. 1. In the good-cavity limit ($\mu \ll 2$, $\mu C \ll 2$) $\Omega \simeq (1-\mu)/4-\mu C$, and in the bad-cavity limit ($\mu \gg 4$, $\mu \gg 8C$) $\Omega \simeq (\mu-1)/4-C$. Here we recover the results previously derived with adiabatic elimination:[5]

$$\lim_{X^2\to 0} g^{(2)}(t) = 1-n_s^{-1}\frac{2C}{1+2C}e^{-\kappa(1+2C)t} \tag{5.7a}$$

and

$$\lim_{X^2\to 0} g^{(2)}(t) = 1-N^{-1}4C\frac{2C}{1+2C}e^{-(1/2)\gamma(1+2C)t} , \tag{5.7b}$$

respectively. Thus, this oscillatory behavior is a new feature made possible by the interplay of near equal cavity and atomic decay rates. The existence of such behavior in a coupled system with competing decay processes needs no explanation as a general phenomenon; certainly an equation such as Eq. (5.3) might be expected to yield complex eigenvalues. However, can a more satisfying physical explanation be given for this specific system? These are not Rabi oscillations in the familiar sense since we have taken $X^2 \to 0$. Neither is this the familiar oscillatory

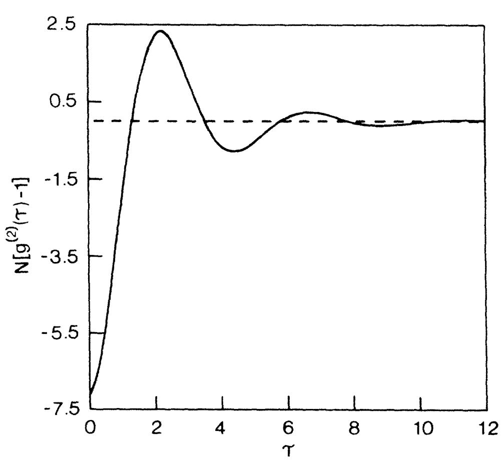

FIG. 1. Second-order correlation function of the transmitted light for $C=4.0$ and $\mu=1.0$.

response in the presence of detuning,[7,24] as both atoms and cavity are driven on resonance. On the other hand, as it turns out, this oscillation is indeed closely related to both of these effects. Its origin lies in the so-called vacuum Rabi splitting,[9,10] as seen from the following simple model.

The correlation functions in Eq. (5.3) are

$$G^{0}_{11}(\tau)=n_s^{-1}\langle\Delta a(\tau)\Delta a(0)\rangle/X^2$$

and

$$G^{0}_{31}(\tau)=n_s^{-1/2}(N/\sqrt{2})^{-1}\langle\Delta J_-(\tau)\Delta a(0)\rangle/X^2 .$$

In dimensioned variables we have

$$\frac{d}{dt}\begin{bmatrix}\langle\Delta a(t)\Delta a(0)\rangle\\ \langle\Delta J_-(t)\Delta a(0)\rangle\end{bmatrix}=\begin{bmatrix}-\kappa & g\\ -Ng & -\frac{1}{2}\gamma\end{bmatrix}\begin{bmatrix}\langle\Delta a(t)\Delta a(0)\rangle\\ \langle\Delta J_-(t)\Delta a(0)\rangle\end{bmatrix}, \tag{5.8}$$

which arises from an underlying dynamic which couples quantized field and polarization oscillators Δa and ΔJ_-. For a simple picture of these oscillator dynamics I return to the original master equation [Eq. (2.4)] and introduce the Schwinger representation:

$$J_+=b^\dagger c,\quad J_-=bc^\dagger , \tag{5.9a}$$

$$J_z=(b^\dagger b-\tfrac{1}{2}N),\quad N=b^\dagger b+c^\dagger c , \tag{5.9b}$$

with boson commutation relations $[b^\dagger,b]=1$, $[c^\dagger,c]=1$. Here $b^\dagger$ and b create and annihilate atoms in their excited state, and $c^\dagger$ and c create and annihilate atoms in their ground state. For weak fields only the first-excited state for the atomic population need be considered and we can make the replacements

$$J_+=\sqrt{N}b^\dagger,\quad J_-=\sqrt{N}b , \tag{5.10a}$$

$$\sum_{j=1}^{N}(2\sigma^j_-\rho\sigma^j_+-\sigma^j_+\sigma^j_-\rho-\rho\sigma^j_+\sigma^j_-)=2b\rho b^\dagger-b^\dagger b\rho-\rho b^\dagger b . \tag{5.10b}$$

Then Eq. (2.4) becomes the master equation for coupled damped harmonic oscillators:

$$\frac{d\rho}{dt}=\mathscr{E}[a^\dagger-a,\rho]+g\sqrt{N}[a^\dagger b-ab^\dagger,\rho]+\tfrac{1}{2}\gamma(2b\rho b^\dagger-b^\dagger b\rho-\rho b^\dagger b)+\kappa(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a) . \tag{5.11}$$

Clearly the equations for $\langle a\rangle$ and $\langle J_-\rangle=\sqrt{N}\langle b\rangle$ lead via the quantum regression theorem to Eq. (5.8). Now in the absence of damping these oscillators are decoupled by the introduction of normal modes

$$A=(a+ib)/\sqrt{2},\quad B=(a-ib)/\sqrt{2} , \tag{5.12}$$

whence

$$g\sqrt{N}[a^\dagger b-ab^\dagger,\rho]=-ig\sqrt{N}[A^\dagger A,\rho]+ig\sqrt{N}[B^\dagger B,\rho] . \tag{5.13}$$

Equation (5.11) is defined in a rotating frame. The frequencies of oscillators A and B in the original frame are $\omega_0+g\sqrt{N}$ and $\omega_0-g\sqrt{N}$, respectively. Therefore, the normal modes are not resonant with the driving field. They are moved from resonance by the energy level splitting, $\pm\hbar g\sqrt{N}$, which lifts the degeneracy that exists when oscillators a and b are uncoupled. For weak fields the first-excited state splitting in this coupled oscillator system corresponds to the first-excited state splitting in the coupled atom-field system. This vacuum Rabi splitting—of course, it is not the vacuum state that is split—is the fundamental origin of oscillation in $g^{(2)}(\tau)$. The frequencies $\pm i\gamma(\frac{1}{2}\mu C)^{1/2}$ given by Eq. (5.6) for $\mu\simeq 1$ are just the frequency shifts $\pm ig\sqrt{N}$. However, weak-field level splitting does not give rise to Rabi oscillations in a radiatively damped two-level atoms. It requires a strong field $X>1/(2\sqrt{2})$ before level splitting is evidenced by complex dynamical eigenvalues. Clearly the dissipation is responsible. What is the role of dissipation here? In terms of normal modes Eq. (5.11) reads

$$\frac{d\rho}{dt}=(\mathscr{E}/\sqrt{2})[(A^\dagger-A)+(B^\dagger-B),\rho]-ig\sqrt{N}[A^\dagger A-B^\dagger B,\rho]+\tfrac{1}{2}(\kappa+\tfrac{1}{2}\gamma)(2A\rho A^\dagger-A^\dagger A\rho-\rho A^\dagger A+2B\rho B^\dagger-B^\dagger B\rho-\rho B^\dagger B)+\tfrac{1}{2}(\kappa-\tfrac{1}{2}\gamma)(2A\rho B^\dagger-A^\dagger B\rho-\rho A^\dagger B+2B\rho A^\dagger-B^\dagger A\rho-\rho B^\dagger A) . \tag{5.14}$$

When $\mu=1$ the normal mode oscillators are decoupled and independently damped with the common damping constant $\kappa=\frac{1}{2}\gamma$. The eigenvalues $-\frac{1}{2}\pm i\sqrt{C/2}$. Entering Eq. (5.5) follow directly. More generally, however, these oscillators remain coupled through their nonindependent decay (normal modes couple to correlated reservoirs). It is this residual interaction which brings the term $(\mu-1)/16$ to Eq. (5.6) and generally masks the evidence of vacuum Rabi splitting. Actually dissipation plays a similar role in the familiar problem of a collisionally broadened transition driven by a classical field. There the dynamical eigenvalues

$$\lambda_{\pm}=-\tfrac{1}{2}(\gamma_{\parallel}+\gamma_{\perp})\pm[\tfrac{1}{4}(\gamma_{\parallel}-\gamma_{\perp})^2-\gamma_{\parallel}\gamma_{\perp}X^2]^{1/2} \quad (5.15)$$

appear, where $\gamma_{\parallel}$ and $\gamma_{\perp}$ are longitudinal and transverse decay rates, respectively. Rabi splitting is evidenced for arbitrarily weak fields when $\gamma_{\parallel}=\gamma_{\perp}$. Of course, here X is a classical amplitude which presupposes the presence of many photons. Eigenvalues showing single-photon state splitting are given by more complicated expressions which must be derived using a quantized field.

I have only considered $g^{(2)}(\tau)$ here; however, the same effects will be evident in the transmitted spectrum. Sidebands will appear in the incoherent spectrum, symmetrically displaced from the driving field frequency by the splitting calculated from Eq. (5.6).

VI. SUMMARY AND CONCLUSIONS

I have given a linearized treatment of quantum fluctuations in absorptive bistability without adiabatically eliminating the atoms or the field. The steady-state covariance matrix has been derived in closed form. From this, earlier results for the ratio of incoherent and coherent intensities, photon antibunching, and squeezing have been generalized to conditions outside the good- and bad-cavity limits. I emphasize that squeezing is not necessarily very small in a bad cavity; rather, large squeezing seems possible in the vicinity of the lower turning point for large values of the bistability parameter C. In the limit of low incident intensities the second-order correlation function for the transmitted light shows an oscillatory decay for moderate values of C and near equal field and polarization decay rates. Corresponding sidebands will appear in the transmitted spectrum. This phenomenon is related to vacuum Rabi splitting, with an oscillation frequency corresponding to the splitting of the degenerate one-photon state by coupling between quantized field and polarization oscillators.

Hopefully the future will see experiments on quantum-statistical effects in absorptive bistability to match the wide theoretical attention they have received.

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