Space-curvature effects in the interaction between atoms and external fields: Zeeman and Stark effects in a space of constant positive curvature

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(Received 3 June 1985)

Space-curvature-induced theoretical modifications in the interaction between atoms and external electromagnetic fields have been investigated in the framework of a "curved-orbital" model: The traditional Euclidean space is replaced by a geometrically simple curved space, i.e., the spherical three-space of radius R. The required solutions of Maxwell's equations on the hypersphere have been obtained in closed form. A novel procedure has been devised in order to obtain analytical expressions of the curved-space Dirac hydrogenic pseudoradial integrals in terms of the Dirac energy parameter $\epsilon=1+\alpha^2E_{nk}$ and of the Dirac quantum number $k=(-1)^{j+l+1/2}(j+1/2)$. The hydrogenic Zeeman and Stark effects have been studied in detail and analytical expressions of the curvature-induced contributions to the Landé g factor and to the Stark matrix elements have been given in terms of ϵ and k. This curved-space model, which will, we hope, provide the basis for an easy extension to the many-electron cases, gives the usual flat-space results in the limit as the radius R of the space increases to infinity.

I. INTRODUCTION

The results of investigating the modification of the atomic spectra induced by the curvature of space has been recently outlined in a series of papers.[1–7] Specifically, in the framework of a "curved-space Dirac-orbital" model we have calculated space-curvature-induced modifications of the fine- and hyperfine-structure energy levels of one-electron atoms.[4] This curved-space model, although defined in the geometrically simplest non-Euclidean space, i.e., the spherical three-space, nevertheless is capable of including, at least roughly, global effects due to the topology of space. It leads to tractable computations and enables one to predict, within the usual framework of theoretical spectroscopy, some curvature modifications of the spectrum. Particularly, it has been shown how the degenerate one-electron fine-structure energy levels are split by an additional space-curvature contribution which vanishes at the asymptotic flat-space limit. In addition to some technical advantages pointed out elsewhere,[3,4,8,9] working in a space with constant positive curvature allows us keeping a direct parallelism between the curved- and flat-space results and permits an easy extension to the many-electron case.

After the study of the space-curvature modifications of the free-atom spectrum,[5,6] another important step is the study of the space-curvature effects in the interaction between atoms and external fields. One of the aims of this paper is the investigation and computation of the Zeeman and Stark splittings in a space of constant positive curvature. Obviously, such an investigation needs a relativistic treatment, i.e., the use of the Dirac equation for stationary states with an external electromagnetic field. For that reason, a brief review of the Dirac equation and of the Dirac-Coulomb orbitals in a spherical three-space is given in Sec. II. Since we consider only the case of weak external fields, the perturbation approach is justified and the investigation implies the computation of matrix elements between the curved-space Dirac-Coulomb orbitals. To our knowledge, closed-form expressions of such matrix elements have not yet been given. We have been able to compute the curved-space Dirac integrals by generalizing a procedure recently devised[10] for calculating the flat-space Dirac-Coulomb r^s radial integrals. Closed-form expressions of the curved-space Dirac radial integrals are obtained in terms of the curved-space Dirac energy parameter $\epsilon=1+E_{nk}/m_0c^2$ and of the Dirac quantum number $k=(-1)^{j+l+1/2}(j+\frac{1}{2})$ (Sec. III). Following the line of treatment sketched by Infeld and Schild[11] in their study of Maxwell equations, the pseudoradial dependence of the static electric and magnetic fields is investigated in comparison with flat-space results in polar coordinates (Sec. IV). The Zeeman and Stark effects of the relativistically bound electron in spherical three-space, where the magnetic and electric field are weak compared to the separation of neighboring fine-structure levels, are considered in Sec. V, and the expressions of the space-curvature modifications (up to the $1/R^2$ contributions) of these effects are given in terms of the quantum numbers.

Before writing down the expression of the Dirac equation in a spherical three-space and before deriving the curved-space form of the interaction of one-electron atoms with external fields, let us recall that the space-time line elements are, in a space of constant positive curvature and in an Euclidean space, respectively,

$$ds^2=c^2dt^2-R^2d\chi^2-R^2\sin^2\chi(d\theta^2+\sin^2\theta\,d\phi^2)\,, \quad (1)$$

and

$$ds^2=c^2dt^2-dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)\,, \quad (2)$$

where $0\leq\chi\leq\pi$, $0\leq r<\infty$ and in both cases $0\leq\theta\leq\pi$ and $0\leq\phi\leq2\pi$. Setting $R\to\infty$, $\chi\to0$ such that $R\chi=r$ remains finite, the spatial part of the line element (1) reduces to that of Euclidean space in which r, θ, ϕ, are the usual polar coordinates. Hence, it can be inferred that the differences between the expressions of the curved- and the flat-space matrix elements will occur in their χ-pseudoradial part and r-radial part while the angular (θ,ϕ) part will remain unchanged: therefore, all the usual coupling schemes and angular selection rules of the traditional atomic-structure calculations will be preserved. Nevertheless, it can be expected[5,12] that the curved-space magnetic and electric pseudoradial matrix elements leading to the same asymptotic flat-space limit will be different.

II. DIRAC-COULOMB ORBITALS IN SPHERICAL THREE-SPACE

Starting from the generally covariant form of the Dirac equation in a Riemannian curved space-time, a convenient choice of the Dirac representation can be made which leads to the usual polar dependence (θ,ϕ) of the Dirac wave function. One gets the following expression for the Dirac equation for stationary states[4,5] with an external electromagnetic field $(V,A_\chi,A_\theta,A_\phi)$,

$$\left[\alpha_\chi\left[p_\chi+i\frac{\beta\hat{K}}{R\sin\chi}\right]+\frac{mc}{\hbar}\beta+W-\frac{1}{\hbar c}(E_T-eV)\right]\Phi(\chi,\theta,\phi)=0\ , \qquad (3)$$

where

$$\beta=\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix},\quad \alpha_u=\begin{bmatrix} 0 & \sigma_u\\ \sigma_u & 0\end{bmatrix},\quad u=\chi,\theta,\phi\ ,$$

$$\sigma_\chi=(\sigma^1\cos\phi+\sigma^2\sin\phi)\sin\theta+\sigma^3\cos\theta\ ,$$

$$\sigma_\theta=(\sigma^1\cos\phi+\sigma^2\sin\phi)\cos\theta-\sigma^3\sin\theta\ ,$$

$$\sigma_\phi=-\sigma^1\sin\phi+\sigma^2\cos\phi\ ,$$

$$p_\chi=-\frac{i\hbar}{R\sin\chi}\frac{\partial}{\partial\chi}\sin\chi\ .$$

in the above, I and $\sigma^1,\sigma^2,\sigma^3$ are the 2×2 unit and Pauli matrices; $\hat{K}=\beta(1+\boldsymbol{\sigma}\cdot\mathbf{l})$; $\boldsymbol{\sigma}=(\sigma^1,\sigma^2,\sigma^3)$; and $\mathbf{l}$ is the classical orbital momentum of the electron. $E_T=mc^2+E$ is the total energy. The magnetic interaction term is

$$W=\frac{e}{\hbar c}\frac{1}{R}\left[\alpha_\chi A_\chi+\frac{\alpha_\theta}{\sin\chi}A_\theta+\frac{\alpha_\phi}{\sin\chi\sin\theta}A_\phi\right]. \qquad (4)$$

When the external electromagnetic field in Eq. (3) reduces to the Coulombic potential $V(\chi)=-(Ze/R)\cot\chi$, a perturbative procedure can be used to obtain the curved-space Dirac-Coulomb orbitals $\Phi(\chi,\theta,\phi)$. Since, at the asymptotic flat-space limit, the function $\Phi(\chi,\theta,\phi)$ must lead to the familiar flat-space Dirac function $\bar{\Phi}(r,\theta,\phi)$, we express $\Phi(\chi,\theta,\phi)$ in the following form:

$$\Phi_{vkm}=\frac{1}{R\sin\chi}\begin{bmatrix} P_{vk}(\chi) & \mathscr{Y}_{ljm}\\ iQ_{vk}(\chi) & \mathscr{Y}_{\bar{l}jm}\end{bmatrix}, \qquad (5)$$

where $\bar{l}=l\pm1$ for $j=l\pm\frac{1}{2}$ and $k=(-1)^{j+l+1/2}(j+\frac{1}{2})$.

The following properties of the $\mathscr{Y}_{ljm}$ and $\mathscr{Y}_{\bar{l}jm}$ spinors hold:

$$(1+\boldsymbol{\sigma}\cdot\mathbf{l})\mathscr{Y}_{ljm}=[j(j+1)-l(l+1)+\tfrac{1}{4}]\mathscr{Y}_{ljm}=-k\mathscr{Y}_{ljm}\ , \qquad (6)$$

$$(1+\boldsymbol{\sigma}\cdot\mathbf{l})\mathscr{Y}_{\bar{l}jm}=k\mathscr{Y}_{\bar{l}jm}\ ,\quad \sigma_\chi\mathscr{Y}_{ljm}=\mathscr{Y}_{\bar{l}jm}\ .$$

Previously,[5] it has been shown how, within the perturbative scheme, a direct parallelism can be kept between the flat-space functions $\bar{\phi}_{vkm}(r,\theta,\phi)$ and the zeroth-order curved-space Dirac functions $\phi^{(0)}(\chi,\theta,\phi)$. This is conveniently achieved by discarding from the Dirac Hamiltonian, a Hermitian perturbation which vanishes at the asymptotic flat-space limit. Finally, one gets the following expressions for the pseudoradial components of the Dirac spinor $\Phi(\chi,\theta,\phi)$ (for details, see Ref. 5):

$$P_{vk}=P_{vk}^{(0)}-\frac{k}{Z^2R^2}\sum_{v'(\neq v)}C_{v'v}P_{v'k}^{(0)}\ ,$$
$$Q_{vk}=Q_{vk}^{(0)}-\frac{k}{Z^2R^2}\sum_{v'(\neq v)}C_{v'v}Q_{v'k}^{(0)}\ , \qquad (7)$$

where

$$C_{v'v}=\frac{(v+|k|)^2(v'+|k|)^2}{(v-v')(v+v'+2|k|)}\times\int_0^\infty c(\bar{P}_{v'k}\bar{Q}_{vk}+Q_{v'k}\bar{P}_{vk})r\,dr\ .$$

The zeroth-order pseudoradial functions $P_{vk}^{(0)}(\chi)$ and $Q_{vk}^{(0)}(\chi)$ are linear combinations of two curved-space Kepler functions $R_S^M(\chi)$

$$P_{vk}^{(0)}=\mathscr{N}\left[(\gamma_2+\gamma_1)\left|\frac{\epsilon^{(0)}k}{\gamma}+1\right|^{1/2}R_S^\gamma-(\gamma_2-\gamma_1)\left|\frac{\epsilon^{(0)}k}{\gamma}-1\right|^{1/2}R_S^{\gamma-1}\right],$$
$$Q_{vk}^{(0)}=\mathscr{N}\left[(\gamma_2-\gamma_1)\left|\frac{\epsilon^{(0)}k}{\gamma}+1\right|^{1/2}R_S^\gamma-(\gamma_2+\gamma_1)\left|\frac{\epsilon^{(0)}k}{\gamma}-1\right|^{1/2}R_S^{\gamma-1}\right], \qquad (8)$$

where

$$\gamma_1=|k+Z\alpha|^{1/2}\ ,\quad \gamma_2=|k-Z\alpha|^{1/2}\ ,$$

$$\gamma=(\mathrm{sgn}k)\gamma_1\gamma_2=(\mathrm{sgn}k)(k^2-Z^2\alpha^2)^{1/2}\ ,$$

$$S=v+|\gamma|-1\ ,$$

$$\mathscr{N}=(\epsilon^{(0)}/8|\gamma|)^{1/2}\left[1+\frac{\alpha^2}{R^2}v(v+2|\gamma|)\right]^{-1/2},$$

$$\epsilon^{(0)}=E_T^{(0)}/mc^2=\left[1+\frac{Z^2\alpha^2}{(v+|\gamma|)^2}\right]^{-1/2}\times\left[1+\frac{\alpha^2}{R^2}v(v+2|\gamma|)\right]^{1/2};$$

and α is the fine-structure constant. The curved-space Kepler functions are

$$R_S^M(\chi)=N_{SM}(\sin\chi)^{S+1}\exp[-q\chi/(S+1)] \times P_v^{(a,a^*)}(-i\cot\chi) , \tag{9}$$

where $a=-(S+1)+iq/(S+1)$, $q=ZR\epsilon$, and, in spite of the presence of the imaginary quantities, the Jacobi polynomial in (9) is a real polynomial in $\cot\chi$ of degree $v=S-M$.

The energy $E_{vk}=E_T-mc^2$, including the curvature contributions (up to the $1/R^2$ terms) has been found[5] and can be written

$$E_{vk}=\frac{1}{\alpha^2}(\epsilon-1) , \tag{10}$$

where

$$\epsilon=\left[1+\frac{Z^2\alpha^2}{(v+|\gamma|)^2}\right]^{-1/2} \times\left[1+\frac{\alpha^2}{2R^2}[(v+|\gamma|)^2+Z^2\alpha^2]\right] -\frac{\alpha^2k}{4R^2}+O\left[\frac{1}{R^4}\right] .$$

Let us remark that the classical Dirac-Coulomb flat-space radial functions $\bar{P}_{vk}(r)$ and $\bar{Q}_{vk}(r)$ are also given by the expression (8) when substituting the curved-space expressions of the Kepler function $R_S^M(\chi)$, of $\epsilon^{(0)}$ and of $\mathcal{N}$ with their flat-space counterparts, i.e.,

$$\begin{aligned}\bar{R}_S^M(r)&=\bar{N}_{SM}r^{M+1}\exp[-\bar{q}r/(S+1)] \times L_v^{2M+1}(2\bar{q}r/(S+1)) ,\\ \mathcal{N}&=(\bar{\epsilon}/8|\gamma|)^{1/2} , \quad \bar{\epsilon}=\left[1+\frac{Z^2\alpha^2}{(v+|\gamma|)^2}\right]^{-1/2} ,\\ \bar{q}&=Z\bar{\epsilon} .\end{aligned} \tag{11}$$

L_v^{2M+1} is a Laguerre polynomial. As will be seen in the following sections, this direct parallelism between the curved- and flat-space Dirac orbitals will be particularly useful when calculating the curved-space pseudoradial Dirac $(R\chi)^s$ integrals. When setting $M=l$, $S=n-1$ ($n=1,2,\dots$), and $\bar{q}=Z$, the expression (11) identifies the Schrödinger hydrogenic radial functions.

III. CLOSED-FORM EXPRESSIONS OF THE PSEUDORADIAL INTEGRALS

In our investigation of the bound states of the electron in external fields, one of the critical parts is the computation of the necessary Dirac pseudoradial integrals in closed form. In Refs. 5 and 12, a procedure has been given, which takes full advantage of the fact that the $R_S^M(\chi)$ are eigenfunctions of a Infeld-Hull[13] factorizable equation and which leads to closed-form expressions of the hyperfine structure parameters in terms of ϵ and of the quantum number k. Nevertheless, the final formula thus obtained is not entirely satisfactory since it involves the rather intricate expressions of the off-diagonal $S'\neq S$, $M'\neq M$ $\langle \bar{R}_{S'}^{M'}|r^s|\bar{R}_S^M\rangle$ flat-space integrals.[14,15] A novel and more direct procedure has been devised recently[10] and is applied in the present paper to calculate the Dirac pseudoradial matrix elements of $(R\chi)^s$ with $s\geq 0$. This procedure is simply based on the fact that the $P_{vk}(\chi)$ and $Q_{vk}(\chi)$ Dirac radial functions are solutions of first-order coupled differential equations and also that some intermediate integrals could be replaced exactly by their asymptotic flat-space values. Since in the present paper, we are concerned just with the computation of the $1/R^2$ contributions to the pseudoradial integrals leading, at the asymptotic flat-space limit, to radial r^s integrals with $s\geq 0$, the knowledge of closed-form expressions of the pseudoradial $(R\chi)^s$ integrals is sufficient. This will be shown in the following sections.

Using Eqs. (3), (5), and (6), setting $W=0$ and introducing the curved-space form of the Coulombic potential, one obtains (in a.u.) the following coupled equations for the Dirac pseudoradial hydrogenic functions $P_{vk}(\chi)$ and $Q_{vk}(\chi)$ in the spherical three-space:

$$\begin{aligned}\frac{1}{R}\left[\frac{d}{d\chi}+\frac{k}{\sin\chi}\right]P_{vk}(\chi) &=-\left[(1+\epsilon)c+\frac{Z\alpha}{R}\cot\chi\right]Q_{vk}(\chi) ,\\ \frac{1}{R}\left[\frac{d}{d\chi}-\frac{k}{\sin\chi}\right]Q_{vk}(\chi) &=-\left[(1-\epsilon)c-\frac{Z\alpha}{R}\cot\chi\right]P_{vk}(\chi) ,\end{aligned} \tag{12}$$

where $Q_{vk}(\chi)$ can be identified with the traditional small component. The associated boundary and normalization conditions are

$$\begin{aligned}&P_{vk}(0)=Q_{vk}(0)=P_{vk}(\pi)=Q_{vk}(\pi)=0 ,\\ &\int_0^\pi(P_{vk}^2+Q_{vk}^2)d\chi=1 .\end{aligned} \tag{13}$$

Then, combining Eq. (12) for $P=P_{vk}$ and $Q=Q_{vk}$ together with their companions for $P'=P_{v'k'}$ and $Q'=Q_{v'k'}$, one can write

$$\begin{aligned}&\frac{1}{R}\frac{d}{d\chi}(P'P+Q'Q)+\frac{k'+k}{R\sin\chi}(P'P-Q'Q) =-2c(P'Q+Q'P)+(\epsilon'-\epsilon)c(P'Q-Q'P) ,\\ &\frac{1}{R}\frac{d}{d\chi}(P'P-Q'Q)+\frac{k'+k}{R\sin\chi}(P'P+Q'Q) =-\left[(\epsilon'+\epsilon)c+\frac{2Z\alpha}{R}\cot\chi\right](P'Q+Q'P) ,\\ &\frac{1}{R}\frac{d}{d\chi}(P'Q+Q'P)+\frac{k'-k}{R\sin\chi}(P'Q-Q'P) =-2c(P'P+Q'Q) +\left[(\epsilon'+\epsilon)c+\frac{2Z\alpha}{R}\cot\chi\right](P'P-Q'Q) ,\end{aligned} \tag{14}$$

$$\frac{1}{R}\frac{d}{d\chi}(P'Q-Q'P)+\frac{k'-k}{R\sin\chi}(P'Q+Q'P)=-(\epsilon'-\epsilon)c(P'P+Q'Q)$$

Let us now consider the determination of the pseudoradial $(R\chi)^s$ integrals and let us set

$$\begin{aligned}
\mathcal{I}_s(v',k';v,k)&=\int_0^{\pi}(R\chi)^s(P'P+Q'Q)d\chi\ ,\\
\mathcal{J}_s(v',k';v,k)&=c\int_0^{\pi}(R\chi)^s(P'Q+Q'P)d\chi\ ,\\
\mathcal{K}_s(v',k';v,k)&=\int_0^{\pi}(R\chi)^s(P'P-Q'Q)d\chi\ ,\\
\mathcal{L}_s(v',k';v,k)&=c\int_0^{\pi}(R\chi)^s(P'Q-Q'P)d\chi\ .
\end{aligned}\tag{15}$$

Multiplying both sides of Eq. (14) by $(R\chi)^{s-1}R\sin\chi$ and introducing the expansions

$$R\sin\chi=(R\chi)-\frac{1}{6R^2}(R\chi)^3+\cdots$$

and

$$\cos\chi=1-\frac{1}{2R^2}(R\chi)^2+\cdots\ ,$$

one gets, after integrating by parts and taking into account the boundary condition (13)

$$\begin{aligned}
&s\mathcal{I}_{s-1}-2\mathcal{J}_s-(k'+k)\mathcal{K}_{s-1}+(\epsilon'-\epsilon)\mathcal{L}_s=\frac{1}{6R^2}X_1(s)+O\left[\frac{1}{R^4}\right],\\
&(k'+k)\mathcal{I}_{s-1}+(\epsilon'+\epsilon)\mathcal{J}_s+2Z\alpha^2\mathcal{J}_{s-1}-s\mathcal{K}_{s-1}=\frac{1}{6R^2}X_2(s)+O\left[\frac{1}{R^4}\right],\\
&2\mathcal{I}_s-s\alpha^2\mathcal{J}_{s-1}-(\epsilon'+\epsilon)\mathcal{K}_s-2Z\alpha^2\mathcal{K}_{s-1}+(k'-k)\alpha^2\mathcal{L}_{s-1}=\frac{1}{6R^2}X_3(s)+O\left[\frac{1}{R^4}\right],\\
&(\epsilon'-\epsilon)\mathcal{I}_s+(k'-k)\alpha^2\mathcal{J}_{s-1}-s\alpha^2\mathcal{L}_{s-1}=\frac{1}{6R^2}X_4(s)+O\left[\frac{1}{R^4}\right],
\end{aligned}\tag{16}$$

where

$$\begin{aligned}
X_1(s)&=(s+2)\mathcal{I}_{s+1}-2\mathcal{J}_{s+2}+(\epsilon'-\epsilon)\mathcal{L}_{s+2}\ ,\\
X_2(s)&=(\epsilon'+\epsilon)\mathcal{J}_{s+2}+6Z\alpha^2\mathcal{J}_{s+1}-(s+2)\mathcal{K}_{s+1}\ ,\\
X_3(s)&=2\mathcal{I}_{s+2}-(s+2)\alpha^2\mathcal{J}_{s+1}-(\epsilon'+\epsilon)\mathcal{K}_{s+2}-6Z\alpha^2\mathcal{K}_{s+1}\ ,\\
X_4(s)&=(\epsilon'-\epsilon)\mathcal{I}_{s+2}-(s+2)\alpha^2\mathcal{L}_{s+1}\ .
\end{aligned}$$

As long as we are concerned solely by the $1/R^2$ contributions in the integrals (15), it is easily inferred that the integrals appearing in the $X_i(s)$ can be replaced by their effective values, i.e., their asymptotic flat-space limits $(R\to\infty,\chi\to 0,R\chi=r)$ which are

$$\begin{aligned}
\bar{\mathcal{I}}_s(v',k';v,k)&=\int_0^{\infty}r^s(\bar{P}'\bar{P}+\bar{Q}'\bar{Q})dr\ ,\\
\bar{\mathcal{J}}_s(v',k';v,k)&=c\int_0^{\infty}r^s(\bar{P}'\bar{Q}+\bar{Q}'\bar{P})dr\ ,\\
\bar{\mathcal{K}}_s(v',k';v,k)&=\int_0^{\infty}r^s(\bar{P}'\bar{P}-\bar{Q}'\bar{Q})dr\ ,\\
\bar{\mathcal{L}}_s(v',k';v,k)&=c\int_0^{\infty}r^s(\bar{P}'\bar{Q}-\bar{Q}'\bar{P})dr\ .
\end{aligned}\tag{17}$$

where $\bar{P}=\bar{P}_{vk}(r)$, $\bar{Q}=\bar{Q}_{vk}(r)$ are the flat-space Dirac radial functions. Moreover, noting that these flat-space integrals verify the flat-space asymptotic limit of Eq. (16), one can replace in Eq. (16), the $X_i(s)$ by the following expressions:

$$\begin{aligned}
\bar{X}_1(s)&=(k'+k)\bar{\mathcal{K}}_{s+1}\ ,\\
\bar{X}_2(s)&=-(k'+k)\bar{\mathcal{I}}_{s+1}+4Z\alpha^2\bar{\mathcal{J}}_{s+1}\ ,\\
\bar{X}_3(s)&=-(k'-k)\alpha^2\bar{\mathcal{L}}_{s+1}-4Z\alpha^2\bar{\mathcal{K}}_{s+1}\ ,\\
\bar{X}_4(s)&=-(k'-k)\alpha^2\bar{\mathcal{J}}_{s+1}\ .
\end{aligned}\tag{18}$$

In the present paper, we shall limit our investigation to the determination of the particular diagonal $(v'=v,k'=k)$ and off-diagonal $(v'=v,k'=-k)$ integrals which occur when studying the Zeeman and Stark effects, respectively. As it is shown in the Appendix A, for these particular cases, the Eqs. (16) become simpler and closed-form expressions of the pseudoradial integrals (15) in terms of ϵ, k, and of the $\bar{X}_i(s)$ are obtainable. Moreover, for these cases, compact expressions of the flat-space integrals (17) have been obtained[10] in terms of $\bar{\epsilon}$ and k, thus leading to closed-form expressions of the $\bar{X}_i(s)$ (see Appendix B). Finally, one gets closed-form expressions of the pseudoradial integrals in terms of ϵ, $\bar{\epsilon}$, and k. Particularly, one finds

$$\begin{aligned}
K_0(v'=v;k'=k)&=\epsilon+\frac{\alpha^2}{R^2}\left[\frac{1}{2}k-\frac{Z^2\alpha^2\bar{\epsilon}}{1-\bar{\epsilon}^2}\right]+O\left[\frac{1}{R^4}\right],\\
J_1(v'=v;k'=k)&=\tfrac{1}{2}(1-2\epsilon k)+\frac{\alpha^2k}{12(1-\bar{\epsilon}^2)R^2}\left[9\bar{\epsilon}^2k+3\bar{\epsilon}k^2-\bar{\epsilon}-6k-\frac{Z^2\alpha^2\bar{\epsilon}(14\bar{\epsilon}^2-9)}{1-\bar{\epsilon}^2}\right]+O\left[\frac{1}{R^4}\right],
\end{aligned}\tag{19}$$

and

$$\mathcal{I}_1(v'=v;k'=-k)=\frac{3Z\alpha^2}{2(1-\epsilon^2)}\left[\epsilon\,\mathcal{I}_0+\frac{\alpha^2\bar{\mathcal{I}}_0}{2R^2(1-\bar{\epsilon}^2)}\left[-\tfrac{5}{9}\bar{\epsilon}+\tfrac{1}{6}k(3+\bar{\epsilon}k)(1+\bar{\epsilon}^2)-\frac{Z^2\alpha^2\bar{\epsilon}}{9(1-\bar{\epsilon}^2)}(4\bar{\epsilon}^2+21)\right]\right],\tag{20}$$

where

$$\bar{\Gamma}_0=-\frac{\gamma}{Z\alpha}\left[\frac{\bar{\epsilon}^2k^2}{\gamma^2}-1\right]^{1/2} \quad \text{and} \quad \Gamma_0=\bar{\Gamma}_0\left[1-\frac{\alpha^2\gamma^2}{R^2(1-\bar{\epsilon}^2)}\right]+O\left[\frac{\alpha^2}{R^2}\right] .$$

When retaining the terms up to α^2 in Eqs. (19) and (20), and introducing the radial quantum number $n=v+|k|$, one gets

$$K_0(v'=v;k'=k)=1-\frac{Z^2\alpha^2}{2n^2}+\frac{\alpha^2}{4R^2}(k-2n^2)+O\left[\frac{1}{R^4}\right] ,$$

$$J_1(v'=v;k'=k)=\tfrac{1}{2}(1-2k)+\frac{Z^2\alpha^2k}{2n^2}$$
$$-\frac{k}{12Z^2R^2}\Bigg[5n^4+n^2-3n^2l(l+1)$$
$$+\frac{Z^2\alpha^2}{2}\left[1-n^2-3k^2+6k-\frac{2n}{|k|}[10n^2-3l(l+1)+1]\right]\Bigg]+O\left[\frac{1}{R^4}\right] , \quad (21)$$

and

$$\Gamma_1(v'=v;k'=-k)=-\frac{3}{2Z}\text{sgn}(k)\left[1-\frac{k^2}{n^2}\right]^{1/2}\left[n^2-\frac{Z^2\alpha^2n(2n+|k|)}{2|k|(n+|k|)}-\frac{n^4}{18Z^2R^2}(7n^2+15k^2+5)\right]+O\left[\frac{\alpha^2}{R^2}\right] . \quad (22)$$

Let us remark that using expression (19) for $K_0(v'=v;k'=k)$ together with the normalization condition $I_0(v'=v;k'=k)=1$, one finds

$$\int_0^{\pi}Q_{vk}^2d\chi=\tfrac{1}{2}(1-\epsilon)-\frac{\alpha^2}{2R^2}\left[\frac{k}{2}-\frac{Z^2\alpha^2\bar{\epsilon}}{1-\bar{\epsilon}^2}\right]+O\left[\frac{1}{R^4}\right] . \quad (23)$$

At the asymptotic flat-space limit, this expression merely reduces to the previous result of Crubellier and Feneuille,[16] obtained by simultaneously using the factorization method and group-theoretical $[O(2,1)]$ considerations, or alternatively found again recently[17] by direct calculations, i.e.,

$$\int_0^{\infty}\bar{Q}_{vk}^2dr=\tfrac{1}{2}(1-\bar{\epsilon}) .$$

Of course, from Eqs. (16), one can derive recurrence relations allowing the determination of closed-form expressions of the $(v'=v,k'=k$ or $k'=-k)$ $(R\chi)^s$ pseudoradial integrals in terms of ϵ, $\bar{\epsilon}$, and k, for any positive value of s. These relations, which are quite similar to the flat-space ones,[10] are not reproduced in the present paper.

IV. PSEUDORADIAL DEPENDENCE OF STATIC FIELDS IN SPHERICAL THREE-SPACE

Before investigating the particular case of static fields in a spherical three-space and deriving the expressions of the interaction terms leading to the Zeeman and Stark effects, let us first briefly recall and comparatively examine the main features of the general solutions of Maxwell's equations in the flat and in the spherical three-space.

A. General solutions of Maxwell's equations

As it is well known (see, for instance, Ref. 18), when working in a curved space-time instead of the usual flat space-time, Maxwell's equations have to be derived from their generally covariant form

$$\frac{\partial}{\partial x^{\nu}}F_{\lambda\mu}+\frac{\partial}{\partial x^{\lambda}}F_{\mu\nu}+\frac{\partial}{\partial x^{\mu}}F_{\nu\lambda}=0 ,$$
$$\frac{\partial}{\partial x^{\mu}}[(-g)^{1/2}g^{\lambda\nu}g^{\mu\xi}F_{\nu\xi}]=0 , \quad (24)$$

where $\mu,\nu,\lambda,\xi=0,1,2,3$, $g_{\mu\nu}$ or $g^{\mu\nu}$ are the covariant or contravariant components of the metric tensor in the line element $ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}$, and the Einstein summation convention is used; $g=\det|g_{\mu\nu}|$.

The electromagnetic-field antisymmetric tensor components $F_{\mu\nu}$ are related to the electromagnetic-potential vector components A_{μ} by the following definition:

$$F_{\mu\nu}=\frac{\partial}{\partial x^{\mu}}A_{\nu}-\frac{\partial}{\partial x^{\nu}}A_{\mu} . \quad (25)$$

Infeld and Schild[11] have already examined the solutions of the Maxwell's equations (24) in a space of constant positive curvature with line-element (1). Particularly, they have given general expressions of the potential vector components A_{μ}

$$A_t=Ne^{\text{inct}}\frac{1}{R}\frac{d\mathscr{B}_n^l}{d\chi}P_l^m\times\begin{cases}\sin(m\phi)\\ \cos(m\phi)\end{cases} ,$$

$$A_{\chi}=iNne^{\text{inct}}\mathscr{B}_n^lP_l^m\times\begin{cases}\sin(m\phi)\\ \cos(m\phi)\end{cases} ,$$

$$A_{\theta}=-Nme^{\text{inct}}\mathscr{B}_n^l\frac{1}{\sin\theta}P_l^m\times\begin{cases}\sin(m\phi)\\ \cos(m\phi)\end{cases} , \quad (26)$$

$$A_{\phi}=-Ne^{\text{inct}}\mathscr{B}_n^l\sin\theta\frac{dP_l^m}{d\theta}\times\begin{cases}\cos(m\phi)\\ -\sin(m\phi)\end{cases} .$$

The associated electromagnetic field components $F_{\mu\nu}$ are

$$
\begin{aligned}
F_{\chi t} &= N l(l+1) e^{inct} \frac{\mathscr{B}_n^l}{R^2 \sin^2\chi} P_l^m \times \begin{Bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}, \\
F_{\theta t} &= N e^{inct} \left[\frac{1}{R} \frac{d\mathscr{B}_n^l}{d\chi} \frac{dP_l^m}{d\theta} + imn \mathscr{B}_n^l \frac{P_l^m}{\sin\theta} \right] \times \begin{Bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}, \\
F_{\phi t} &= N e^{inct} \left[\frac{m}{R} \frac{d\mathscr{B}_n^l}{d\chi} P_l^m + in \mathscr{B}_n^l \sin\theta \frac{dP_l^m}{d\theta} \right] \times \begin{Bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{Bmatrix}, \\
F_{\theta\phi} &= N l(l+1) e^{inct} \mathscr{B}_n^l \sin\theta P_l^m \times \begin{Bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{Bmatrix}, \\
F_{\phi\chi} &= N e^{inct} \left[\frac{1}{R} \frac{d\mathscr{B}_n^l}{d\chi} \sin\theta \frac{dP_l^m}{d\theta} + imn \mathscr{B}_n^l P_l^m \right] \times \begin{Bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{Bmatrix}, \\
F_{\chi\theta} &= -N e^{inct} \left[m \frac{1}{R} \frac{d\mathscr{B}_n^l}{d\chi} \frac{1}{\sin\theta} P_l^m + in \mathscr{B}_n^l \frac{dP_l^m}{d\theta} \right] \times \begin{Bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}.
\end{aligned} \tag{27}
$$

In Eqs. (26) and (27) $N = N_{nlm}$ is a free arbitrary complex constant, $P_l^m = P_l^m(\cos\theta)$ are associated Legendre functions, $|m| \leq l$ in order to ensure the uniqueness of the field components in physical space. The $\mathscr{B}_n^l = \mathscr{B}_n^l(\chi)$ functions are solutions of the differential equation

$$
\left[\frac{d^2}{d\chi^2} - \frac{l(l+1)}{\sin^2\chi} + n^2 \right] \mathscr{B}_n^l = 0 . \tag{28}
$$

Since this differential equation (28) is a Infeld-Hull[13] (type A or E) factorizable equation, the $\mathscr{B}_n^l$ functions, satisfying the boundary conditions of the problem under consideration, can be obtained from the knowledge of one of them ($\mathscr{B}_n^0$ or $\mathscr{B}_n^n$, for instance) by means of the up and down ladder operators

$$
\begin{aligned}
\mathscr{B}_n^{l-1} &= \left[l \cot\chi + \frac{d}{d\chi} \right] \mathscr{B}_n^l , \\
\mathscr{B}_n^{l+1} &= \left[(l+1)\cot\chi - \frac{d}{d\chi} \right] \mathscr{B}_n^l .
\end{aligned} \tag{29}
$$

In flat space, when using polar coordinates, the electromagnetic-potential and -field tensor components are also given by Eqs. (26) and (27) where the pseudoradial functions $\mathscr{B}_n^l(\chi)$ have to be replaced by the $\overline{\mathscr{B}}_n^l = \overline{\mathscr{B}}_n^l(r)$ radial functions which are solutions of the equations

$$
\begin{aligned}
&\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + n^2 \right] \overline{\mathscr{B}}_n^l = 0 . \\
&\overline{\mathscr{B}}_n^{l-1} = \left[\frac{l}{r} + \frac{d}{dr} \right] \overline{\mathscr{B}}_n^l , \\
&\overline{\mathscr{B}}_n^{l+1} = \left[\frac{l+1}{r} - \frac{d}{dr} \right] \overline{\mathscr{B}}_n^l .
\end{aligned} \tag{30}
$$

This parallelism between the expressions of the electromagnetic fields when switching from the flat-space description (in r, θ, ϕ) to the spherical-space description (in χ,θ,ϕ) and conversely, will be particularly useful for investigating the Zeeman and Stark effects in the spherical three-space.

B. Static fields

For $n=0$, the expression (26)–(30) yield the static fields in spherical three-space and in flat space. From the asymptotic flat-space limit of Eqs. (27) ($R \to \infty, \chi \to 0$, $R\chi = r$), one finds again the already known expressions of the static electromagnetic field components in polar coordinates[19]

$$
\begin{aligned}
\bar{F}_{rt} &= N l(l+1) \frac{1}{r^2} \overline{\mathscr{B}}_0^l P_l^m \times \begin{Bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}, \\
\bar{F}_{\theta t} &= N \frac{d\overline{\mathscr{B}}_0^l}{dr} \frac{dP_l^m}{d\theta} \times \begin{Bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}, \\
\bar{F}_{\phi t} &= N m \frac{d\overline{\mathscr{B}}_0^l}{dr} P_l^m \times \begin{Bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{Bmatrix}, \\
\bar{F}_{\theta\phi} &= N l(l+1) \overline{\mathscr{B}}_0^l \sin\theta P_l^m \times \begin{Bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{Bmatrix}, \\
\bar{F}_{\phi r} &= N \frac{d\overline{\mathscr{B}}_0^l}{dr} \sin\theta \frac{dP_l^m}{d\theta} \times \begin{Bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{Bmatrix}, \\
\bar{F}_{r\theta} &= -N m \frac{d\overline{\mathscr{B}}_0^l}{dr} \frac{1}{\sin\theta} P_l^m \times \begin{Bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{Bmatrix},
\end{aligned} \tag{31}
$$

where, according to the boundary conditions to be satisfied for the problem under consideration, the function $\overline{\mathscr{B}}_0^l$ is one (or a combination) of the two following radial functions:

$$
\bar{f}_l(r) = 1/r^l , \quad \bar{g}_l(r) = r^{l+1} . \tag{32}
$$

The first one corresponds to static field with singularities at the origin and the other to static fields vanishing at the origin. They can be generated by iterative use of Eqs. (30) when starting from the particular ($n=0, l=0$) solutions $\bar{f}_0 = 1$ and $\bar{g}_0 = r$.

In spherical three-space, the two particular ($n=0, l=0$) solutions of Eq. (28) which, at the asymptotic flat-space limit lead to the $\bar{f}_0$ and $\bar{g}_0$ functions are found to be $f_0 = 1$ and $g_0 = R\chi$, respectively. They generate the two families

$$f_0=1\ ,\quad f_1=\frac{1}{R}\cot\chi\ ,\quad f_2=(3\cot^2\chi+1)/3R^2\ ,$$
$$f_3=(15\cot^3\chi+9\cot\chi)/15R^3\ ,\quad \ldots\ ,$$
$$g_0=R\chi\ ,\quad g_1=3R^2(1-\chi\cot\chi)\ , \tag{33}$$
$$g_2=\tfrac{15}{4}R^3[\chi(3\cot^2\chi+1)-3\cot\chi]\ ,$$
$$g_3=\tfrac{35}{12}R^4[\chi(15\cot^3\chi+9\cot\chi)-15\cot^2\chi-4]\ .$$

From the expression (26) of the electromagnetic potential vector, it is seen that the pseudoradial parts of the magnetostatic potentials $(0,A_\chi,A_\theta,A_\phi)$ are just the functions $\mathscr{B}^l_0(\chi)$ while the the pseudoradial parts of the electrostatic potentials $(A_t,0,0;0)$ are the derivatives $(1/R)(d\mathscr{B}^l_0/d\chi)$. One gets the following (electrostatic) families:

$$F_l(\chi)\simeq\frac{1}{R}\frac{d}{d\chi}f_l \quad\text{and}\quad G_l(\chi)\simeq\frac{1}{R}\frac{d}{d\chi}g_l\ ,$$

leading to the asymptotic flat-space limits $(1/r)^{l+1}$ and r^l, respectively,

$$F_1=(1+\cot^2\chi)/R^2\ ,\quad F_2=\cot\chi(1+\cot^2\chi)/R^3\ ,$$
$$F_3=\tfrac{1}{5}(5\cot^4\chi+6\cot^2\chi+1)/R^4\ ,\quad \ldots\ ,$$
$$G_1=\tfrac{3}{2}R[\chi(\cot^2\chi+1)-\cot\chi]\ , \tag{34}$$
$$G_2=-\tfrac{5}{2}R^2[3\chi\cot\chi(\cot^2\chi+1)-3\cot^2\chi-2]\ ,$$
$$G_3=\tfrac{35}{16}[\chi(15\cot^4\chi+18\cot^2\chi+3)$$
$$-15\cot^3\chi-13\cot\chi]\ ,\quad \ldots\ .$$

Thus, it can be expected that the pseudoradial integrals leading to the same flat-space limit will have, in spherical three-space, different space-curvature dependence according to their magnetostatic or electrostatic origin. This result is to be compared in some respects to the differentiation occurring between the relativistic expressions of some magnetic and electric parameters leading to the same nonrelativistic limit such as the dipolar magnetic and quadrupolar electric hyperfine $\langle r^{-3}\rangle$ parameters.

Let us remark that the $F_l(\chi)$ and $G_l(\chi)$ functions have been found by a quite different way, when establishing the curved-space form of the multipolar expansion of the bielectronic Coulomb potential

$$\frac{1}{R}\cot\omega_{ij}=\frac{1}{R}\cot\chi_>+\sum_l(C_i^{(l)}C_j^{(l)})F_l(\chi_>)G_l(\chi_<)\ , \tag{35}$$

where ω_{ij} is the "angular separation" on the hypersphere between electrons i and j and $C_i^{(l)}=C^{(l)}(\theta_i,\phi_i)$ is a spherical tensor. As already pointed out,[3] this expansion (35) is the curved-space homolog of the Laplace multipole expansion

$$\frac{1}{r_{ij}}=\frac{1}{r_>}+\sum_l(C_i^{(l)}C_j^{(l)})(r_<^l/r_>^{l+1})\ . \tag{36}$$

V. EXTERNAL UNIFORM STATIC FIELDS

In order to write down the interaction terms leading to the Zeeman and Stark effects in spherical three-space, let us first recall that, in flat space, the Cartesian components of the electromagnetic field are related to the antisymmetric field tensor by the following equations:

$$\tilde F^{tx}=E_x\ ,\quad \tilde F^{ty}=E_y\ ,\quad \tilde F^{tz}=E_z\ ,$$
$$\tilde F^{xy}=B_z\ ,\quad \tilde F^{yz}=B_x\ ,\quad \tilde F^{zx}=B_y\ . \tag{37}$$

A. Zeeman effect in spherical three-space

Let us assume that the uniform magnetic field $\mathscr{H}$ is directed along the z axis. Then, in Cartesian coordinates, the associated flat-space antisymmetric field tensor components are

$$\tilde F^{xy}=-\tilde F^{yx}=\mathscr{H}\ ,\quad \text{all other } \tilde F^{\xi\eta}\equiv0\ . \tag{38}$$

Consequently, in polar coordinates $(x^\mu,x^\nu=t,r,\theta,\phi)$, the field tensor components are given by the expression

$$\bar F^{\mu\nu}=\left[\frac{\partial x^\mu}{\partial x}\frac{\partial x^\nu}{\partial y}-\frac{\partial x^\nu}{\partial x}\frac{\partial x^\mu}{\partial y}\right]\mathscr{H}\ ,$$

and one gets

$$\bar F^{\phi r}=\mathscr{H}/r\ ,\quad \bar F^{\theta\phi}=-(\mathscr{H}/r^2)\cot\theta\ ,$$
$$\text{all other } \bar F^{\mu\nu}\equiv0\ . \tag{39}$$

Using metric (2), one gets the covariant field tensor components

$$\bar F_{\phi r}=\mathscr{H}r\sin^2\theta\ ,\quad \bar F_{\theta\phi}=-(\mathscr{H}r^2)\sin\theta\cos\theta\ ,$$
$$\text{all other } \bar F_{\mu\nu}\equiv0\ . \tag{40}$$

It can be noted that these expressions are a particular case of expressions (31) with $l=1$, $m=0$, $N=\frac{3}{2}\mathscr{H}$, and $\bar B^1_0=\frac{1}{3}r^2=\frac{1}{3}\bar g_1(r)$ and correspond to a magnetic potential with components $\bar A_\phi=-\frac{1}{2}\mathscr{H}r^2\sin^2\theta$, $\bar A_r=\bar A_\theta=0$.

Keeping in mind that in spherical three-space, the pseudoradial parts of the magnetostatic fields are given by Eqs. (33) and owing to the above remark, one deduces that the uniform magnetic static potential and field components are given by Eqs. (26) and (27) when setting $n=0$, $l=1$, $m=0$, and

$$B^1_0=\tfrac{1}{3}g_1(\chi)=R^2(1-\chi\cot\chi)\ .$$

One gets

$$A_\phi=\tfrac{3}{2}\mathscr{H}R^2(\chi\cot\chi-1)\sin^2\theta\ ,\quad A_\chi=A_\theta=0\ ,$$
$$F_{\theta\phi}=3\mathscr{H}R^2(\chi\cot\chi-1)\sin\theta\cos\theta\ , \tag{41}$$
$$F_{\phi\chi}=\tfrac{3}{2}\mathscr{H}R[\chi(1+\cot^2\chi)-\cot\chi]\sin^2\theta\ .$$

Substituting in Eq. (4) the expression (41) for A_ϕ, one gets the interaction term in the Dirac Hamiltonian associated with a magnetic field in the z direction

$$W_z=\frac{e}{\hbar c}\tfrac{3}{2}\mathscr{H}R\frac{\chi\cot\chi-1}{\sin\chi}\sin\theta\begin{bmatrix}0 & \sigma_\phi\\ \sigma_\phi & 0\end{bmatrix}\ . \tag{42}$$

Since

$$(\sin\theta)\sigma_\phi=-i\sqrt{2}(C^{(1)}\sigma^{(1)})^{(1)}_0\ ,$$

it is easily inferred that the interaction term associated with an arbitrarily directed field is

$$\mathscr{W}=\frac{3eR}{\hbar c\sqrt{2}}\frac{(1-\chi\cot\chi)}{\sin\chi}\times\begin{bmatrix}0 & i(C^{(1)}\sigma^{(1)})^{(1)}\\ i(C^{(1)}\sigma^{(1)})^{(1)} & 0\end{bmatrix}\cdot\mathscr{H}\ . \tag{43}$$

In order to investigate the space-curvature-induced modifications of the Zeeman splitting, it is convenient to derive the curved-space form of the Landé g factor which is defined by the relation

$$\langle\phi_{vk}\,|\,\mathscr{W}\,|\,\phi_{vk}\rangle=\beta_e g\langle\phi_{vk}\,|\,\mathscr{H}\cdot\mathbf{j}\,|\,\phi_{vk}\rangle\ , \tag{44}$$

where $\beta_e=e\hbar/2m_0c$ is the Bohr magneton. Using the expression (43) for $\mathscr{W}$ together with the expressions for the spin-angular reduced matrix elements,[20] i.e.,

$$\langle l\tfrac{1}{2}j\,||(C^{(1)}\sigma^{(1)})^{(1)}||\,\bar{l}\tfrac{1}{2}j\rangle=(-1)^{l+j+1/2}[(j+\tfrac{1}{2})2j(j+1)]^{-1/2},$$

$$\langle l\tfrac{1}{2}j\,||\mathbf{j}||\,l\tfrac{1}{2}j\rangle=[j(j+1)(2j+1)]^{1/2}\ ,$$

one gets

$$g=(-1)^{l+j-1/2}\frac{(2j+1)}{j(j+1)}c\times\int_0^\pi P_{vk}Q_{vk}\frac{3R(\chi\cot\chi-1)}{\sin\chi}d\chi\ . \tag{45}$$

Noting that

$$3R(\chi\cot\chi-1)/\sin\chi=-(R\chi)-\frac{7}{30R^2}(R\chi)^3+\cdots\ ,$$

one can express the pseudoradial integral in Eq. (45) in terms of the already known integrals J_1 and $\bar{J}_3$ [see Eqs. (19) and (B2), respectively], and one gets

$$g=-\frac{k}{j(j+1)}\left[J_1+\frac{7}{30R^2}\bar{J}_3\right]+O\left[\frac{1}{R^4}\right]\ , \tag{46}$$

or, in terms of ϵ, $\bar{\epsilon}$, and k, after noting that $j(j+1)=k^2-\frac{1}{4}$,

$$g=-\frac{k}{k^2-\frac{1}{4}}\left[\tfrac{1}{2}(1-2\epsilon k)+\frac{\alpha^2}{40R^2(1-\bar{\epsilon}^2)}\left[3k^2(10\bar{\epsilon}^2-9)+\bar{\epsilon}k(24k^2-29)+7+\frac{Z^2\alpha^2\bar{\epsilon}k}{1-\bar{\epsilon}^2}[7(1+4\bar{\epsilon}^2)+8\bar{\epsilon}k(2-7\bar{\epsilon}^2)]\right]\right]\ . \tag{47}$$

At the asymptotic flat-space limit, expression (47) reduces to the relativistic expressions of the flat-space Landé g factor, i.e.,

$$\bar{g}=-k(1-2\bar{\epsilon}k)/(2k^2-\tfrac{1}{2})\ ,$$

including the Breit-Margenau correction.

When retaining in g the terms up to α^2 and since

$$-k=j(j+1)-l(l+1)+\tfrac{1}{4}\ ,$$

the expression (47) reduces to

$$g=1+\frac{j(j+1)-l(l+1)+\frac{3}{4}}{2j(j+1)}-\frac{Z^2\alpha^2k^2}{2n^2j(j+1)}-\frac{k(7-8k)[5n^4+n^2-3n^2l(l+1)]}{40Z^2R^2j(j+1)}\ . \tag{48}$$

The two first terms in Eq. (48) correspond to the well-known nonrelativistic expression of the Landé factor, the second term to the Breit-Margenau correction and the remaining terms correspond to additional curvature contributions which vanish at the asymptotic flat-space limit. Let us remark that while the Breit-Margenau correction is the same for the two degenerate states $l=j-\frac{1}{2}$ and $l=j+\frac{1}{2}$ of each fine-structure energy level n,j, this is not the case for the space-curvature modifications which depend on the sign of k. Moreover, as n increases the Breit-Margenau correction decreases (as $1/n^2$) while the space-curvature modifications increase (as n^4). Of course, these curvature modifications should be detectable only in regions where the local curvature could be really important. For the case of highly excited states $n\simeq100$, these curvature effects should be comparable to the Breit-Margenau effect in regions where the local curvature radius is of the order of magnitude of few centimeters. For illustrative purpose, space-curvature contributions to the Landé-g factor are reproduced in Table I.

TABLE I. Space-curvature contributions to the Landé g factor.

State	k	Flat space	Breit-Margenau	Curvature effects
$nd_{5/2}$	-3	$\frac{6}{5}$	$-\frac{18Z^2\alpha^2}{35n^2}$	$\frac{93n^2(5n^2-17)}{350Z^2R^2}$
$nd_{3/2}$	2	$\frac{4}{5}$	$-\frac{8Z^2\alpha^2}{15n^2}$	$\frac{3n^2(5n^2-17)}{25Z^2R^2}$
$np_{3/2}$	-2	$\frac{4}{3}$		$\frac{23n^2(n^2-1)}{15Z^2R^2}$
$np_{1/2}$	1	$\frac{2}{3}$	$-\frac{2Z^2\alpha^2}{3n^2}$	$\frac{n^2(n^2-1)}{6Z^2R^2}$
$ns_{1/2}$	-1	2		$\frac{n^2(5n^2+1)}{2Z^2R^2}$

B. Stark effect in spherical three-space

The main part of this section is in complete analogy with the preceeding section. Let us assume that the uniform electric field $\mathscr{E}$ is directed along the z axis. The, the associated flat-space antisymmetric field tensor components are, in Cartesian coordinates

$$\bar{F}^{tz} = -\bar{F}^{zt} = \mathscr{E} \ , \quad \text{all other } \bar{F}^{\xi\nu} \equiv 0 \ . \tag{49}$$

The field tensor contravariant components are given in polar coordinates $(x^\mu, x^\nu = t, r, \theta, \phi)$ by the expression

$$\bar{F}^{\mu\nu} = \left[\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial z} - \frac{\partial x^\nu}{\partial t}\frac{\partial x^\mu}{\partial z} \right] \mathscr{E} \ .$$

One gets

$$\bar{F}^{rt} = -\mathscr{E}\cos\theta \ , \quad \bar{F}^{\theta t} = (1/r)\mathscr{E}\sin\theta \ , \quad \text{all other } \bar{F}^{\mu\nu} \equiv 0 \ . \tag{50}$$

The covariant components are

$$\bar{F}_{rt} = \mathscr{E}\cos\theta \ , \quad \bar{F}_{\theta t} = -\mathscr{E} r \sin\theta \ , \quad \text{all other } \bar{F}_{\mu\nu} \equiv 0 \ . \tag{51}$$

These expressions are particular cases of expressions (31) with $n=0$, $l=1$, $m=0$, $N=\frac{3}{2}\mathscr{E}$, and $\overline{\mathscr{B}}{}_0^1 = \frac{1}{3}r^2$. The associated electric potential is

$$A_t = \mathscr{E} r\cos\theta \ , \quad A_\chi = A_\theta = A_\phi = 0 \ . \tag{52}$$

Then, in spherical three-space, the uniform electric field components and electric potential follow from Eqs. (27) and (26) with $n=0$, $l=1$, $m=0$, and $\mathscr{B}_0^1 = R^2(1-\chi\cot\chi)$. One gets

$$A_t = \tfrac{3}{2}\mathscr{E}R[\chi(\cot^2\chi+1)-\cot\chi]\cos\theta \ ,$$

$$F_{\chi t} = 3\mathscr{E}R\left[\frac{1-\chi\cot\chi}{\sin^2\chi}\right]\cos\theta \ , \tag{53}$$

$$F_{\theta t} = -\tfrac{3}{2}\mathscr{E}R[\chi(\cot^2\chi+1)-\cot\chi]\sin\theta \ .$$

Keeping in mind that, in terms of the spherical tensor components, $\cos\theta = C_0^{(1)}$, it is easily inferred that the electric potential associated with a uniform, arbitrarily directed electric field is

$$\mathscr{U} = \tfrac{3}{2}R[\chi(\cot^2\chi+1)-\cot\chi]C^{(1)}\cdot\mathscr{E} \ . \tag{54}$$

In the Stark effect where the electric field is weak compared with the separation of neighboring fine-structure levels, the unperturbed states are assumed to be the solutions of the Dirac-Coulomb equation (3) and the determination of the Stark splittings amounts to evaluation of the matrix elements

$$\langle \phi_{v'k'm'} | \mathscr{U} | \phi_{vkm} \rangle = \mathscr{E}\langle l'j'm' | C_q^{(1)} | ljm \rangle \tfrac{3}{2}R\int_0^\pi (P_{v'k'}P_{vk} + Q_{v'k'}Q_{vk})[\chi(\cot^2\chi+1)-\cot\chi]d\chi \ . \tag{55}$$

Application of standard angular momentum techniques[20] leads to

$$\langle \mathscr{Y}_{l'j'm'} | C_q^{(1)} | \mathscr{Y}_{ljm} \rangle = \langle \mathscr{Y}_{\bar{l}'j'm'} | C_q^{(1)} | \mathscr{Y}_{\bar{l}jm} \rangle = (-1)^{j'-m'} \begin{pmatrix} j' & 1 & j \\ -m' & q & m \end{pmatrix} \langle l'j' || C^{(1)} || lj \rangle \ ,$$

with the associated selection rule $l' = l \pm 1$.

After noting that

$$\tfrac{3}{2}R[\chi(\cot^2\chi+1)-\cot\chi] = (R\chi) + \frac{2}{15R^2}(R\chi)^3 + \cdots \ ,$$

the pseudoradial integral in Eq. (55) can be expressed in terms of the pseudoradial integral $\mathscr{I}_1(v',k';v,k)$ and of the flat-space integral $\bar{\mathscr{I}}_3(v',k';v,k)$. Finally, one gets

$$\langle \phi_{v'k'm'} | \mathscr{U} | \phi_{vkm} \rangle = (-1)^{m'+1/2} \begin{pmatrix} j' & 1 & j \\ -m' & q & m \end{pmatrix} [(2j+1)(2j'+1)]^{1/2} \begin{pmatrix} j' & 1 & j \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \mathscr{E}\left[\mathscr{I}_1 + \frac{2}{15R^2}\bar{\mathscr{I}}_3\right] + O\left[\frac{1}{R^4}\right] . \tag{56}$$

For illustrative purpose, let us investigate the low-field Stark splittings of a given fine-structure energy level n,j. From the selection rule $l' = l \pm 1$, matrix elements (55) vanish except for $k' = -k$. When the electric field is assumed to be parallel to the quantization axis, one gets

$$\langle \phi_{v,-k,m} | \mathscr{U} | \phi_{vkm} \rangle = -\frac{m}{2j(j+1)}\mathscr{E}\left[\mathrm{I}_1 + \frac{2}{15R^2}\bar{\mathrm{I}}_3 \right] . \tag{57}$$

Closed-form expressions of the pseudoradial integral I_1 and of the flat-space integral $\bar{\mathrm{I}}_3$ have been obtained in terms of ϵ, $\bar{\epsilon}$, and k [see Eqs. (20) and (B3)] and one gets

$$\langle \phi_{v,-k,m} | \mathscr{U} | \phi_{v,k,m} \rangle = \frac{m\mathscr{E}}{4j(j+1)}\frac{\alpha\gamma}{(1-\epsilon^2)}\left[\frac{\epsilon^2k^2}{\gamma^2}-1\right]^{1/2} \times \left[3\epsilon - \frac{\alpha^2}{4R^2(1-\bar{\epsilon}^2)}\left[14\bar{\epsilon}k^2 - k(3+\bar{\epsilon}k)(1+\bar{\epsilon}^2) + \frac{12Z^2\alpha^2\bar{\epsilon}^3}{1-\bar{\epsilon}^2}\right]\right] , \tag{58}$$

or, when retaining the terms up to α^2

$$\langle \phi_{v,-k,m} \mid \mathscr{U} \mid \phi_{vkm} \rangle = \mathrm{sgn}(k)\frac{3m\mathscr{E}}{Z(4k^2-1)}\left[1-\frac{k^2}{n^2}\right]^{1/2} n^2\left[1-\frac{Z^2\alpha^2(2n+|k|)}{2n|k|(n+|k|)}-\frac{n^2k^2}{Z^2R^2}\right]. \quad (59)$$

In Eq. (58), the first term corresponds to the flat-space relativistic expression of the Stark matrix element—while the last terms (in $1/R^2$) correspond to additional space-curvature contributions which vanish at the asymptotic flat-space limit. From expression (59), it is seen that space-curvature corrections increase with n (as n^4) while relativistic effects decrease with n. The study of the hydrogenic Stark effect can be pursued in a spherical three-space in the same way as in flat-space using expressions (58) and (59) instead of their flat-space counterpart.

VI. CONCLUSION

Continuing our study of space-curvature effects in atomic-structure calculations, the space-curvature-induced modifications in the interactions between atoms and external electromagnetic fields have been investigated in the framework of the curved-space Dirac-orbital model. Closed-form expressions of the solutions of the Maxwell equations in a spherical three-space have been obtained. A novel method for computing the required Dirac pseudoradial integrals in terms of the quantum numbers has been devised. Particularly, space-curvature modifications of the hydrogenic low-field Zeeman and Stark splittings have been given in terms of the Dirac energy parameter ϵ and of the Dirac quantum number k. This simplistic and heuristic model which includes global effects coming from the topology of space leads to results which are not especially difficult to obtain or to use in parallel with the traditional flat-space ones: as already pointed out, the later are easily found again at the asymptotic limit $R\to\infty$. All the basic computational material has been prepared for further extension of the model; when considering a space of constant negative curvature (open space), instead of a space of positive curvature (closed space), the procedure of calculation is formally analogous. Abandoning Euclidean geometry in favor of the simplest non-Euclidean geometry may be also considered a preliminary step for further investigations of the gravitational modifications of the spectrum involving more elaborate metrics.

ACKNOWLEDGMENTS

The Laboratoire de Physique des Lasers is "associé au Centre National de la Recherche Scientifique no. 282."

APPENDIX A: CALCULATIONS OF THE $v'=v$, $k'=\pm k$ PSEUDORADIAL INTEGRALS

1. Determination of the $k'=k$ pseudoradial integrals

For $k'=k$, $\epsilon'=\epsilon$, the relations (16) reduce to

$$sI_{s-1}-2J_s-2kK_{s-1}=\frac{k}{3R^2}\overline{K}_{s+1}+O\left[\frac{1}{R^4}\right],$$

$$2kI_{s-1}+2\epsilon J_s+2Z\alpha^2J_{s-1}-sK_{s-1} =\frac{1}{3R^2}(-k\overline{I}_{s+1}+2Z\alpha^2\overline{J}_{s+1})+O\left[\frac{1}{R^4}\right], \quad \text{(A1)}$$

$$2I_s-s\alpha^2J_{s-1}-2\epsilon K_s-2Z\alpha^2K_{s-1} =-\frac{2Z\alpha^2}{3R^2}\overline{K}_{s+1}+O\left[\frac{1}{R^4}\right],$$

where the shortened notation $I_s=\mathscr{I}_s(v,k;v,k),\ldots$ is used. Setting $s=0$ in the first and third Eq. (A1) and $s=1$ in the first and second Eq. (A1), one gets the following system of linear equations allowing the determination of the integrals J_0, J_1, K_0, and K_{-1} in terms of I_0 and of the $v'=v$, $k'=k$ flat-space integrals (17):

$$J_0+kK_{-1}=-\frac{k}{6R^2}\overline{K}_1+O\left[\frac{1}{R^4}\right],$$

$$\epsilon K_0+Z\alpha^2K_{-1}=I_0+\frac{Z\alpha^2}{3R^2}\overline{K}_1+O\left[\frac{1}{R^4}\right],$$

$$J_1+kK_0=\tfrac{1}{2}I_0-\frac{k}{6R^2}\overline{K}_2+O\left[\frac{1}{R^4}\right], \quad \text{(A2)}$$

$$\epsilon J_1+Z\alpha^2J_0-\tfrac{1}{2}K_0=-kI_0-\frac{1}{6R^2}(k\overline{I}_2-2Z\alpha^2\overline{J}_2)+O\left[\frac{1}{R^4}\right].$$

Closed-form expressions of the flat-space integrals appearing in Eq. (A2) have been obtained[10] in terms of

$$\overline{\epsilon}=[1+Z^2\alpha^2/(v+|\gamma|)^2]^{-1/2}$$

and of the quantum number k (see Appendix B). Owing to the normalization condition (13), $I_0=1$ hence the above system can be solved. Particularly, one obtains the expression (19) of $J_1=\mathscr{J}_1(v,k;v,k)$ and $K_0=\mathscr{K}_0(v,k;v,k)$ together with the following expressions:

$$K_{-1}=\frac{1-\epsilon^2}{Z\alpha^2}-\frac{1}{6ZR^2}\left[k(k+4\overline{\epsilon})-\frac{Z^2\alpha^2(1+8\overline{\epsilon}^2)}{1-\overline{\epsilon}^2}\right]+O\left[\frac{1}{R^4}\right],$$

$$J_0=\frac{k(\epsilon^2-1)}{Z\alpha^2}+\frac{k}{4ZR^2}\left[k(k+3\overline{\epsilon})-\frac{Z^2\alpha^2(1+6\overline{\epsilon}^2)}{1-\overline{\epsilon}^2}\right]+O\left[\frac{1}{R^4}\right]. \quad \text{(A3)}$$

When retaining in Eq. (19), the terms up to α^2 and introducing the usual radial quantum number $n=v+|\gamma|$, one gets the expression (21) of $J_1=\mathscr{J}_1(v,k;v,k)$ and of $K_0=\mathscr{K}_0(v,k;v,k)$ together with the following expressions:

$$K_{-1}\simeq\frac{Z}{n^2}-\frac{Z^3\alpha^2}{n^3}\left[\frac{1}{n}-\frac{1}{|k|}\right]+\frac{1}{6ZR^2}\left[3n^2-l(l+1)+Z^2\alpha^2\left[1-\frac{3n}{|k|}+\frac{k}{2n^2}\right]\right],$$

$$J_0\simeq-\frac{kZ}{n^2}+\frac{Z^3\alpha^2k}{n^3}\left[\frac{1}{n}-\frac{1}{|k|}\right]-\frac{k}{4ZR^2}\left[3n^2-l(l+1)+Z^2\alpha^2\left[1-\frac{3n}{|k|}+\frac{k}{2n^2}\right]\right]. \qquad \text{(A4)}$$

2. Determination of the $k'=-k$ pseudoradial integrals

For $k'=-k$, it is easily checked that $\epsilon'-\epsilon=\alpha^2k/2R^2$ and that the relations (16) reduce to

$$s\,\mathsf{I}_{s-1}-2\,\mathsf{J}_s=-\frac{\alpha^2}{2R^2}k\,\bar{\mathsf{L}}_s+O\left[\frac{1}{R^4}\right],$$

$$2\epsilon\,\mathsf{J}_s+2Z\alpha^2\,\mathsf{J}_{s-1}-s\,\mathsf{K}_{s-1}=-\frac{\alpha^2}{2R^2}(k\,\bar{\mathsf{J}}_s-\tfrac{4}{3}Z\,\bar{\mathsf{J}}_{s+1})+O\left[\frac{1}{R^4}\right],$$

$$2\,\mathsf{I}_s-s\alpha^2\,\mathsf{J}_{s-1}-2\epsilon\,\mathsf{K}_s-2Z\alpha^2\,\mathsf{K}_{s-1}-2k\alpha^2\,\mathsf{L}_{s-1}=\frac{\alpha^2}{2R^2}(k\,\bar{\mathsf{K}}_s+\tfrac{2}{3}k\,\bar{\mathsf{L}}_{s+1}-\tfrac{4}{3}Z\,\bar{\mathsf{K}}_{s+1})+O\left[\frac{1}{R^4}\right],$$

$$2k\alpha^2\,\mathsf{J}_{s-1}+s\alpha^2\,\mathsf{L}_{s-1}=\frac{\alpha^2}{2R^2}(k\,\bar{\mathsf{I}}_s-\tfrac{2}{3}k\,\bar{\mathsf{J}}_{s+1})+O\left[\frac{1}{R^4}\right]. \qquad \text{(A5)}$$

where the shortened notation $\mathsf{I}_s=\mathscr{I}_s(v,-k;v,k)$ is used.

Setting $s=0$ into the first equation (A5), $s=1$ into the four Eqs. (A5), $s=2$ into the first equations, one gets the following system of equations, including the curvature contributions up to the $1/R^2$ terms

$$\mathsf{J}_0=\frac{\alpha^2k}{4R^2}\bar{\mathsf{L}}_0,$$

$$\mathsf{I}_0-2\,\mathsf{J}_1=-\frac{\alpha^2k}{2R^2}\bar{\mathsf{L}}_1,$$

$$2\epsilon\,\mathsf{J}_1+2Z\alpha^2\,\mathsf{J}_0-\mathsf{K}_0=\frac{\alpha^2}{R^2}(\tfrac{2}{3}Z\,\bar{\mathsf{J}}_2-\tfrac{1}{2}k\,\bar{\mathsf{J}}_1),$$

$$2\,\mathsf{I}_1-\alpha^2\,\mathsf{J}_0-2\epsilon\,\mathsf{K}_1-2Z\alpha^2\,\mathsf{K}_0-2k\alpha^2\,\mathsf{L}_0=\frac{\alpha^2}{2R^2}(k\,\bar{\mathsf{K}}_1+\tfrac{2}{3}k\,\bar{\mathsf{K}}_2-\tfrac{4}{3}Z\,\bar{\mathsf{K}}_2),$$

$$2k\,\mathsf{J}_0+\mathsf{L}_0=\frac{k}{R^2}(\tfrac{1}{2}\bar{\mathsf{I}}_1-\tfrac{1}{3}\bar{\mathsf{J}}_2),$$

$$\mathsf{I}_1-\mathsf{J}_2=-\frac{\alpha^2k}{4R^2}\bar{\mathsf{L}}_2,$$

$$\epsilon\,\mathsf{J}_2+Z\alpha^2\mathsf{J}_1-\mathsf{K}_1=\frac{\alpha^2}{2R^2}(\tfrac{2}{3}Z\,\bar{\mathsf{J}}_3-\tfrac{1}{2}k\,\bar{\mathsf{J}}_2). \qquad \text{(A6)}$$

Solving this system, one gets closed-form expressions of the pseudoradial integrals I_1, J_0, J_1, J_2, K_0, K_1, and L_0 in terms of one of them, say I_0, and of the $(v'=v;k'=-k)$ flat-space radial integrals. In a last step, these flat-space integrals can all be expressed in terms of $\bar{\epsilon}$, k, and $\bar{\mathsf{I}}_0$ (see Appendix B). One gets the expression (20) of I_1 together with the following expressions:

$$\mathsf{J}_0=0,$$

$$\mathsf{J}_1=\tfrac{1}{2}\,\mathsf{I}_0-\frac{\alpha^2k^2}{8R^2}\bar{\mathsf{I}}_0,$$

$$\mathsf{J}_2=\frac{3Z\alpha^2\epsilon}{2(1-\epsilon^2)}\mathsf{I}_0+\frac{3Z\alpha^4}{4R^2(1-\bar{\epsilon}^2)^2}\left[-\tfrac{5}{9}\bar{\epsilon}+\tfrac{1}{6}k(1+\bar{\epsilon}^2)(3-\bar{\epsilon}k)-\frac{Z^2\alpha^2\bar{\epsilon}(21+4\bar{\epsilon}^2)}{9(1-\bar{\epsilon}^2)}\right]\bar{\mathsf{I}}_0, \qquad \text{(A7)}$$

$$\mathcal{K}_0=\epsilon\mathcal{I}_0+\frac{\alpha^2}{4R^2}\left[k(1-\bar{\epsilon}k)-\frac{4Z^2\alpha^2\bar{\epsilon}}{(1-\bar{\epsilon}^2)}\right]\bar{\mathcal{I}}_0\,,$$

$$\mathcal{K}_1=\frac{3Z\alpha^2}{2(1-\epsilon^2)}\left[\tfrac{1}{3}(1+2\epsilon^2)\mathcal{I}_0+\frac{\alpha^2}{2R^2(1-\bar{\epsilon}^2)}\left[-\tfrac{1}{9}(3+2\bar{\epsilon}^2)+\bar{\epsilon}k+\tfrac{1}{6}k^2(1-\bar{\epsilon}^2+2\bar{\epsilon}^4)-\frac{Z^2\alpha^2}{1-\bar{\epsilon}^2}(3+30\bar{\epsilon}^2-8\bar{\epsilon}^4)\right]\bar{\mathcal{I}}_0\right],$$

$$\mathcal{L}_0=\frac{\alpha^2}{4R^2}\frac{Z\bar{\epsilon}k}{(1-\bar{\epsilon}^2)}\bar{\mathcal{I}}_0\,,$$

where

$$\bar{\mathcal{I}}_0=-\frac{\gamma}{Z\alpha}\left[\frac{\bar{\epsilon}^2k^2}{\gamma^2}-1\right]^{1/2}.$$

One has to compute the pseudoradial integral $\mathcal{I}_0$; using the expression (7) of the pseudoradial functions P_{vk} and Q_{vk} in terms of the zeroth-order functions $P_{vk}^{(0)}$ and $Q_{vk}^{(0)}$ and introducing $n=v+|k|$, one can write

$$\mathcal{I}_0=\mathcal{I}_0^{(0)}(v,-k;v,k)+\frac{k}{Z^2R^2}\sum_{v'(\neq v)}[C_{v',v}(-k)\bar{\mathcal{I}}_0(v',-k;v,k)-C_{v',v}(k)\bar{\mathcal{I}}_0(v',k;v,-k)]\,, \tag{A8}$$

where

$$\mathcal{I}_0^{(0)}=\int_0^{\pi}(P_{v,-k}^{(0)}P_{v,k}^{(0)}+Q_{v,-k}^{(0)}Q_{v,k}^{(0)})d\chi\,,$$

$$C_{v'v}(k)=\frac{n^2n'^2}{n^2-n'^2}\bar{J}_1(v',k;v,k)\,.$$

Let us compute the off-diagonal $v'\neq v$ flat-space integrals (when truncated at the α^2 terms) and consider first the determination of $\bar{J}_1(v',k;v,k)$. Setting

$$\mathcal{I}_s=\tilde{I}_s+O(\alpha^2),\ldots,\quad \bar{\epsilon}'-\bar{\epsilon}=\frac{Z^2\alpha^2}{2}\left[\frac{1}{n^2}-\frac{1}{n'^2}\right],$$

$$\bar{\epsilon}'+\bar{\epsilon}=2-\frac{Z^2\alpha^2}{2}\left[\frac{1}{n^2}+\frac{1}{n'^2}\right]$$

in Eq. (B1), the principal parts $\tilde{I}_s,\tilde{J}_s,\ldots,$ of the flat-space integrals are found to be solutions of the following system:

$$\begin{aligned}&s\tilde{\mathcal{I}}_{s-1}-2\tilde{\mathcal{J}}_s-(k'+k)\tilde{\mathcal{K}}_s=0\,,\\&(k'+k)\tilde{\mathcal{I}}_{s-1}+2\tilde{\mathcal{J}}_s-s\tilde{\mathcal{K}}_{s-1}=0\,,\\&2\tilde{\mathcal{I}}_s-2\tilde{\mathcal{K}}_s=0\,,\\&\frac{Z^2}{2}\left[\frac{1}{n^2}-\frac{1}{n'^2}\right]\tilde{\mathcal{I}}_s+(k'-k)\tilde{\mathcal{J}}_{s-1}-s\tilde{\mathcal{L}}_{s-1}=0\,.\end{aligned} \tag{A9}$$

For $k'=k$, after setting $s=0$ in the fourth and third Eq. (A9) and $s=1$ in the first one, one gets, as expected from the orthogonality property of the nonrelativistic part of the radial functions, $\tilde{I}_0=0$ and also $\tilde{K}_0=\tilde{I}_0$, $\tilde{J}_0-2\tilde{J}_1-2k\tilde{K}_0=0$. As a consequence,

$$\tilde{J}_1=0\,,\ \bar{J}_1(v',k;v,k)=O(\alpha^2)$$

and the pseudoradial integral (A8) reduces to

$$\mathcal{I}_0=\mathcal{I}_0^{(0)}(v,-k;v,k)+O\left[\frac{\alpha^2}{R^2}\right].$$

Keeping in mind that the Dirac $P_{vk}^{(0)}$ and $Q_{vk}^{(0)}$ radial functions are linear combinations of the Kepler functions $R_S^{\gamma}(\chi)$ and $R_S^{\gamma-1}(\chi)$ [see Eq. (8)] and also that $\epsilon=\epsilon^{(0)}+O(\alpha^2/R^2)$, one gets

$$\mathcal{I}_0^{(0)}(v,-k;v,k)=\epsilon\langle S\gamma\,|\,S\gamma-1\rangle\,. \tag{A10}$$

An analytical expression of the overlap integral in (A10) can be obtained when keeping in mind that the following relation holds for curved-space Kepler matrix elements of any derivable function $f(\chi)$ (for details, see Appendix B of Ref. 4)

$$\frac{1}{\alpha}\left[\frac{\epsilon^2k^2}{\gamma^2}-1\right]^{1/2}\langle S\gamma\,|f\,|S\gamma-1\rangle=\left\langle S\gamma\left|\left[\frac{\gamma}{R}\cot\chi-\frac{Z\epsilon}{\gamma}-\frac{1}{2R}\frac{d}{d\chi}\right]f\right|S\gamma\right\rangle. \tag{A11}$$

Using Eq. (B11) of Ref. 4, one gets

$$\langle S\gamma\,|\cot\chi\,|S\gamma\rangle=\frac{Z\epsilon R}{(v+|\gamma|)^2}\,,$$

and one can write in terms of ϵ [see Eq. (10)]

$$\left\langle S\gamma\left|\frac{\cot\chi}{R}\right|S\gamma\right\rangle=\frac{1-\epsilon^2}{Z\alpha\epsilon}+\frac{1}{R^2}\frac{(\bar{\epsilon}^2k^2-\gamma^2)}{Z\bar{\epsilon}(1-\bar{\epsilon}^2)}+O\left[\frac{1}{R^4}\right]. \tag{A12}$$

Setting $f=1$, $\langle S\gamma\,|\,S\gamma\rangle=1$ in Eq. (A11) and using Eq. (A12), one obtains

$$\langle S\gamma\,|\,S\gamma-1\rangle=-\frac{\gamma}{Z\epsilon\alpha}\left[\frac{\epsilon^2k^2}{\gamma^2}-1\right]^{1/2}\left[1-\frac{\alpha^2}{R^2}\frac{\gamma^2}{(1-\bar{\epsilon}^2)}\right] \tag{A13}$$

and, therefore, the expression (20) of $\mathcal{I}_0$.

At the nonrelativistic limit, i.e., when retaining the terms up to $Z^2\alpha^2$ and introducing the usual radial quantum number $n=v+|k|$, the expression (A13) reduces to

$$\langle S\gamma\,|\,S\gamma-1\rangle\simeq-\mathrm{sgn}(k)\left[1-\frac{k^2}{n^2}\right]^{1/2}\left[1+\frac{n^2k^2}{Z^2R^2}\right], \tag{A14}$$

and one obtains the expression (22) of $\mathcal{I}_1$.

APPENDIX B: CLOSED-FORM EXPRESSIONS OF THE $v'=v$, $k'=k$, AND $k'=-k$ FLAT-SPACE RADIAL INTEGRALS

As it has been shown previously,[10] analytical expressions of the flat-space integrals (17) in terms of $\bar{\epsilon}$ and k can be obtained by solving the asymptotic flat-space limit of Eqs. (16), i.e.,

$$\begin{aligned} &s\overline{\mathscr{I}}_{s-1}-2\overline{\mathscr{J}}_s-(k'+k)\overline{\mathscr{K}}_{s-1}+(\bar{\epsilon}'-\bar{\epsilon})\overline{\mathscr{L}}_s=0\ ,\\ &(k'+k)\overline{\mathscr{I}}_{s-1}+(\bar{\epsilon}'+\bar{\epsilon})\overline{\mathscr{J}}_s+2Z\alpha^2\overline{\mathscr{J}}_{s-1}-s\overline{\mathscr{K}}_{s-1}=0\ ,\\ &2\overline{\mathscr{I}}_s-s\alpha^2\overline{\mathscr{J}}_{s-1}-(\bar{\epsilon}'+\epsilon)\overline{\mathscr{K}}_s-2Z\alpha^2\overline{\mathscr{K}}_{s-1}+(k'-k)\alpha^2\overline{\mathscr{L}}_{s-1}=0\ ,\\ &(\bar{\epsilon}'-\bar{\epsilon})\overline{\mathscr{I}}_s+(k'-k)\alpha^2\overline{\mathscr{J}}_{s-1}-S\alpha^2\overline{\mathscr{L}}_{s-1}=0\ . \end{aligned} \tag{B1}$$

For $k'=k$, one gets the following expressions for the integrals encountered when calculating the space-curvature modifications of the Landé g factor:

$$\begin{aligned} \bar{I}_2&=\frac{Z^2\alpha^4(1+4\bar{\epsilon}^2)}{2(1-\bar{\epsilon}^2)^2}-\frac{k^2(1+2\bar{\epsilon}^2)+3\bar{\epsilon}k-1}{2(1-\bar{\epsilon}^2)}\ ,\\ \bar{J}_2&=\frac{1}{2Z}\left[\frac{Z^2\alpha^2[3\bar{\epsilon}-k(1+2\bar{\epsilon}^2)]}{1-\bar{\epsilon}^2}-k(1-k^2)\right],\\ \bar{J}_3&=\frac{1}{4Z^2}\left[\frac{Z^4\alpha^4[3(1+4\bar{\epsilon}^2)-2k\bar{\epsilon}(3+2\bar{\epsilon}^2)]}{(1-\bar{\epsilon}^2)^2}+\frac{Z^2\alpha^2(6\bar{\epsilon}k^3-3k^2-11\bar{\epsilon}k+3)}{1-\bar{\epsilon}^2}\right],\\ \bar{K}_1&=\frac{1}{2Z}\left[\frac{Z^2\alpha^2(1+2\bar{\epsilon}^2)}{1-\bar{\epsilon}^2}-k(k+\bar{\epsilon})\right],\\ \bar{K}_2&=\frac{Z^2\alpha^4\bar{\epsilon}(3+2\bar{\epsilon}^2)}{2(1-\bar{\epsilon}^2)^2}-\frac{\alpha^2\bar{\epsilon}(3k^2+3\bar{\epsilon}k-1)}{2(1-\bar{\epsilon}^2)}\ . \end{aligned} \tag{B2}$$

For $k'=-k$, one gets the following expressions for the integrals encountered when studying the Stark effect:

$$\begin{aligned} \bar{\mathrm{I}}_0&=-\frac{\gamma}{Z\alpha}\left[\frac{\bar{\epsilon}^2k^2}{\gamma^2}-1\right]^{1/2},\\ \bar{\mathrm{I}}_1&=\frac{3Z\alpha^2\bar{\epsilon}}{2(1-\bar{\epsilon}^2)}\bar{\mathrm{I}}_0\ ,\\ \bar{\mathrm{I}}_2&=\frac{\alpha^2}{2(1-\bar{\epsilon}^2)}\left[1-k^2+\frac{Z^2\alpha^2}{1-\bar{\epsilon}^2}(1+4\bar{\epsilon}^2)\right]\bar{\mathrm{I}}_0\ ,\\ \bar{\mathrm{I}}_3&=\frac{5Z\alpha^4\bar{\epsilon}}{8(1-\bar{\epsilon}^2)^2}\left[5-3k^2+\frac{Z^2\alpha^2}{1-\bar{\epsilon}^2}(3+4\bar{\epsilon}^2)\right]\bar{\mathrm{I}}_0\ ,\\ \bar{\mathrm{J}}_0&=0\ ,\quad \bar{\mathrm{J}}_t=\tfrac{1}{2}t\,\bar{\mathrm{I}}_{t-1}\ ,\\ \bar{\mathrm{K}}_t&=\bar{\epsilon}\bar{\mathrm{I}}_t+Z\alpha^2\frac{t}{t+1}\bar{\mathrm{I}}_{t-1}\ ,\\ \bar{\mathrm{L}}_0&=0\ ,\quad \bar{\mathrm{L}}_t=-2k\frac{t}{t+1}\bar{\mathrm{I}}_{t-1}\ . \end{aligned} \tag{B3}$$

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