Novel method for the calculation of Čerenkov free-electron-laser gain

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Čerenkov free-electron-laser gain has been calculated in a number of ways by many authors. Often, the method employed depends upon the operational regime, either single-particle or collective. A technique is presented here which is largely regime independent and offers an intuitive picture of the stimulated-emission mechanism. The technique employs the "averaged Lagrangian," and the reader is referred to Whitham for a more thorough discussion of that method.

I. INTRODUCTION

The Čerenkov free-electron laser uses a relativistic electron beam coupled to a dielectric-lined waveguide mode (slow-wave structure) to produce and amplify radiation. The electrons enter the waveguide and spontaneous Čerenkov radiation is emitted. Some radiation will be emitted into a waveguide mode propagating with a phase velocity approximately equal to the electron-beam velocity. This mode will interact with the beam and induce a longitudinal bunching of the electrons on the order of a modal wavelength. Energy can be transferred from the beam to the radiation field, thereby increasing its amplitude, and the coherence of the enhanced radiation is ensured by the bunching. This describes the stimulated emission mechanism of the device and it is this rate of energy transfer which determines the single-pass gain.[1,2] The gain is dependent upon the strength of the radiation-beam interaction as well as the inherent dispersive properties of the waveguide. Since the beam interacts most strongly with radiation which has an electric field component in the same direction, it is the fundamental TM (or TM-like) mode which is most easily amplified.

The first task in the gain analysis is the evaluation of the TM-mode dispersion relation for the waveguide geometry of interest with an electron beam propagating in the longitudinal direction. This is done simply by describing the beam as a region of effective dielectric "constant" ϵ_b and solving Maxwell's equations in the beam, vacuum, and dielectric regions. Continuity of the appropriate fields at each interface yields the desired dispersion relation as a function of the frequency ω, the wavenumber k, and ϵ_b. Some dispersion relations for common geometries are given in the Appendix.

The role of the dispersion relation is twofold. First, it is a characteristic equation relating ω to k. Thus, for radiation traveling synchronously with the electron beam, the dispersion relation supplies the functional dependence of output wavelength on beam energy. Second, and this is the focus of this paper, it can be used to solve for the single-pass gain in both the Compton (single-particle) and collective limits. Furthermore, the functional form of the solutions provides insight into the relation between these limits.

II. THE AVERAGED LAGRANGIAN

A linear dispersive system is characterized by solutions of the form

$$\psi = a e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} .$$

Whitham[3] has developed a formalism which examines the slow variations of these wavetrains for a variety of different physical systems. In this paper the method is adapted and extended to calculate the resultant radiation amplitude growth due to interaction with an electron beam.

In general, the Lagrangian density can be written[4]

$$L = L(t,\psi,\psi^*,\dot{\psi},\dot{\psi}^*,\nabla\psi,\nabla\psi^*) .$$

Here $\psi = a(\mathbf{x})e^{i(kz-\omega t)}$, where z is the propagation direction, $\mathbf{x}$ the transverse directions, and ψ represents the longitudinal electric field component (for TM modes). Furthermore, for ω complex, ψ can be written as the product of a time-dependent amplitude and the phase $e^{i(kz-\omega_0 t)}$ where $\omega = \omega_0 + i\omega''$. That is,

$$\psi = a(\mathbf{x})e^{\omega'' t}e^{i(kz-\omega_0 t)} = A(\mathbf{x},t)e^{i\phi} .$$

If A, ω, and k are slowly varying (with respect to ϕ), then an average Lagrangian density can be described as

$$\mathscr{L} = \frac{1}{2\pi}\int_0^{2\pi} L(t,A,A^*,\dot{\phi},\phi_z,e^{i\phi})\,d\phi .$$

The averaged variational principle is then proposed:

$$\delta\int dt\,dz\,\mathscr{L}(t,A,A^*,\dot{\phi},\phi_z) = 0 .$$

Variations in A and A^* yield the two equations

$$\mathscr{L}_A = 0 \tag{1a}$$

and

$$\mathscr{L}_{A^*} = 0 . \tag{1b}$$

Variation in ϕ yields the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\mathscr{L}_\omega - \frac{\partial}{\partial z}\mathscr{L}_k = 0$$

where (2)

$$\dot{\phi}=-\omega_0,\quad \phi_z=k\ .$$

For linear systems, the Lagrangian density is quadratic in amplitude, which allows one to write

$$\mathscr{L}=|A|^2 f(t,\omega,k)\ .$$

However, evaluation of Eqs. (1) yields

$$f(t,\omega,k)=0=D(t,\omega,k)\,,$$

where D is just the dispersion relation for the system.

An energy conservation equation is derived from the Euler-Lagrange equations by an application of Noether's theorem.[5] The final result becomes

$$\frac{\partial}{\partial t}(\omega\mathscr{L}_\omega)-\frac{\partial}{\partial z}(\omega\mathscr{L}_k)=-\mathscr{L}_t\ . \tag{3}$$

The energy density E is just $\omega\mathscr{L}_\omega$, while the flux density F is $-(\partial/\partial z)(\omega\mathscr{L}_k)$. Thus, an integral over the waveguide length of Eq. (3) is an expression of Poynting's theorem

$$\frac{\partial}{\partial t}\left[\int dz\,E\right]+F=\int dz\,\mathscr{L}_t\ . \tag{4}$$

For the case where all of the energy given up by the beam is converted to radiation energy, i.e., no loss mechanisms in the waveguide, this becomes

$$\frac{\partial|A|^2}{\partial t}\int dz\,\omega D_\omega=|A|^2\left[\int dz\,D_t+\int dz\frac{\partial}{\partial t}(\omega D_\omega)-\int dz\frac{\partial}{\partial z}(\omega D_k)\right]. \tag{5}$$

In order to proceed further, the explicit form of ϵ_b must be determined. It is this term which distinguishes the single-particle and the collective regimes.

III. THE BEAM DIELECTRIC FUNCTION ϵ_b

The dielectric function is related to the conductivity by

$$\epsilon_b=1+\frac{4\pi i\sigma}{\omega}\,,$$

where σ is found from $\delta J=\sigma Ez$. δJ is the induced longitudinal perturbed current density which is a result of the beam's interaction with the radiation field. It can be found from the momentum-distribution-function perturbation δf via

$$\delta J=-e\int dp\,v\,\delta f\,, \tag{6}$$

where δf can, in turn, be found from the linearized Vlasov equation

$$\frac{d}{dt}\delta f=-\dot{p}\frac{\partial F_0}{\partial p}\ . \tag{7}$$

In the single-particle regime, ω is approximated as ω_0, i.e., the frequency shift and growth rate are small compared to ω_0. A zeroth-order integration is performed over Eq. (7) resulting in

$$\delta f=eE_z(\mathbf{x})\frac{\partial F_0}{\partial p}\left[\frac{e^{i(kv-\omega)t}-1}{i(kv-\omega)}\right]. \tag{8}$$

When substituted into Eq. (6), the following expression for the conductivity σ results:

$$\sigma=e^2e^{-i(kz-\omega t)}\int dp\,v\frac{\partial F_0}{\partial p}\left[\frac{e^{i(kv-\omega)t}-1}{i(kv-\omega)}\right], \tag{9}$$

and consequently,

$$\epsilon_b=1-\frac{4\pi e^2}{\omega}e^{-i(kz-\omega t)}\int dp\,v\frac{\partial F_0}{\partial p}\left[\frac{e^{i(kv-\omega)t}-1}{i(kv-\omega)}\right]. \tag{10}$$

In the collective regime, the interaction wavelength is long enough and the beam dense enough so that the induced modulation propagates with the beam, i.e.,

$$\delta f\propto e^{i(kv-\omega)t}\ .$$

Therefore, the linearized Vlasov equation becomes

$$i(kv-\omega)\delta F=eE_z\frac{\partial F_0}{\partial p}$$

and the induced current density is thus

$$\delta J=ie^2\int dp\frac{\partial F_0}{\partial p}\frac{v}{(kv-\omega)}E_z\ . \tag{11}$$

The beam dielectric function is then

$$\epsilon_b=1-\frac{4\pi e^2}{\omega}\int dp\frac{\partial F_0}{\partial p}\frac{v}{(kv-\omega)}\ . \tag{12}$$

Since the growth rate in the collective regime can be relatively large, ω cannot be approximated by its real limit and remains a complex quantity.

Thus, two major distinctions are made between the operational regimes. In the single-particle case, the dispersion relation is explicitly complex and time dependent where both characteristics are a result of the form of ϵ_b. In the collective case, the dispersion relation is implicitly complex due to the complex form of ω and is time independent. With these expressions for ϵ_b, the gain per pass can now be calculated for both regimes.

IV. GAIN

A. Single particle

Referring to Eq. (5), one makes the further assumption that the beam interacts with the empty waveguide radiation fields, i.e., Eq. (5) is evaluated in the limit as $\epsilon_b\to 1$. This becomes

$$\left.\frac{\partial|A|^2}{\partial t}\omega LD_\omega\right|_{\epsilon_b=1}=2|A|^2\int_0^L dz\,4\pi\sigma_{sp}\left.D_{\epsilon_b}\right|_{\epsilon_b=1}, \tag{13}$$

where L is the waveguide length. Note that the left-hand side is now purely real since the only complex term in D was ϵ_b, thus the growth rate can be written

$$\Gamma_{sp} \equiv \frac{1}{|A|^2}\frac{\partial |A|^2}{\partial t} = \frac{8\pi}{L}\mathrm{Re}\int_0^L dz\,\sigma_{sp}\frac{D_{\epsilon_b}}{\omega D_\omega}\bigg|_{\epsilon_b=1} . \tag{14}$$

The integration over σ_{sp} is performed by parts with the (cold beam) assumption that $f_0=\delta(p-p_0)$. Thus,

$$\Gamma_{sp} = -\frac{2\omega_p^2}{\gamma^3}\left[\frac{L}{v}\right]^2\frac{D_{\epsilon_b}}{D_\omega}G'(\theta)\bigg|_{\epsilon_b=1} , \tag{15}$$

where

$$G'(\theta) = \frac{\partial}{\partial\theta}\left[\frac{1-\cos\theta}{\theta^2}\right],\quad \theta=(kv-\omega)L/v ,$$

is the familiar single-particle line shape.

B. Collective

In the collective regime, Eq. (5), evaluated in the limit $\epsilon_b=1$, reduces to just

$$D_\omega = \frac{\partial D}{\partial \omega} + D_{\epsilon_b}\left[\frac{\partial \epsilon_b}{\partial\omega}\right] = 0 . \tag{16}$$

Integrating Eq. 12, one finds that

$$\epsilon_b = 1 - \frac{\omega_p^2}{\gamma^3}\frac{1}{(kv-\omega)^2} ,$$

therefore

$$D_\omega = \frac{2\omega_p^2}{\gamma^3}\frac{D_{\epsilon_b}}{(kv-\omega)^3}\bigg|_{\epsilon_b=1}$$

or (17)

$$(kv-\omega)^3 = \frac{2\omega_p^2}{\gamma^3}\frac{D_{\epsilon_b}}{D_\omega}\bigg|_{\epsilon_b=1} .$$

The right-hand side is evaluated at $\omega=\omega_0$ and the synchronous condition $kv=\omega_0$ is assumed. A cubic equation in the complex quantity $(\omega_s+i\omega'')$ results. The imaginary part of this equation yields the energy growth rate

$$\Gamma_{\mathrm{col}} \equiv 2\omega'' = \sqrt{3}\left[\frac{2\omega_p^2}{\gamma^3}\frac{D_{\epsilon_b}}{D_\omega}\right]^{1/3}\bigg|_{\epsilon_b=1} . \tag{18}$$

The gain per pass is defined as the product of the growth rate and the one-way transit time L/v. Thus,

$$\mathcal{G}_{sp} = \frac{-2\omega_p^2}{\gamma^3}\left[\frac{L}{v}\right]^3\frac{D_{\epsilon_b}}{D_\omega}G'(\theta)\bigg|_{\epsilon_b=1} \tag{19}$$

and

$$\mathcal{G}_{\mathrm{col}} = \sqrt{3}\left[\frac{2\omega_p^2}{\gamma^3}\left[\frac{L}{v}\right]^3\frac{D_{\epsilon_b}}{D_\omega}\right]^{1/3}\Bigg|_{\epsilon_b=1}$$

$$= \sqrt{3}\left[\frac{\mathcal{G}_{sp}}{-G'(\theta)}\right]^{1/3} . \tag{20}$$

Thus, the single-particle gain is a linear function of the beam current ($\propto\omega_p^2$) while the collective gain depends on its cube root. Furthermore, Eq. (20) provides a condition which defines the intersection of the two operating regimes.

That is, when $\mathcal{G}_{\mathrm{col}}=\mathcal{G}_{sp}=\mathcal{G}$, then

$$\mathcal{G} = \left[\frac{(\sqrt{3})^3}{-G'(\theta)}\right]^{1/2} .$$

When $|G'(\theta)|$ is evaluated at, for example, its maximum, then

$$\mathcal{G} \sim 2\pi . \tag{21}$$

Therefore, while Eqs. (19) and (20) allow one to evaluate the gain in either the single-particle or collective limits, Eq. (21) provides a criterion applicable in the neighborhood of the intersection of the two regimes.

V. ATTENUATION

The formalism described in Sec. II can easily be extended to include the effects of attenuation in the dielectric. One assumes that the dielectric liner material can be described by a complex dielectric constant. The dispersion relation, in the absence of the beam, is identical to that of the passive (empty) waveguide with the substitution

$$\epsilon \to \tilde{\epsilon} = \epsilon + \frac{4\pi i\tilde{\sigma}}{\omega} .$$

An evaluation of Eq. (5), without the beam, yields the following result for low losses:

$$\frac{1}{|A|^2}\frac{\partial |A|^2}{\partial z} = \frac{-8\pi\tilde{\sigma}}{\omega}\frac{D_{\tilde{\epsilon}}}{D_k}\bigg|_{\tilde{\epsilon}=\epsilon} . \tag{22}$$

It is convenient to identify the conductivity with the imaginary part of the dielectric constant, i.e., $\tilde{\sigma}=\omega\epsilon''/4\pi$. ϵ'' shall be assumed constant in the dielectric, a liberal restriction when one considers the dimensions of the dielectric films which are used. The value of ϵ'' for the appropriate wavelength region can be found in material handbooks. It is usually given in terms of the "loss tangent," defined as the ratio of the imaginary to the real part of the dielectric function: ϵ''/ϵ. With this substitution, Eq. (22) becomes

$$\alpha \equiv \frac{1}{|A|^2}\frac{\partial |A|^2}{\partial z} = -2\epsilon''\frac{D\tilde{\epsilon}}{D_k}\bigg|_{\tilde{\epsilon}=\epsilon} . \tag{23}$$

Thus, a wave of initial intensity I_0 will be reduced in intensity to $I=I_0e^{-\alpha z}$ after a distance z.

The change in the temporal growth rate due to the attenuation in the dielectric can now be easily evaluated. If the dispersion relation contains both an electron beam and a dielectric of complex $\tilde{\epsilon}$, then Eq. (5), in the single-particle regime, becomes

$$\frac{\partial|A|^2}{\partial t}\omega D_\omega-\frac{\partial|A|^2}{\partial z}\omega D_k=\frac{2|A|^2}{L}4\pi D_{\epsilon_b}\mathrm{Re}\int_0^L dz\,\sigma_b\bigg|_{\substack{\epsilon_b=1\\ \tilde\epsilon=\epsilon}}$$

or (24)

$$\Gamma_{sp}=\frac{-2\omega_p^2}{\gamma^3}\left[\frac{L}{v}\right]^2\frac{D_{\epsilon_b}}{D_\omega}G'(\theta)-2\epsilon''\frac{D_{\tilde\epsilon}}{D_\omega}\bigg|_{\substack{\epsilon_b=1\\ \tilde\epsilon=\epsilon}}$$

$$=\Gamma_{sp}^{(0)}-\Gamma_{\rm att}\ .$$

In the collective regime, Eq. (5) becomes approximately

$$\frac{1}{|A|^2}\frac{\partial|A|^2}{\partial t}\left[D_\omega-\frac{2\omega_p^2}{\gamma^3}\frac{D_{\epsilon_b}}{(kv-\omega)^3}\right]=-\frac{1}{|A|^2}\frac{\partial|A|^2}{\partial z}D_k\bigg|_{\substack{\epsilon_b=1\\ \tilde\epsilon=\epsilon}}\ . \tag{25}$$

This is now a complex equation in $\omega_s+i\omega''$ and ω''. When one solves for the imaginary part of the equation, the following cubic equation in ω'' results:

$$\Gamma^3+\Gamma_{\rm att}\Gamma^2-\Gamma_{\rm col}^{3(0)}=0,\quad \Gamma_i=2\omega_i''\ . \tag{26}$$

The only real positive root of this equation yields the approximate solution $(\Gamma_{\rm att}/\Gamma_{\rm col}^{(0)}\ll 1)$,

$$\Gamma_{\rm col}=\Gamma_{\rm col}^{(0)}-\tfrac{1}{3}\Gamma_{\rm att}\,, \tag{27}$$

a well-known result from the traveling-wave tube literature.[6]

VI. CONCLUSIONS

It is worthwhile, at this point, to review the assumptions and limitations of the preceding formalism. First, the existence of a phase averaged Lagrangian density was postulated. These averages were performed under the assumption that the system undergoes slow modulations and derivatives of amplitude, frequency, and wave number are thus ignorable.

Second, it is assumed, for the single-particle case, that $\omega\sim\omega_0$. That is, the frequency shift and growth rate are quite small compared to ω_0. In the collective case, ω remains complex and one solves for the imaginary part of the quantity $kv-\omega$.

In both cases, an expression is derived for the net rate of energy transfer from the beam to the radiation field minus the loss due to energy attenuation in the dielectric.

It should be apparent that the single-particle and collective growth rates presented here can both be derived as limiting cases of a more general formalism. The utility of the Lagrangian technique is that knowledge of the evolution of the system is obtained from the dispersion relation and its derivatives. Furthermore, direct comparison of the growth rates in the two regimes is greatly facilitated.

The formalism described here can also be used to examine the growth rate for systems where the beam cannot be considered negligible. That is, instead of evaluating the expressions in the no-beam limit, one could determine, self-consistently, the effect of high-current beams on the radiation amplitude. It should be noted that the growth rates derived in this paper are only valid in the small signal regime where the growth can be considered linear. The technique can also be employed to evaluate large signal, nonlinear growth rates with the difference that the averaged Lagrangian is no longer simply proportional to the dispersion relation.

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APPENDIX

Dispersion relations for some common waveguides loaded with a dielectric of constant ϵ and an electron beam.

(1) The double slab with a vacuum width of $2d$, film thickness of a, and beam width of $2b$ is

$$D=\tanh(qd)\tan(pa)-\frac{\epsilon q}{p}-i\Omega\left[\frac{\epsilon q}{p}\tanh(qd)-\tan(pa)\right]=0\,,$$

$$\Omega(\epsilon_b)=i\frac{\tanh(qb)-\sqrt{\epsilon_b}\tanh(sb)}{1-\sqrt{\epsilon_b}\tanh(qb)\tanh(sb)}\ .$$

(2) The single slab with parallel conducting plate with vacuum width of δ, film thickness of a, beam width of b, and film-beam gap of $\delta-b$ is

$$D=\left[\tan(pa)-\frac{\epsilon q}{p}\tanh(q\delta)\right]+\frac{\tanh(qb)-\tanh(sb)/\sqrt{\epsilon_b}}{1-\tanh(qb)\tanh(sb)/\sqrt{\epsilon_b}}\left[\frac{\epsilon q}{p}-\tan(pa)\tanh(q\delta)\right]=0\ .$$

(3) The single slab with film thickness of a, beam width of b, and film-beam gap of δ is

$$D=\left[\tan(pa)-\frac{\epsilon q}{p}\right]+\frac{\tanh(sb)}{\sqrt{\epsilon_b}}\frac{\epsilon_b+\tanh(q\delta)}{1+\tanh(q\delta)}\left[\tan(pa)-\frac{\epsilon q}{p}\frac{1+\epsilon_b\tanh(q\delta)}{\epsilon_b+\tanh(q\delta)}\right].$$

$$q^2=k^2-\omega^2/c^2,\ \ p^2=\omega^2\epsilon/c^2-k^2,\ \ s^2=\epsilon_b q^2 .$$

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