Squeezed states from nondegenerate four-wave mixers

J. R. Klauder, S. L. McCall, and B. Yurke
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
(Received 18 October 1985)

By closely following the work of Reid and Walls, a fully quantum-mechanical treatment of an atomic radiation-field interaction is developed with the aim of deriving basic equations for a nondegenerate four-wave mixer. The resultant equations are examined in forward and cavity geometries for their implications regarding squeezed states. Good agreement between our theory and the experiments of Slusher *et al.* is demonstrated.

I. INTRODUCTION

Squeezed states of the radiation field, which may have potential application in low-noise precision measurement and detection,[1] offer an intriguing alternative to conventional coherent states. Both single-mode[2] and two-mode[3] squeezed states have been discussed from a theoretical viewpoint, with emphasis in recent years focusing on an analysis of two-mode squeezed-state generation by parametric amplifiers and four-wave mixers. The level of discussion has ranged from (semi-) classical and quantum phenomenology to a fully quantum-mechanical analysis. In particular, Reid and Walls[4] have presented a fully quantum-mechanical treatment including a two-level atomic system for a degenerate four-wave mixer in which signal and idler frequencies are equal, while Yurke[5] has developed a phenomenological quantum model by directly adopting a four-photon interaction for a nondegenerate four-wave mixer with unequal signal and idler frequencies. The predicted squeezing and its dependence on model parameters has been obtained for both models.

From an experimental point of view the creation of a squeezed state by a nondegenerate four-wave mixer, as opposed to a degenerate one, is in some aspects simpler. Indeed, the reduced variance in one of the field quadrature components that is characteristic of a squeezed state has been observed recently for the first time in just such a system.[6] Consequently, it becomes important to ask whether the theoretical analysis of a fully quantum-mechanical, nondegenerate four-wave mixer holds any surprises regarding the properties of its squeezed states. In this paper we address this question, and conclude, within the framework of the model and standard approximations to its solution, that the squeezed-state properties are essentially those which one would expect from previous fully quantum-mechanical models.[4] However, in contrast with phenomenological quantum models,[5] it is noteworthy that our results show a marked limitation to squeezing at higher pump intensities for atomic-beam-induced signal generation. It is likely that alternative mechanisms to induce nonlinearities may lead to greater squeezing.

We base our analysis quite closely on that of Reid and Walls[4] (hereafter referred to as RW), and as much as possible we choose to follow their notation. We do not repeat their treatment fully, but instead implicitly refer to their paper to fill any gaps in our presentation. When we must deviate from the treatment of RW then our presentation becomes more complete. Our conclusions are in good agreement with the experimental results reported in Ref. 6.

II. THEORETICAL FOUNDATIONS

The model Hamiltonian corresponds to N atoms in a total volume V and is taken to be $H=\int d^3r\, H_r/\delta V$, where

$$H_r=\sum_{j=1}^{4}\hbar\omega_j a_j^\dagger a_j+\sum_{i=1}^{N_0}\tfrac{1}{2}\hbar\omega_0\sigma_{zi}+ig\sum_{i=1}^{N_0}(\sigma_i a_r^\dagger-\sigma_i^\dagger a_r)+\sum_{i=1}^{N_0}(\sigma_i\Gamma^\dagger+\sigma_i^\dagger\Gamma)\,,$$

$$a_r=\sum_{j=1}^{4}a_j e^{i\mathbf{k}_j\cdot\mathbf{r}}\,. \tag{1}$$

Here the medium is modeled by a localized system of N_0 two-level atoms in a volume δV with ω_0 being the atomic resonance frequency, while a_j and $a_j^\dagger$ are standard annihilation and creation operators, and Γ and $\Gamma^\dagger$ denote atomic reservoir operators. The radiation-field frequencies are given by

$$\begin{aligned}\omega_1&=\omega_2=\omega\,,\\ \omega_3&=\omega+\zeta\,,\\ \omega_4&=\omega-\zeta\,,\end{aligned} \tag{2}$$

where $0\le\zeta<\omega$. For a degenerate four-wave mixer $\zeta=0$, and (1) reduces to the Hamiltonian adopted by RW. We proceed to study this model following the pattern of RW.

The by-now standard way to analyze such problems is first to derive a c-number Fokker-Planck equation for the amplitude f of a generalized form of P representation of the density operator in which all complex variables are treated as independent;[7] e.g., if v' and $v'^\dagger$ appear, then the latter is not the complex conjugate of the former. In this treatment, the c-number$\leftrightarrow q$-number correspondence $v'\leftrightarrow\sum\sigma_i$, $v'^\dagger\leftrightarrow\sum\sigma_i^\dagger$, $D\leftrightarrow\sum\sigma_{zi}$, $\bar\alpha'\leftrightarrow a_r$, $\bar\alpha'^\dagger\leftrightarrow a_r^\dagger$ is set up, an initially infinite-order differential equation is approximated by a diffusion equation, and the relations $\bar\alpha=\bar\alpha' e^{i\omega t}$ and $v=v'e^{-i\omega t}$ (plus "conjugates") are introduced. The resultant Fokker-Planck equation becomes

$$\frac{\partial f}{\partial t}=\left[i\zeta\left[\frac{\partial}{\partial\alpha_3}\alpha_3-\frac{\partial}{\partial\alpha_4}\alpha_4\right]+\left[-\frac{\partial}{\partial\bar{\alpha}}gv-\frac{\partial}{\partial v}(-\gamma v+gD\bar{\alpha})-\frac{\partial}{\partial D}[-\gamma_{\parallel}(D+N_0)-2g(v^{\dagger}\bar{\alpha}+\bar{\alpha}^{\dagger}v)]\right.\right.$$
$$\left.\left.+\frac{1}{2}\frac{\partial^2}{\partial v^2}(2gv\bar{\alpha})+\frac{1}{2}\frac{\partial^2}{\partial D^2}[2\gamma_{\parallel}(D+N_0)-4g(v^{\dagger}\bar{\alpha}+v\bar{\alpha}^{\dagger})]\right]+\text{c.c.}\right]f\,. \tag{3}$$

Here c.c stands for a similar term with $\alpha_3\to\alpha_3^{\dagger}$, $v\to v^{\dagger}$, etc., $\delta=(\omega_0-\omega)/\gamma_{\perp}$ is the normalized detuning from line center, $\gamma=\gamma_{\perp}(1+i\delta)$, and $\gamma_{\perp}$ and $\gamma_{\parallel}$ are the atomic transverse and longitudinal relaxation parameters. If $\zeta=0$, then (3) reduces to the RW Fokker-Planck equation.

Next one passes from the Fokker-Planck equation to an equivalent c-number Langevin equation to find an expression for the polarization v. Assuming that $\gamma_{\perp},\gamma_{\parallel}\gg\gamma_F$, a field damping rate, then one may set $\dot{v}=\dot{D}=0$ and solve these equations for the "steady-state" values of v and D. It is important to observe that the drift terms in the Langevin equations for v and D are proportional to the coefficients of $\partial/\partial v$ and $\partial/\partial D$ in (3), respectively, and do not involve ζ. Consequently, the equations $\dot{v}=0$ and $\dot{D}=0$ are *independent* of ζ and are formally identical to the RW equations for the degenerate case. As customary at this point the pump fields are assumed large and nondepleting, while the signal and idler field are treated to the lowest appropriate order. In symbols, $\bar{\alpha}=\epsilon+\alpha$, $\epsilon=\epsilon_1e^{i\mathbf{k}_1\cdot\mathbf{r}}+\epsilon_2e^{i\mathbf{k}_2\cdot\mathbf{r}}$, $\alpha=\alpha_3e^{i\mathbf{k}_3\cdot\mathbf{r}}+\alpha_4e^{i\mathbf{k}_4\cdot\mathbf{r}}$, with ϵ_1 and ϵ_2 fixed. The result for the atomic polarization at $\mathbf{r}$ is given by the RW expression

$$v_r=-g\frac{N_0}{\gamma\pi}(\epsilon+\alpha)\left[1-\frac{\epsilon^{\dagger}\alpha+\epsilon\alpha^{\dagger}}{I_s\Pi}\right]+\Gamma_r\,, \tag{4}$$

where $n_0=\gamma_{\perp}\gamma_{\parallel}/4g^2$ is the line-center saturation intensity, $I_s=n_0(1+\delta^2)$, and $\Pi=1+|\epsilon|^2/I_s$. For purely radiative damping ($\gamma_{\parallel}=2\gamma_{\perp}$), the dominant, nonzero noise correlations read

$$\begin{aligned}&\langle\Gamma_r(t)\Gamma_r(t')\rangle=D_1\delta(t'-t)\,,\\&D_1=\frac{-N_0\epsilon^2}{\gamma_{\perp}I_s(1+\delta^2)^2\Pi^3}\left[(1-i\delta)^3+\frac{|\epsilon|^4}{2n_0^2}\right],\\&\langle\Gamma_r^{\dagger}(t)\Gamma_r(t')\rangle=D_2\delta(t'-t)\,,\\&D_2=\frac{N_0|\epsilon|^2}{\gamma_{\perp}I_s(1+\delta^2)^2\Pi^3}\left[\frac{2}{n_0}|\epsilon|^2+\frac{|\epsilon|^4}{2n_0^2}\right].\end{aligned} \tag{5}$$

The equation for the field amplitudes α_j, $j=3,4$, involves ζ and reads

$$\dot{\alpha}_j(t)=i\zeta_j\alpha_j(t)+(\delta V)^{-1}\int d^3r\,e^{-i\mathbf{k}_j\cdot\mathbf{r}}gv_r(t)\,, \tag{6}$$

where $\zeta_3=\zeta=-\zeta_4$. The latter integral favors the phase-matching condition $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4$ for sufficiently long interaction lengths L. Since ζ appears separately the integral in (6) may be approximately evaluated as in RW. The final phase-matched coupled equations are

$$\begin{aligned}&\dot{\alpha}_3=(\gamma'+i\zeta)\alpha_3+\chi\alpha_4^{\dagger}+F_3(t)\,,\\&\dot{\alpha}_4=(\gamma'+i\zeta)\alpha_4+\chi\alpha_3^{\dagger}+F_4(t)\,,\end{aligned} \tag{7}$$

with similar equations for $\dot{\alpha}_3^{\dagger}$ and $\dot{\alpha}_4^{\dagger}$, where, with $\epsilon_1=\epsilon_2$ and $I=|\epsilon_1|^2$,

$$\begin{aligned}&\gamma'=\frac{-2C'(1+2I/I_s)}{(1+i\delta)(1+4I/I_s)^{3/2}}\equiv-\gamma_R+i\gamma_I\,,\\&\chi=\frac{2C'(2I/I_s)}{(1+i\delta)(1+4I/I_s)^{3/2}}\equiv\chi_R+i\chi_I\,,\\&2C'=g^2N/\gamma_{\perp};\end{aligned} \tag{8}$$

alternatively $2C'=\alpha_0c$, where α_0 is the line-center small-signal field-attenuation coefficient and c is the speed of light. Here F_3 and F_4 denote integrated noise terms which, exactly as in RW, exhibit the nonzero noise correlations given by

$$\begin{aligned}&\langle F_3^{\dagger}(t)F_4^{\dagger}(t')\rangle=\langle F_3(t)F_4(t')\rangle^*=R\delta(t-t')\,,\\&\langle F_3(t)F_3^{\dagger}(t')\rangle=\langle F_4(t)F_4^{\dagger}(t')\rangle=\Lambda\delta(t-t')\,,\end{aligned} \tag{9}$$

where

$$\begin{aligned}&R=R_R+iR_I\,,\\&R_R=\frac{-2C'}{(1+\delta^2)^2(1+4I/I_s)^{5/2}}\\&\quad\times\left\{(1-3\delta^2)\frac{2I}{I_s}\left[1+\frac{I}{I_s}\right]+(1+\delta^2)^2\left[\frac{1}{2}\left[1+\frac{4I}{I_s}\right]^{5/2}-\frac{1}{2}-\frac{5I}{I_s}-\frac{15I^2}{I_s^2}\right]\right\},\\&R_I=\frac{-2C'(3\delta-\delta^3)}{(1+\delta^2)^2(1+4I/I_s)^{5/2}}\frac{2I}{I_s}\left[1+\frac{I}{I_s}\right],\\&\Lambda=\frac{2C'}{(1+\delta^2)^2(1+4I/I_s)^{5/2}}\\&\quad\times\left\{12(1+\delta^2)\frac{I^2}{I_s^2}+(1+\delta^2)^2\left[\frac{1}{2}\left[1+\frac{4I}{I_s}\right]^{5/2}-\frac{1}{2}-\frac{5I}{I_s}-\frac{15I^2}{I_s^2}\right]\right\}.\end{aligned} \tag{10}$$

For $\zeta=0$, Eq. (7) reduces to that of RW, which indeed becomes their basic equation for application to degenerate four-wave mixing in backward, forward, and cavity geometries. Thus, for $\zeta>0$, Eq. (7) becomes our basic result. Observe that if $\alpha_3(t)$ and $\alpha_4(t)$ denote a solution for $\zeta=0$, then

$$\alpha_3(t,\zeta)\equiv\alpha_3(t)e^{i\zeta t},\quad \alpha_4(t,\zeta)\equiv\alpha_4(t)e^{-i\zeta t} \tag{11}$$

denote a solution for $\zeta>0$, albeit for another noise sample. This remark follows because (7) implies that

$$\dot\alpha_3(t,\zeta)=\gamma'\alpha_3(t,\zeta)+\chi\alpha_4^\dagger(t,\zeta)+F_3(t,\zeta),\quad \dot\alpha_4(t,\zeta)=\gamma'\alpha_4(t,\zeta)+\chi\alpha_3^\dagger(t,\zeta)+F_4(t,\zeta), \tag{12}$$

plus similar equations for $\dot\alpha_3^\dagger$ and $\dot\alpha_4^\dagger$, where

$$F_3(t,\zeta)\equiv e^{-i\zeta t}F_3(t),\quad F_4(t,\zeta)\equiv e^{i\zeta t}F_4(t) \tag{13}$$

are statistically equivalent complex white noises for all ζ. Clearly all solutions of (7) for $\zeta>0$ are related to solutions for $\zeta=0$ in this way, and this connection makes relatively easy the extension of results for four-wave mixers from the degenerate to the nondegenerate case.

The fact that ζ makes so little imprint on our results derives from the choice of the starting model in Eq. (1), and the several approximations made in its analysis. For sufficiently large ζ it is apparent that one or more of these assumptions breaks down. However, as we shall see, in the range of the present experiments[6] the equations derived above seem to be quite satisfactory, in part because the pump intensity is small compared to the saturation intensity.

III. FOUR-WAVE MIXERS

A. Forward four-wave mixing

Reid and Walls[4] have given a rather complete discussion and comparison of their work to previous work for various geometries. For convenience we first consider the case of forward four-wave mixing. Following RW the relevant equations can be modeled quantum mechanically if we make a change from ct to z as independent variable in the temporal equations for α_3 and α_4. For forward four-wave mixing the resultant equations follow from (12) as

$$\begin{aligned}\frac{d\alpha_3(z)}{dz}&=-\alpha\alpha_3(z)+\bar\chi\alpha_4^\dagger(z)+G_1(z),\\ \frac{d\alpha_4^\dagger(z)}{dz}&=-\alpha\alpha_4^\dagger(z)+\bar\chi^*\alpha_3(z)+G_2^\dagger(z),\\ \langle G_1^\dagger(z)G_2^\dagger(z')\rangle&=\langle G_1(z)G_2(z')\rangle^*=\bar R\delta(z-z'),\\ \langle G_1(z)G_1^\dagger(z')\rangle&=\langle G_2(z)G_2^\dagger(z')\rangle=\bar\Lambda\delta(z-z').\end{aligned} \tag{14}$$

Here, following RW, we have introduced

$$\begin{aligned}\alpha&=\gamma_R n/[c\cos(\phi/2)],\\ \bar\chi&=\chi n/[c\cos(\phi/2)],\\ \bar\Lambda&=\Lambda n/[c\cos(\phi/2)],\\ \bar R&=Rn/[c\cos(\phi/2)],\end{aligned} \tag{15}$$

where ϕ denotes the angle of propagation separating the two weak fields, α_3 and α_4. The imaginary term relating to γ_I contributes an intensity-dependent addition to the refractive index n, and does not affect the squeezing; indeed γ_I may be effectively eliminated by working with $\alpha_j'=\alpha_j\exp(i\gamma_I z/c)$, $j=3,4$. With the change of dependent variable from $\alpha_j(t,\zeta)$ to $\alpha_j(z)$, and with ζ being understood, Eqs. (14) are *identical* to that obtained by RW. Consequently their solution applies without change, and so, given that the initial states for α_3 and α_4 at $z=0$ are coherent states, it follows that at $z=L$

$$\begin{aligned}\langle\alpha_3\alpha_4\rangle&=\langle\alpha_3^\dagger\alpha_4^\dagger\rangle^*=\tfrac12(\Sigma_{12}+2\Sigma_{23}+\Sigma_{34}),\\ \langle\alpha_3\alpha_3^\dagger\rangle&=\langle\alpha_4\alpha_4^\dagger\rangle=\tfrac12 a(\Sigma_{34}-\Sigma_{12}),\end{aligned} \tag{16}$$

where

$$\begin{aligned}a&=-|\bar\chi|/\bar\chi=-\bar\chi^*/|\bar\chi|,\\ \Sigma_{ij}&=-(\Pi_{ij}/\lambda_{ij})[1-\exp(-\lambda_{ij}L)],\\ \lambda_{12}&=2(\alpha+|\bar\chi|),\quad \lambda_{23}=2\alpha,\quad \lambda_{34}=2(\alpha-|\bar\chi|),\\ \Pi_{12}&=\tfrac12 a^{-2}(2a\bar\Lambda+\bar R+a^2\bar R^*),\\ \Pi_{23}&=\tfrac12 a^{-2}(-\bar R+a^2\bar R^*),\\ \Pi_{34}&=\tfrac12 a^{-2}(-2a\bar\Lambda+\bar R+a^2\bar R^*).\end{aligned} \tag{17}$$

This solution for the covariance is just that of RW.

At this point in the calculation we must depart slightly from the analysis of Reid and Walls to account for the fact that generally $\omega_3=\omega+\zeta>\omega_4=\omega-\zeta$. The quadrature components X_1 and X_2 of interest to us are defined by the relation[8]

$$X_1+iX_2\equiv(\sqrt{\omega_3/\omega}\,a_3+e^{-i\psi}\sqrt{\omega_4/\omega}\,a_4^\dagger)/\sqrt2. \tag{18}$$

If $\zeta=0$ this prescription reduces to that of RW. The variance of X_1 or X_2 involves correlations of the operators a_3,a_4, and their adjoints, and when normally ordered these correlations are identical to the expressions given in (16). Consequently, it follows that

$$\Delta X_{1,2}^2-\tfrac14=\tfrac14[(\omega_3/\omega)\langle\alpha_3\alpha_3^\dagger\rangle+(\omega_4/\omega)\langle\alpha_4\alpha_4^\dagger\rangle \pm(\sqrt{\omega_3\omega_4}/\omega)2\,\mathrm{Re}(e^{i\psi}\langle\alpha_3\alpha_4\rangle)]. \tag{19}$$

As ψ varies the minimum of the right-hand side, for X_1, say, occurs for the same value of ψ independently of ζ. We shall discuss two choices of ψ.

Following RW we first choose $e^{i\psi}=a$, which leads to the final expression for squeezing given by

$$\Delta X_1^2-\frac14=\frac18\left[(1+r)A\frac{1-e^{-2(\alpha+|\bar\chi|)L}}{\alpha+|\bar\chi|}+(1-r)B\frac{1-e^{-2(\alpha-|\bar\chi|)L}}{\alpha-|\bar\chi|}\right], \tag{20}$$

$$\begin{aligned}r&\equiv(1-\zeta^2/\omega^2)^{1/2},\\ A&\equiv\bar\Lambda-(\bar\chi_R\bar R_R-\bar\chi_I\bar R_I)/|\bar\chi|,\\ B&\equiv\bar\Lambda+(\bar\chi_R\bar R_R-\bar\chi_I\bar R_I)/|\bar\chi|.\end{aligned}$$

When $r=1$ ($\zeta=0$) this expression reduces to that of RW, namely,

$$\Delta X_1^2 - \frac{1}{4} = \frac{1}{4} A \frac{1-e^{-2(\alpha+|\bar{X}|)L}}{\alpha+|\bar{X}|} . \tag{21}$$

In ideal circumstances $(\alpha+|\bar{X}|)L \gg 1$, so that the exponential term is negligible, and thus

$$\Delta X_1^2 - \frac{1}{4} = \frac{1}{4} \frac{A}{\alpha+|\bar{X}|} . \tag{22}$$

In the ideal noise limit, where $\bar{\Lambda}=\bar{R}_R=\bar{X}_R=0$ and $\bar{R}_I=|\bar{X}|=-\bar{X}_I$, it follows that $A=-|\bar{X}|$ and so

$$\Delta X_1^2 - \frac{1}{4} = -\frac{1}{4} \frac{|\bar{X}|}{\alpha+|\bar{X}|} . \tag{23}$$

The conditions for the ideal noise limit to hold are $\delta \gg 1$, $I \ll I_s$, and $10I^2 \ll I_s^2/\delta$. Under these conditions

$$\frac{\alpha}{|\bar{X}|} = \frac{1+2I/I_s}{(2I/I_s)(1+\delta^2)^{1/2}} \simeq \frac{I_s}{2I\delta} , \tag{24}$$

which can be widely adjusted within the range of validity of the ideal noise limit. Below threshold squeezing is spoiled by loss, however as $\alpha/|\bar{X}| \to 0$ it would seem that $\Delta X_1^2 \to 0$ leading to maximal squeezing.

Now let us return to the case $r<1$ ($\zeta>0$) given by (20). At first glance it appears that squeezing is spoiled by the second term for in the ideal noise limit $B=|\bar{X}|$ and the exponential term swamps the other terms whenever $(|\bar{X}|-\alpha)L \gg 1$. That would be true and cause for concern save for the fact that in present-day experiments[6] $(\alpha+|\bar{X}|)L \ll 1$, and as a consequence $||\bar{X}|-\alpha|L \ll 1$ as well. Thus the exponential factors in (20) are in fact of order unity. Moreover, in practice $1-r \approx 10^{-12}$ so that the second term in (20) can really be dropped, which again leads to (21). However, squeezing remains minimal now, even in the ideal noise limit, since (by expanding the exponent)

$$\Delta X_1^2 - \frac{1}{4} = -\frac{1}{4} \frac{|X|}{\alpha+|\bar{X}|} (1-e^{-2(\alpha+|\bar{X}|)L}) \approx -\tfrac{1}{2} |\bar{X}| L . \tag{25}$$

When operating conditions near the ideal noise limit are attainable the preceding analysis is suitable for the case of forward four-wave mixing. However, in the absence of the ideal noise limit we must reexamine the choice of $e^{i\psi}$ in (19). It is self-evident that to minimize ΔX_1^2 an optimal choice of $e^{i\psi}$ is one for which

$$\Delta X_1^2 - \tfrac{1}{4} = \tfrac{1}{4}[(\omega_3/\omega)\langle \alpha_3\alpha_3^\dagger\rangle + (\omega_4/\omega)\langle \alpha_4\alpha_4^\dagger\rangle - 2(\sqrt{\omega_3\omega_4}/\omega)|\langle \alpha_3\alpha_4\rangle|] . \tag{26}$$

When the various terms are substituted in this expression it follows that

$$\Delta X_1^2 - \frac{1}{4} = \frac{1}{8} \frac{A}{\alpha+|\bar{X}|}(1-e^{-\lambda_{12}L}) + \frac{1}{8}\frac{B}{\alpha-|\bar{X}|}(1-e^{-\lambda_{34}L}) - \frac{r}{8}\left[\left[\frac{A}{\alpha+|\bar{X}|}(1-e^{-\lambda_{12}L}) - \frac{B}{\alpha-|\bar{X}|}(1-e^{-\lambda_{34}L})\right]^2 + \left[\frac{C}{\alpha}(1-e^{-2\alpha L})\right]^2\right]^{1/2} . \tag{27}$$

Here, as before, $\lambda_{12}=2(\alpha+|\bar{X}|)$, $\lambda_{34}=2(\alpha-|\bar{X}|)$, and A and B are given in (20); the new factor is

$$C \equiv (\bar{X}_R\bar{R}_I + \bar{X}_I\bar{R}_R)/|\bar{X}| . \tag{28}$$

In the ideal noise limit $C=0$ and this expression reduces to (20), but not in general. When $(\alpha+|\bar{X}|)L \ll 1$ and $r \approx 1$, then (27) may be replaced by

$$\Delta X_1^2 - \tfrac{1}{4} = \tfrac{1}{4} L\{(A+B) - [(B-A)^2 + C^2]^{1/2}\} . \tag{29}$$

In this case the variance of the other quadrature is given by

$$\Delta X_2^2 - \tfrac{1}{4} = \tfrac{1}{4} L\{(A+B) + [(B-A)^2 + C^2]^{1/2}\} . \tag{30}$$

As will become apparent later, the formulas developed here for forward four-wave mixing when $(\alpha+|\bar{X}|)L \ll 1$ imply negligible squeezing for the realizable situation of Ref. 6. Backward four-wave mixing is not substantially different than the forward case whenever $\zeta \ll c/L$, an inequality that will be well fulfilled. Under these conditions the analysis for a cavity geometry discussed below can make direct use of the analysis for the forward case carried out in the present section.

B. Cavity geometry

A cavity geometry has the potential to increase squeezing since the radiation repeatedly passes through the active medium before exiting the cavity. Since a detailed analysis of a cavity geometry has been treated elsewhere[5] we shall only outline the basic ideas. In addition to the medium equations (14), which relate the initial and final mode amplitudes, $\alpha_j(0)$ and $\alpha_j(L)$, $j=3,4$ respectively, we have two sets of boundary conditions involving two additional mode sets $\alpha_j(\text{in})$ and $\alpha_j(\text{out})$, $j=3,4$. These boundary conditions read

$$\begin{aligned} \alpha_j(\text{out}) &= -\sin\theta\,\alpha_j(\text{in}) + \cos\theta\,\alpha_j(L) , \\ \alpha_j(0) &= \cos\theta\,\alpha_j(\text{in}) + \sin\theta\,\alpha_j(L) , \end{aligned} \tag{31}$$

for $j=3,4$. Here $\sin\theta$ (≈ 1) represents the reflectivity and $\cos\theta$ the transmissivity of a partially reflecting mirror at one end of the cavity. Interest centers on the cavity out modes a_j(out), $j=3,4$ and the field-quadrature components they engender in the manner of (18). In turn, the expression of interest is given by

$$\Delta X_{\substack{1\\2}}^2-\tfrac{1}{4}=\tfrac{1}{4}[(\omega_3/\omega)\langle\beta_3\beta_3^\dagger\rangle+(\omega_4/\omega)\langle\beta_4\beta_4^\dagger\rangle \pm(\sqrt{\omega_3\omega_4}/\omega)2\,\mathrm{Re}(e^{i\psi}\langle\beta_3\beta_4\rangle)], \quad (32)$$

where we have set $\beta_j\equiv\alpha_j$(out). It follows, in a lengthy but straightforward fashion, that

$$\begin{aligned}\langle\beta_3\beta_3^\dagger\rangle&=\langle\beta_4\beta_4^\dagger\rangle\\ &=\tfrac{1}{4}A_c(1-e^{-2(\alpha+|\bar{X}|)L})/(\alpha+|\bar{X}|)\\ &\quad+\tfrac{1}{4}B_c(1-e^{-2(\alpha-|\bar{X}|)L})/(\alpha-|\bar{X}|)\end{aligned} \quad (33)$$

and, with $a^*=-|\bar{X}|/\bar{X}$ as before,

$$\begin{aligned}a^*\langle\beta_3\beta_4\rangle=&-\tfrac{1}{4}A_c(1-e^{-2(\alpha+|\bar{X}|)L})/(\alpha+|\bar{X}|)\\ &+\tfrac{1}{4}B_c(1-e^{-2(\alpha-|\bar{X}|)L})/(\alpha-|\bar{X}|)\\ &+iC_c(1-e^{-2\alpha L})/\alpha\ .\end{aligned} \quad (34)$$

Here we have introduced

$$\begin{aligned}A_c&\equiv EA(1-\sin\theta\, e^{-(\alpha-|\bar{X}|)L})^2,\\ B_c&\equiv EB(1-\sin\theta\, e^{-(\alpha+|\bar{X}|)L})^2,\\ C_c&\equiv EC[1-2e^{-\alpha L}\sin\theta\cosh(|\bar{X}|L)+e^{-2\alpha L}\sin^2\theta],\\ E&\equiv\frac{\cos^2\theta}{[1-2e^{-\alpha L}\sin\theta\cosh(|\bar{X}|L)+e^{-2\alpha L}\sin^2\theta]^2}\ .\end{aligned} \quad (35)$$

Note that for no cavity ($\sin\theta=0$) these formulas just reduce to the ones given previously.

As before we shall discuss two choices of $e^{i\psi}$. First if $e^{i\psi}=a$ it follows that the maximum squeezing is given by

$$\begin{aligned}\Delta X_1^2-\tfrac{1}{4}=&\tfrac{1}{8}(1+r)A_c(1-e^{-2(\alpha+|\bar{X}|)L})/(\alpha+|\bar{X}|)\\ &+\tfrac{1}{8}(1-r)B_c(1-e^{-2(\alpha-|\bar{X}|)L})/(\alpha-|\bar{X}|)\ .\end{aligned} \quad (36)$$

For $(\alpha+|\bar{X}|)L\ll 1$ and $r\approx 1$ this equation simplifies considerably. It is even heuristically useful to consider the simplification that arises when $(\alpha+|\bar{X}|)L\ll 1-\sin\theta$, in which case it follows that

$$\begin{aligned}A_c&=\frac{1+\sin\theta}{1-\sin\theta}A\,,\\ B_c&=\frac{1+\sin\theta}{1-\sin\theta}B\,,\\ C_c&=\frac{1+\sin\theta}{1-\sin\theta}C\,,\end{aligned} \quad (37)$$

and thus

$$\Delta X_1^2-\frac{1}{4}=\frac{1}{2}\frac{1+\sin\theta}{1-\sin\theta}AL\equiv\tfrac{1}{2}AL^*\ . \quad (38)$$

Here

$$L^*\equiv\frac{1+\sin\theta}{1-\sin\theta}L \quad (39)$$

denotes an increased, effective active medium length, which when $A<0$, tends to enhance squeezing. The factor $(1+\sin\theta)/(1-\sin\theta)$ is the normal intensity enhancement factor for a passive cavity or for an active cavity restricted to the linear regime.

With the alternative convention for $e^{i\psi}$ it follows that

$$\Delta X_1^2-\tfrac{1}{4}=\tfrac{1}{2}(\langle\beta_3\beta_3^\dagger\rangle-r\,|\langle\beta_3\beta_4\rangle|)\,, \quad (40)$$

which leads in the general case to

$$\begin{aligned}\Delta X_1^2-\frac{1}{4}=&\frac{1}{8}\frac{A_c}{\alpha+|\bar{X}|}(1-e^{-\lambda_{12}L})+\frac{1}{8}\frac{B_c}{\alpha-|\bar{X}|}(1-e^{-\lambda_{34}L})\\ &-\frac{r}{8}\left[\left[\frac{A_c}{\alpha+|\bar{X}|}(1-e^{-\lambda_{12}L})-\frac{B_c}{\alpha-|\bar{X}|}(1-e^{-\lambda_{34}L})\right]^2+\left[\frac{C_c}{\alpha}(1-e^{-2\alpha L})\right]^2\right]^{1/2}\ .\end{aligned} \quad (41)$$

When $(\alpha+|\bar{X}|)L\ll 1-\sin\theta$ and $r\approx 1$ this equation simplifies to become

$$\Delta X_1^2-\tfrac{1}{4}=\tfrac{1}{4}L^*\{(A+B)-[(B-A)^2+C^2]^{1/2}\}\,, \quad (42)$$

with L^* given as before. Again this equation shows an enhanced squeezing over that of (30). The quadrature component variance is given by

$$\Delta X_2^2-\tfrac{1}{4}=\tfrac{1}{4}L^*\{(A+B)+[(B-A)^2+C^2]^{1/2}\}\ . \quad (43)$$

For small amounts of squeezing, where (42) can be trusted, it is apparent that the effects of the cavity can be replaced simply by considering an active medium of effective length L^*. Of course, if $(\alpha+|\bar{X}|)L\ll 1-\sin\theta$ is not a valid approximation then the cavity effects are more complicated and the full expression (41) should be used.

C. Relation to experiment

Lastly we turn our attention to a comparison with the experimental results reported in Ref. 6. In these experiments laser light ($\lambda=5980$ Å) is used to pump Na atoms in a four-wave-mixing fashion with $\omega/2\pi=5\times 10^{14}$ Hz and $\zeta/2\pi=1.4\times 10^8$–$4.2\times 10^8$ Hz; hence $1-r\le 7$

$\times 10^{-13}$. In line with the experimental setup we have chosen the line-center small-signal field-attenuation coefficient $\alpha_0 = 7\ \text{cm}^{-1}$, the line-center saturation intensity $n_0 = 10\ \text{mW/cm}^2$, and the detuning parameter $\delta = 200$. The active medium length is 1 cm. For a forward (or backward) geometry choosing $L = 1$ cm is appropriate; this is also true for a ring cavity having a single pass through the active medium per cavity round trip. However, for a simple cavity in which the radiation travels back and forth through the active medium twice per cavity round trip it is appropriate to take $L = 2$ cm. This latter case reflects the experimental geometry of Ref. 6. For the cavity in question $\cos^2\theta = 0.02$, which leads to an effective cavity enhancement factor $(1+\sin\theta)/(1-\sin\theta) = 198$.

It follows from (41) that if $I/I_s = 0.03$, then $\Delta X_1^2 - \frac{1}{4} = -0.0386$, i.e., a 15.4% squeezing is predicted. If $I/I_s = 0.025$, then $\Delta X_1^2 - \frac{1}{4} = -0.0701$, or a 28.0% squeezing. For $I/I_s = 0.02$, then $\Delta X_1^2 - \frac{1}{4} = -0.0900$ or a 36% squeezing. For the three chosen cases $\alpha L \leq 3.24\times 10^{-4}$, $|\bar{\chi}|L \leq 3.54\times 10^{-3}$, and so $(\alpha + |\bar{\chi}|)L \leq 3.86\times 10^{-3}$, which is about 2.6 times smaller than $1-\sin\theta = 1.01\times 10^{-2}$. Thus we may expect that (42) yields similar predictions for squeezing accurate to within 40–50 %; this is indeed the case. Incidentally, for $I/I_s = 0.025$, for example, $A = -5.57\times 10^{-4}$, $B = 2.26\times 10^{-3}$, and $C = 8.37\times 10^{-4}$ showing that formulas such as (36) and (38) that neglect C can lead to significantly different predictions.

Two additional effects need to be considered. For convenience they are illustrated for (42) while the changes for (41) are simply indicated. First, if ψ is not chosen exactly optimal then (42) should be replaced by

$$\Delta X_1^2 - \tfrac{1}{4} = \tfrac{1}{4}L^*\{(A+B) - f[(A-B)^2 + C^2]^{1/2}\}, \tag{44}$$

where $f = \cos(\Delta\psi)$. If there is a phase jitter then $f = \overline{\cos(\Delta\psi)}$ denoting an average over the distribution of $\Delta\psi$. For a uniform distribution with $|\Delta\psi| \leq \Psi$ then $f = \sin(\Psi)/\Psi$. In the case of (41) the appropriate formula is obtained if r is replaced by fr. We note for the case $I/I_s = 0.02$ a uniform phase jitter with $|\Delta\psi| \leq \Psi = 1$ rad still yields a 16.4% squeezing. Second, if the photodetectors used to measure ΔX^2 have a quantum efficiency $\eta < 1$, then (42) should be replaced by

$$\Delta X_1^2 - \tfrac{1}{4} = \frac{\eta}{4}L^*\{(A+B) - [(A-B)^2 + C^2]^{1/2}\}; \tag{45}$$

for (41), the right-hand side is multiplied by η. For the experiments reported in Ref. 6, $\eta = 0.50$; thus for the case $I/I_s = 0.02$ a 18% squeezing still persists. Finally, for $I/I_s = 0.02$, the combination of these two effects leads to a 8.2% squeezing. Although some parameters of the experiment are less certain than we have implied, the various predictions we have derived compare favorably with the experimental results of Ref. 6 where a 7% squeezing was reported for comparable intensities.

We observe that the change of squeezing as I/I_s decreases in the chosen regime arises principally from a change of the factor A. As can be seen from (10) when $\delta \gg 1$ both $\bar{\Lambda}$ and $\bar{R}_R$ are rapidly changing functions of I/I_s in the interval $0.01 \lesssim I/I_s \lesssim 0.1$. Since $|\chi_I/\chi_R| \approx \delta^{-1}$, the principal change in A then comes from the change in $\bar{\Lambda}$, the factor that reflects spontaneous emission, and this change comes, in turn, just about at the *onset* of the third condition for the ideal noise limit; in particular for $I/I_s = 0.025$ we have $10(I/I_s)^2 = 0.006 \approx \delta^{-1} = 0.005$.

Note added. On completing this paper we received a copy of work by Reid and Walls[9] that relates to the same subject.

ACKNOWLEDGMENTS

We are pleased to thank R. E. Slusher for his constant interest in and encouragement of this work. J.R.K. thanks B. L. Schumaker for a helpful discussion.

[1]See, e.g., D. F. Walls, Nature **306**, 141 (1983).
[2]D. Stoler, Phys. Rev. D **1**, 3217 (1970).
[3]C. M. Caves, Phys. Rev. D **26**, 1817 (1982).
[4]M. D. Reid and D. F. Walls, Phys. Rev. A **31**, 1622 (1985).
[5]B. Yurke, Phys. Rev. A **32**, 300 (1985).
[6]R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985).
[7]P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980).
[8]C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); B. L. Schumaker and C. M. Caves, *ibid.* **31**, 3093 (1985).
[9]M. D. Reid and D. F. Walls (unpublished).