Potential scattering of electrons in a quantized radiation field

J. Bergou
Central Research Institute for Physics, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest, Hungary

F. Ehlotzky
Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria
(Received 26 August 1985)

Potential scattering of electrons in a strong laser field is reconsidered. The laser beam is described by a quantized single-mode plane-wave field with a finite number of quanta in the mode. The scattering amplitude is expanded in powers of the potential, and the first two Born terms are considered. It is shown that in the limit of an infinite number of field quanta, the Kroll-Watson approximation is recovered. Additional insight is gained into the validity of this low-frequency theorem. The approach rests on the introduction of electron-dressed quantized-field states. Relations to earlier work are indicated.

I. INTRODUCTION

Electron scattering in a strong laser field has been investigated by many authors as one of the main mechanisms in laser heating of plasmas. Moreover, the process is of general interest for a deeper understanding of laser-matter interaction. Several reviews have appeared on this subject in recent years.[1] Much theoretical work has also been devoted to this problem in view of the experiments by Weingartshofer *et al.*[2] on induced and inverse bremsstrahlung. From the fundamental point of view particular interest has been devoted to the low-frequency behavior of these scattering phenomena starting with the work of Kroll and Watson.[3] According to these authors, a simple relation can be found in the low-frequency domain between laser-assisted scattering and the corresponding cross sections of the elastic process. The domain of validity of this low-frequency theorem is also of some practical interest and has been checked for particular scattering potentials by Shakeshaft.[4] It is the purpose of the present paper to gain some further insight into the physical meaning and range of validity of the Kroll-Watson theorem.

For simplicity we consider nonresonant potential scattering of electrons in a monochromatic plane-wave field. The potential is taken to have finite range and we expand the transition matrix into a Born series in U. The radiation field is treated in dipole approximation, which is permitted for all standard laser frequencies. First, we present an elementary derivation of the Kroll-Watson formula following the procedure of Choudhury.[5] Here, the laser source is treated as a classical external field. Then we consider electron scattering in a quantized single-mode field. It is assumed that the number of field quanta is finite in the initial and final state. For the description of the scattering process, "electron-dressed" field states are introduced following a method of Bergou and Varró.[6] These coupled electron-field states have many features in common with coherent states but they are orthogonal and complete and turn out to be very convenient for the description of laser assisted phenomena. By letting the number of quanta in the field mode go to infinity, we recover the Kroll-Watson result together with some additional information.

From the structure of the consecutive terms of the Born-series expansion of the transition matrix we can draw the following main conclusions. (i) If the density of photons in the radiation mode is macroscopic, which may be considered as a quantized version of a laser field, then we can recover the Kroll-Watson type of result: the cross section of a scattering process accompanied by the emission or absorption of n photons can be described by the cross section of elastic scattering multiplied by a characteristic Bessel function factor which modifies the angular distribution of the elastic process. At the same time the particle momenta and energies have to be renormalized in a specific manner. (ii) This result can now be regarded as the leading term of an expansion in terms of inverse powers of the initial photon number n_i in the field mode. The next correction is of the order of $n_i^{-3/4}$ from which follows that for macroscopic intensities of the field the Kroll-Watson formula cannot be significantly improved. (iii) It should be noted, however, that the elastic cross sections involved have to be taken between renormalized electron momenta, with the general renormalization prescription given below. Only in the low-frequency limit does this prescription reduce to the result of Kroll and Watson and, therefore, to the usual elastic cross section.

In passing, we note that it is also interesting to investigate the other extreme within the same scheme. Here, the number of photons in the field mode is taken to be small. Then we have to include a macroscopic number of modes into our considerations in order to obtain a finite result. In the special case of single-photon processes, we can thus reproduce the formulas for spontaneous bremsstrahlung and for the one-quantum inverse process.

II. ELEMENTARY THEORY

For the purpose of later references we present here an elementary treatment of electron scattering in an external radiation field and we derive those formulas required for our discussions in the subsequent sections. A more rigorous analysis of the semiclassical theory can be found in the work of Mittleman[7] and Leone *et al.*[8]

We describe the laser field by a classical monochromatic wave in dipole-approximation, for which the vector potential is given by

$$\mathbf{A}(t)=A_0\boldsymbol{\epsilon}\cos(\omega t)\ . \tag{2.1}$$

The Schrödinger equation for a particle in this field

$$i\hbar\partial_t\psi=(2m)^{-1}[-i\hbar\nabla-(e/c)\mathbf{A}(t)]^2\psi \tag{2.2}$$

has solutions of the form

$$\psi_{\mathbf{p}}=V^{-1/2}\exp((i\hbar)^{-1}\{E_{\mathbf{p}}^{(0)}t-\mathbf{p}[\mathbf{x}-\mathbf{x}_c(t)]\})\ , \tag{2.3}$$

where $\mathbf{x}_c(t)=-(\mu c/\omega)\boldsymbol{\epsilon}\sin(\omega t)$ is the solution of the classical Lorentz equation of motion. $\mu^2=(eA_0/mc^2)^2$ is the intensity parameter and the time-dependent contribution of the A^2 term has been dropped since it yields an overall phase shift that cancels in our calculations. From the set of modified plane-wave solutions (2.3) with $E_{\mathbf{p}}^{(0)}=\mathbf{p}^2/2m$ we can immediately write down the retarded Green's function

$$G^{(+)}(\mathbf{x}',\mathbf{x};t',t)=\Theta(t'-t)(2\pi\hbar)^{-3}\int d^3p\,\exp\{(i\hbar)^{-1}[E_{\mathbf{p}}^{(0)}(t'-t)-\mathbf{p}(\mathbf{x}'-\mathbf{x}-\mathbf{x}_c(t')+\mathbf{x}_c(t))]\}\ . \tag{2.4}$$

We expand the transition matrix element of electron scattering in the field (2.1) in powers of the potential U and write $T_{fi}=T_{fi}^{(1)}+T_{fi}^{(2)}+\cdots$. Choosing two states ψ_{p_i} and $\psi_{p_f}^*$ for the ingoing and outgoing particle, respectively, we obtain for the first Born term

$$T_{fi}^{(1)}=(i\hbar)^{-1}\int d^4x\,\psi_{\mathbf{p}_f}^*U\psi_{\mathbf{p}_i}=\sum_n T_n^{(1)}\ , \tag{2.5}$$

with

$$T_n^{(1)}=-2\pi iV^{-1}U(\mathbf{Q})J_n(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon})\delta(E_f^{(0)}-E_i^{(0)}-n\hbar\omega)\ , \tag{2.6}$$

where $U(\mathbf{Q})$ is the Fourier transform of the scattering potential as a function of the momentum transfer $\mathbf{Q}=\mathbf{p}_f-\mathbf{p}_i$. The J_n are Bessel functions of integer order with $n>0$ for induced absorption and $n<0$ for emission. Moreover, we have $\rho=\mu c/\hbar\omega$ and, consequently, $\rho\,|\,\mathbf{Q}\,|$ is dimensionless.

For the second Born term we get with the Green's function (2.4)

$$T_{fi}^{(2)}=(i\hbar)^{-1}\int d^4x\int d^4x\,\psi_{\mathbf{p}_f}^*(\mathbf{x}',t')U(\mathbf{x}')G^{(+)}(\mathbf{x}',\mathbf{x};t',t)U(\mathbf{x})\psi_{\mathbf{p}_i}(\mathbf{x},t)=\sum_n T_n^{(2)}\ , \tag{2.7}$$

with

$$T_n^{(2)}=-2\pi iV^{-1}\sum_{k=-\infty}^{+\infty}(2\pi\hbar)^{-3}\int d^3p'\,U(\mathbf{p}_f-\mathbf{p}')U(\mathbf{p}'-\mathbf{p}_i)\times\frac{J_{n-k}(\rho(\mathbf{p}_f-\mathbf{p}')\cdot\boldsymbol{\epsilon})J_k(\rho(\mathbf{p}'-\mathbf{p}_i)\cdot\boldsymbol{\epsilon})}{E'-E_i^{(0)}-k\hbar\omega+i\eta}\delta(E_f^{(0)}-E_i^{(0)}-n\hbar\omega)\ . \tag{2.8}$$

In (2.8) we evaluate the sum over k by the following approximation.[5] On account of Graf's addition theorem for Bessel functions we may write[9]

$$J_{n-k}(x)J_k(y)=(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi\,\exp(ik\varphi)J_n(w)\{[x+y\exp(i\varphi)][x+y\exp(-i\varphi)]^{-1}\}^{n/2}\ , \tag{2.9}$$

where $w=(x^2+y^2+2xy\cos\varphi)^{1/2}$. We now assume that the integral (2.9) is dominated by contributions from $-\epsilon<\varphi<\epsilon$, where $|\,k\,|^{-1}>\epsilon>0$, since for large values of φ the integrand is rapidly oscillating. We therefore make the approximations $\cos\varphi\simeq1$ and $\sin\varphi\simeq\varphi$ and extend the range of integration to infinity. Then (2.9) yields

$$J_{n-k}(x)J_k(y)=J_n(x+y)\delta(k-ny(x+y)^{-1})\ . \tag{2.10}$$

Essentially, we have made a crude application of the method of stationary phase in going from (2.9) to (2.10) and the validity of this procedure is not so obvious, but has its justification through the results of the more rigorous work of Mittleman[7] and Leone *et al.*[8] On account of (2.10) we can replace the photon number k by its effective value in the denominator of (2.8),

$$k_{\text{eff}}=n(\mathbf{p}'-\mathbf{p}_i)\cdot\boldsymbol{\epsilon}(\mathbf{Q}\cdot\boldsymbol{\epsilon})^{-1}\ . \tag{2.11}$$

Equation (2.8) will, therefore, read

$$T_n^{(2)}=-2\pi iV^{-1}(2\pi\hbar)^{-3}\times\int d^3p'\frac{U(\mathbf{p}_f-\mathbf{p}')U(\mathbf{p}'-\mathbf{p}_i)}{E'-E_i^{(0)}-k_{\text{eff}}\hbar\omega+i\eta}\times J_n(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon})\delta(E_f^{(0)}-E_i^{(0)}-n\hbar\omega)\ . \tag{2.12}$$

Now we use (2.11) to rewrite the denominator of (2.12) in the form

$$E'-E_i^{(0)}-k_{\text{eff}}\hbar\omega=\widetilde{E}'-\widetilde{E}_i \tag{2.13}$$

and, similarly, to write for the arguments of the δ functions in (2.6) and (2.8)

$$E_f^{(0)}-E_i^{(0)}-n\hbar\omega=\widetilde{E}_f-\widetilde{E}_i\ , \tag{2.14}$$

by defining renormalized particle momenta and energies through the relations

$$\widetilde{\mathbf{p}}_n=\mathbf{p}-n\hbar\omega m\boldsymbol{\epsilon}(\mathbf{Q}\cdot\boldsymbol{\epsilon})^{-1},\ \ \widetilde{E}_n=\widetilde{\mathbf{p}}_n^{\,2}/2m\ . \tag{2.15}$$

The same renormalization can be carried out for all higher-order Born terms. Consequently, the transition matrix element T_n for the nth-order nonlinear process may be written as

$$T_n=T_{\rm el}(\widetilde{E}_n,\mathbf{Q})J_n(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon})\ , \tag{2.16}$$

where $T_{\rm el}$ is the matrix element of the corresponding elastic process at energy $\widetilde{E}_n$. The relation (2.16) is the essence of the Kroll-Watson approximation.

III. PARTICLE-DRESSED FIELD STATES

For the description of electron scattering in a quantized single-mode radiation field in the dipole approximation, it is convenient to use particle-dressed field states in the form introduced by Bergou and Varró.[6] These states, which may also be considered as generalized coherent states of the radiation field, form a complete and orthogonal set, and the photon distribution of these states is determined by the momenta of the particles embedded in the field. We shall present here only so much information on these states as is required for the description of potential scattering to be discussed below. For more details on these states we refer the reader to the work of Bergou and Varró.[6]

We consider the Schrödinger equation for a particle in a quantized radiation field

$$i\hbar\partial_t\psi=H\psi,\ \ H=H_e+H_f+H_I\ , \tag{3.1}$$

where $H_e=\hat{\mathbf{p}}^2/2m$ is the Hamiltonian of the free electron, $H_f=\hbar\omega(a^\dagger a+\frac{1}{2})$ the Hamiltonian of the free field mode, and the interaction term reads

$$H_I=-(e/mc)\mathbf{A}\cdot\hat{\mathbf{p}}+(e^2/2mc^2)\mathbf{A}^2\ . \tag{3.2}$$

Here we write for the vector potential of the field in the dipole approximation and for linear polarization,

$$\mathbf{A}=\alpha\boldsymbol{\epsilon}(a^\dagger+a),\ \ \alpha=c(2\pi\hbar/\omega V)^{1/2}\ . \tag{3.3}$$

In the following, we shall neglect contributions from the $\mathbf{A}^2$ part of the interaction since, as in our semiclassical treatment in Sec. II,[6] this will have no consequences.

In the Schrödinger picture, the Hamiltonian (3.1) is time independent and we can look for stationary states of the form

$$\psi=\varphi\exp(-iEt/\hbar)\ , \tag{3.4}$$

where the solutions φ are found to be given by

$$|\varphi\rangle\equiv|\varphi_{\mathbf{p},n}\rangle=|\mathbf{p}\rangle D_{\sigma_p}|n\rangle\ . \tag{3.5}$$

Here, $|\mathbf{p}\rangle$ is a momentum eigenstate of the electron, and $|n\rangle$ is a number state of the field oscillator. D_{σ_p} denotes a unitary displacement operator, which is defined by

$$D_{\sigma_p}=\exp[\sigma_p(a^\dagger-a)],\ \ \sigma_p=(e\alpha/mc\hbar\omega)\mathbf{p}\cdot\boldsymbol{\epsilon}\ . \tag{3.6}$$

The energy eigenvalues belonging to the states (3.5) and (3.6) are evaluated to be

$$E\equiv E_{pn}=(\mathbf{p}^2/2m)+\hbar\omega(n+\tfrac{1}{2}-\sigma_p^2)\ , \tag{3.7}$$

and on account of the properties of $|\mathbf{p}\rangle$, $|n\rangle$, and D_{σ_p}, these states form a complete orthonormal set. A more general form of coherent (3.5) and (3.6) has been introduced by Rosenberg.[10]

IV. POTENTIAL SCATTERING

Potential scattering of electrons in a quantized radiation field has been investigated before in a series of papers by Kelsey and Rosenberg[11] and by Rosenberg.[12] These authors use for their analysis the methods of formal scattering theory, and they represent the radiation field by number states for which depletion is neglected. The latter approximation, however, requires from the outset a large number of quanta in the field mode and, therefore, the final results of the calculations of these authors turn out to be identical with those of the quasiclassical theory.[7,8] Kelsey and Rosenberg[11] explicitly refer to this equivalence in the last paragraph of Sec. III of their paper. In our approach, this requirement of a large number of quanta in the field mode is not necessary, except for the semiclassical limit. In another paper by Rosenberg,[10] mentioned earlier, the same scattering problem is reinvestigated by introducing generalized coherent states, however somewhat different approximations are made for the evaluation of the T-matrix elements to be discussed below.

As in the elementary treatment in Sec. II, we expand the transition matrix element in powers of the scattering potential U, and we consider the first and second Born term. If during the scattering process the number of quanta in the field mode changes by $n\gtrless 0$, then the matrix element in first Born approximation reads

$$T_n^{(1)}=(i\hbar)^{-1}\int dt\,\exp[(i\hbar)^{-1}(E_f^{(n)}-E_i)t]\times\langle\mathbf{p}_f|U|\mathbf{p}_i\rangle\langle n_i-n|D^*_{\sigma_f}D_{\sigma_i}|n_i\rangle\ , \tag{4.1}$$

where the ingoing and outgoing particles have been described by states of the form (3.4), (3.5), (3.6), and (3.7). n_i and $n_f=n_i-n$ are the numbers of field quanta in the initial and final states, respectively. The energies in the initial and final states of the particle-field system are given by

$$E_f^{(n)}=(p_f^2/2m)+(n_i-n)\hbar\omega,\ \ E_i=(p_i^2/2m)+n_i\hbar\omega\ . \tag{4.2}$$

Here we have neglected the zero-point energy $\hbar\omega/2$ and the particle-dependent corrections $\sigma_i^2\hbar\omega$ and $\sigma_f^2\hbar\omega$, since σ^2 is usually small for nonrelativistic electrons, in particular in the limit $V\to\infty$.[6]

Carrying out the integration and introducing the Fourier-transform $U(\mathbf{Q})$ of the scattering potential, we obtain from (4.1)

$$T_n^{(1)}=-2\pi iU(\mathbf{Q})\langle n_i-n|D_{\sigma_i-\sigma_f}|n_i\rangle\delta(E_f^{(n)}-E_i)\ , \tag{4.3}$$

where we have taken into account the multiplication rule of the D operator, $D^*_{\sigma_1}D_{\sigma_2}=D_{-\sigma_1}D_{\sigma_2}=D_{\sigma_2-\sigma_1}$.[6] In (4.3) the matrix element of $D_{\sigma_i-\sigma_f}=D_{\sigma_{fi}}$ can be expressed in terms of Laguerre polynomials in the following form

$$\langle n_f \mid D_{\sigma_{fi}} \mid n_i \rangle = \{[\min(n_i,n_f)]!/[\max(n_i,n_f)]!\}^{1/2} \times \exp(\sigma_{fi}^2/2)\sigma_{fi}^n L^n_{\min(n_i,n_f)}(\sigma_{fi}^2) \,, \quad (4.4)$$

and in the limit $n_i \to \infty$, with $(n_i/V)=\text{const}=\rho_{\text{ph}}$, we obtain for this finite photon density in the radiation mode[9]

$$\langle n_i - n \mid D_{\sigma_{fi}} \mid n_i \rangle = J_{-n}(2\sigma_{fi} n_i^{1/2}) \,. \quad (4.5)$$

However, in this limit we may identify $\hbar\omega n_i/V$ with the energy density $\omega^2 A_0^2/8\pi c^2$ in our classical wave (2.1), and so we obtain from the definitions of μ, ρ, α, and σ,

$$2\sigma_{fi} n_i^{1/2} = -\rho \mathbf{Q}\cdot\boldsymbol{\epsilon} \,. \quad (4.6)$$

Hence, we recover from (4.3) and (4.5) the result of our semiclassical calculation (2.6).

Our result (4.3) for the T-matrix element in Born approximation is essentially identical with formula (2.24) of Rosenberg,[10] except that this author uses a more general composition of radiation field modes and he does not consider the quasiclassical limit yielding our formulas (4.5) and (4.6).

Next, we evaluate the second-order Born term for the same transitions. This yields the matrix element

$$T_n^{(2)} = (i\hbar)^{-2} \int dt' \int dt \langle \psi_f(t') \mid U G^{(+)}(t',t) U \mid \psi_i(t) \rangle \,, \quad (4.7)$$

where the retarded Green's function $G^{(+)}(t',t)$ can be easily obtained from the states (3.4) and (3.5) to be

$$G^{(+)}(t',t) = \theta(t'-t) \sum_{n'} \int d^3p' \exp[(i\hbar)^{-1} E_{p',n'}(t'-t)] \times \mid p' \rangle \langle p' \mid D_{\sigma_{p'}} \mid n' \rangle \langle n' \mid D^*_{\sigma_{p'}} \,. \quad (4.8)$$

If we insert (4.8) into (4.7), and carry out the integration with respect to time, we obtain

$$T_n^{(2)} = -2\pi i \sum_{n'} \int d^3p' \frac{U(\mathbf{p}_f-\mathbf{p}')U(\mathbf{p}'-\mathbf{p}_i)}{E_{p',n'}-E_i+i\eta} \delta(E_f^{(n)}-E_i) \times \langle n_f \mid D^*_{\sigma_f} D_{\sigma_{p'}} \mid n' \rangle \langle n' \mid D^*_{\sigma_{p'}} D_{\sigma_i} \mid n_i \rangle \,. \quad (4.9)$$

Let us write out the denominator in (4.9) explicitly,

$$\begin{aligned} E_{p',n'} - E_i &= E' + n\hbar\omega - E_i^{(0)} - n_i\hbar\omega \\ &= E' - E_i^{(0)} - k\hbar\omega \,, \end{aligned} \quad (4.10)$$

where $k = n_i - n'$ is the number of virtually absorbed $(k>0)$ or emitted $(k<0)$ quanta in the intermediate states. If it is possible to replace k, as in the elementary treatment, by an appropriately chosen fixed k_{eff}, then the denominator of (4.9) becomes independent of n', and the summation in the numerator can be easily carried out on account of $\sum_{n'} \mid n' \rangle \langle n' \mid = I$ and the unitarity of the D's. Thus we get

$$T_n^{(2)} = -2\pi i \int d^3p' \frac{U(\mathbf{p}_f-\mathbf{p}')U(\mathbf{p}'-\mathbf{p}_i)}{E'-E_i^{(0)}-k_{\text{eff}}\hbar\omega+i\eta} \times \langle n_f \mid D_{\sigma_{fi}} \mid n_i \rangle \delta(E_f^{(n)}-E_i) \,. \quad (4.11)$$

In order to obtain an estimate for k_{eff}, we have to investigate in (4.9) the expression

$$\langle n_f \mid D^*_{\sigma_f} D_{\sigma_{p'}} \mid n' \rangle \langle n' \mid D^*_{\sigma_{p'}} D_{\sigma_i} \mid n_i \rangle \,.$$

This expression is the product of two terms of similar structure. We shall therefore consider the matrix element $\langle n' \mid D^*_{\sigma_{p'}} D_{\sigma_i} \mid n_i \rangle$ in some detail. The other one will yield analogous results with an appropriate change of variables. In our analysis of the above matrix element we observe the following facts:

(i) Because of the properties of the D operator and with our definition for $k=n_i-n'$, this matrix element can be written as

$$\langle n' \mid D^*_{\sigma_{p'}} D_{\sigma_i} \mid n_i \rangle = \langle n_i - k \mid D_{\sigma_i-\sigma_{p'}} \mid n_i \rangle \,. \quad (4.12)$$

(ii) $D_{\sigma_i-\sigma_{p'}} \mid n_i \rangle$ is some kind of general coherent quantum state of the quantized-field mode which develops from the number state $\mid n_i \rangle$ upon application of the unitary operator $D_{\sigma_i-\sigma_{p'}}$. Consequently, (4.12) represents the probability amplitude for finding (n_i-k) photons in the state $D_{\sigma_i-\sigma_{p'}} \mid n_i \rangle$. Therefore, our task reduces to the problem of finding those k values for which the probability $\mid \langle n_i - k \mid D_{\sigma_i-\sigma_{p'}} \mid n_i \rangle \mid^2$ is significant.

(iii) In view of this, we shall investigate the statistical properties of a general state $D_\sigma \mid n \rangle$ of this type. Since D_σ has the displacement properties $D^*_\sigma a D_\sigma = a + \sigma$ and $D^*_\sigma a^\dagger D_\sigma = a^\dagger + \sigma$, we obtain for the mean photon number in such a state[6]

$$\begin{aligned} \langle n \rangle &= \langle n \mid D^*_\sigma a^\dagger a D_\sigma \mid n \rangle \\ &= \langle n \mid (a^\dagger+\sigma)(a+\sigma) \mid n \rangle = n + \sigma^2 \,, \end{aligned} \quad (4.13)$$

and for the mean-square deviation, we get

$$\begin{aligned} \langle (n - \langle n \rangle)^2 \rangle &= \langle n^2 \rangle - \langle n \rangle^2 \\ &= \langle n \mid D^*_\sigma (a^+ a)^2 D_\sigma \mid n \rangle - (n+\sigma^2)^2 \\ &= 2\sigma^2(n+\tfrac{1}{2}) \,. \end{aligned} \quad (4.14)$$

According to the definitions of α and σ in (3.3) and (3.6), we obtain $\sigma^2 \sim V^{-1}$ and therefore in the limit of large n and large V with a finite photon density $\rho_{\text{ph}}=n/V$, we conclude from (4.13) and (4.14) in this limit:

(a) In the state $D_\sigma \mid n \rangle$ the mean photon number $\langle n \rangle$ becomes n, and, therefore, in this limit our representation of particle-dressed field states will merge in the number-state representation chosen by Kelsey and Rosenberg[11] and Rosenberg.[12] In our single-mode approximation this becomes particularly apparent from the representation (4.4) for the matrix elements $\langle n_f \mid D_{\sigma_{fi}} \mid n_i \rangle$ and their

Hilb-type asymptotic form (4.5) to which we again refer to in (5.7).

(b) The root-mean-square deviation $(\langle n^2\rangle - \langle n\rangle^2)^{1/2} = \Delta n$ becomes $(2n)^{1/2}\sigma \sim \rho_{\rm ph}^{1/2}$ and $(\Delta n/\langle n\rangle)$ approaches to zero.

(iv) By applying these results to (4.12), we infer that only those k values will yield significant contributions to the magnitude of this matrix element for which the deviation of $n_i - k$ from n_i is not significantly more than the root-mean-square deviation. Consequently, we can write

$$k_{\rm eff} = C(2n_i)^{1/2}(\sigma_{p'} - \sigma_i) , \tag{4.15}$$

where C is a constant of proportionality. The meaning and order of magnitude estimate of the other matrix element in (4.9) can be analyzed in a similar way. If we write $n_f = n_i - n$, then we see that only those values of k will yield a significant contribution to the order of magnitude of this matrix element for which the deviation of $n_i - k$ from $n_i - n$ is of the order of $[2(n_i - n)]^{1/2}(\sigma_f - \sigma_{p'})$. Therefore, we can write down the relation

$$n - k_{\rm eff} = C[2(n_i - n)]^{1/2}(\sigma_f - \sigma_{p'}) , \tag{4.16}$$

where C has to be the same constant of proportionality as in (4.15), since we should choose a universally defined measure for the deviation from the mean value of the photon number. In fact the relations (4.15) and (4.16) determine the width of two virtual photon distributions which have their maximum overlap at $k_{\rm eff}$. This overlap is essentially responsible for the accuracy of the Kroll-Watson approximation.

If we consider sufficiently large photon numbers n_i in the initial state of the field mode (which is well justified for a laser), then we may neglect in (4.16) the depletion n. In that case, (4.15) and (4.16) furnish two simple equations for the evaluation of C and $k_{\rm eff}$. We obtain by means of (3.6)

$$k_{\rm eff} = n(\mathbf{p}' - \mathbf{p}_i)\cdot\boldsymbol{\epsilon}(\mathbf{Q}\cdot\boldsymbol{\epsilon})^{-1} , \tag{4.17}$$

which is identical to our result (2.11) of the elementary theory. For the universal constant C, we evaluate the expression

$$C = n(2n_i)^{-1/2}(\sigma_f - \sigma_i)^{-1} . \tag{4.18}$$

If we consider the limit $n_i \to \infty$, $V \to \infty$ with $(n_i/V) = \text{const} = \rho_{\rm ph}$, then we get an account of (4.6),

$$C = n\,2^{1/2}(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon})^{-1} . \tag{4.19}$$

In the same limit, however, the order of nonlinearity of a laser-induced scattering process is measured by $J_n^2(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon})$ of (2.16). It is well known that Bessel functions will yield maximum contributions if $|n| \lesssim |\rho\mathbf{Q}\cdot\boldsymbol{\epsilon}|$ and we therefore conclude that $|C| \simeq 1$. This lends some additional support to our arguments that $k_{\rm eff}$ is determined by the widths of the virtual photon distributions in the intermediate states of (4.9).

In the aforementioned work of Rosenberg[10] a Green's function is derived, formula (2.23), which essentially agrees with our Green's function (4.8), except that we only consider a single-mode radiation field. Rosenberg then considers a low-frequency approximation yielding his formulas (3.2), (3.3), (3.5), and (3.7) which, however, in his following investigations do not yield our generalized renormalization conditions (4.15) and (4.16) even in the single-mode case. Consequently, we also derive a different renormalized low-frequency T-matrix element (5.6) which will be discussed below.

V. CONCLUDING REMARKS

On account of the results of our treatment of potential scattering of electrons in a quantized radiation field, the following conclusions can be drawn concerning the range of validity of the Kroll-Watson renormalization scheme, which we have outlined in Sec. II for a classical single-mode field.

(i) The Kroll-Watson analysis has been carried out here in a more general frame work of particle-dressed field states.[6] We were able to relate $k_{\rm eff}$ to the widths of the virtual photon distributions in the intermediate states of (4.9). For finite initial photon number n_i a generalized expression for $k_{\rm eff}$ can be derived from (4.15) and (4.16). This reads

$$k_{\rm eff} = n(\mathbf{p}' - \mathbf{p}_i)\boldsymbol{\epsilon}\{\mathbf{Q}\cdot\boldsymbol{\epsilon} + [(1 + n/n_i)^{1/2} - 1](\mathbf{p}_f - \mathbf{p}')\cdot\boldsymbol{\epsilon}\}^{-1} . \tag{5.1}$$

If $|n/n_i| < 1$, the second term in the denominator of (5.1) will represent a small correction to the first term $\mathbf{Q}\cdot\boldsymbol{\epsilon}$. For low radiation frequencies ω, we may, therefore, write on account of (4.16) and (4.18), $(\mathbf{p}_f - \mathbf{p}')\cdot\boldsymbol{\epsilon} \simeq \gamma\mathbf{Q}\cdot\boldsymbol{\epsilon}$, where γ is a parameter of the order of unity. By only retaining linear terms in (n/n_i), we may thus replace (5.1) by

$$k_{\rm eff} = n(\mathbf{p}' - \mathbf{p}_i)\cdot\boldsymbol{\epsilon}[\mathbf{Q}\cdot\boldsymbol{\epsilon}(1 + \gamma n/2n_i)]^{-1} . \tag{5.2}$$

If this approximation is taken to be sufficiently accurate, then we are able to demonstrate that the Kroll-Watson scheme can be carried through even for moderate values of the photon number n_i in the initial field state. In accordance with (5.2), we can define new renormalized momenta by

$$\widetilde{\mathbf{p}}_n = \mathbf{p} - n\hbar\omega m\boldsymbol{\epsilon}[\mathbf{Q}\cdot\boldsymbol{\epsilon}(1 + \gamma n/2n_i)]^{-1} , \tag{5.3}$$

and thus we can put the denominator of (4.11) into the form $\widetilde{E}' - \widetilde{E}_i$ as in (2.13). However, for the energy conservation relation we would get instead of (2.14),

$$\widetilde{E}_f - \widetilde{E}_i = E_f^{(0)} - E_i^{(0)} - n\hbar\omega(1 + \gamma n/2n_i)^{-1} , \tag{5.4}$$

which does not agree with the arguments of the δ functions in (4.3) and (4.11). Consequently, the renormalization procedure has to be slightly modified. To this end, we rewrite (5.4) such as to have on the left-hand side the correct argument of the δ functions of (4.3) and (4.11). Then we make the expansion

$$\begin{aligned}\delta(E_f^{(0)} - E_i^{(0)} - n\hbar\omega) &\\ = \delta(\widetilde{E}_f - \widetilde{E}_i) - n\hbar\omega[1 - (1 + \gamma n/2n_i)^{-1}]& \\ \times \delta'(\widetilde{E}_f - \widetilde{E}_i) + \cdots ,&\end{aligned} \tag{5.5}$$

and we introduce this into the total renormalized transition amplitude T_n, which may be obtained by evaluating all higher Born terms $T_n^{(\nu)}$, $\nu \geqq 3$, by means of our generalized renormalization scheme (5.2) and (5.3). If we write

$$T_n = -2\pi i t_n(\widetilde{E}_i,\mathbf{Q})\delta(E_f^{(0)}-E_i^{(0)}-n\hbar\omega)$$

and use the expansion (5.5), we get

$$T_n = -2\pi i[t_n(\widetilde{E}_i,\mathbf{Q}) + n\hbar\omega(\gamma n/2n_i)(\partial t_n/\partial\widetilde{E}_i)]\delta(\widetilde{E}_f-\widetilde{E}_i) , \quad (5.6)$$

where again we have only retained the linear term in (n/n_i). From (4.3) and (4.11) we conclude that $t_n = T_{\rm el}\langle n_f | D_{\sigma_{fi}} | n_i\rangle$, where $t_{\rm el}(\widetilde{E}_i,\mathbf{Q})$ is the renormalized amplitude of the elastic process. From (5.6), we recover in the limit $(n/n_i)\rightarrow 0$ the Kroll-Watson approximation and we now see why this approximation works so well for laser-field intensities.[4] In deriving the relation (5.6) for a finite photon number n_i in the initial state $| n_i\rangle$, we were able to go beyond the results obtained by Kelsey and Rosenberg.[11] This can be traced back to the properties of the generalized coherent states which do not require in their derivation the neglect of depletion. Moreover, in the work of Rosenberg[10] on scattering in a low-frequency multimode field, described by generalized coherent states, formulas different from (5.6) have been derived on account of another approach to the low-frequency approximation.

(ii) In our investigation of scattering in a quantized radiation field, the nonlinearities of field-induced processes are characterized by the factors $|\langle n_f | D_{\sigma_{fi}} | n_i\rangle|^2$ instead of the Bessel function terms $J_n^2(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon})$ of the semiclassical theory. The matrix elements $\langle n_f | D_{\sigma_{fi}} | n_i\rangle$ can be expressed in terms of Laguerre polynomials (4.4). This more general expression has to be considered if there is a finite number n_i of quanta in the initial field state. In the limit $n_i\rightarrow\infty$, $V\rightarrow\infty$ with $(n_i/V)=\rho_{\rm ph}$ we can use the asymptotic relation of Hilb[9]

$$\lim_{n_i\rightarrow\infty} [\exp(-x/2)(x/n_i)^{n/2}L_{n_i}^{(n)}(x)] = J_n[2(n_i x)^{1/2}] + O(n_i^{-3/4}) , \quad (5.7)$$

with $x=\sigma_{fi}^2$. By means of (4.6) we thus recover from (4.3) and (4.11) the Kroll-Watson formula

$$d\sigma_n = d\sigma_{\rm el}(\widetilde{E}_n,\mathbf{Q})J_n^2(\rho\mathbf{Q}\cdot\boldsymbol{\epsilon}) , \quad (5.8)$$

with

$$\widetilde{E}_n = \widetilde{p}_n^{\,2}/2m \text{ and } \widetilde{\mathbf{p}}_n = \mathbf{p}_i - n\hbar\omega m\boldsymbol{\epsilon}(\mathbf{Q}\cdot\boldsymbol{\epsilon})^{-1} .$$

(iii) In our picture of the scattering problem, we have evaluated the probability amplitudes for finding $n_f = n_i - n$ photons in the quantized-field mode, if this state develops from an initial state $| n_i\rangle$ during the simultaneous scattering of an electron, embedded in the field, by the potential U with momentum transfer $\mathbf{Q}$. This result of our quantum-mechanical treatment of the radiation field confirms the intuitive photon picture associated with calculations based on a classical external field as described in Sec. II and discussed in the more rigorous work of Mittleman[7] and Leone *et al.*[8]

In conclusion we should like to point out that the main purpose of our paper has been to show how the generalized coherent states in the form introduced by Bergou and Varró[6] can be used to conveniently and elegantly treat scattering processes in strong radiation fields, yielding results which go beyond those of earlier investigations. We are aware that our present problem could also be treated within the framework of formal scattering theory but we did not intend to follow this line here. Instead, as a first step we tried to get some more intuitive insight into the validity of the Kroll-Watson approximation, using the formalism of particle-dressed field states.

ACKNOWLEDGMENT

This work was supported by the Österreichische Forschungsgemeinschaft under Project No. 06/0400.

[1] M. Gavrila, *Atomic Physics 9*, edited by N. Fortsan (Plenum, New York, 1985); G. Ferrante, *Physics of Ionized Gases*, edited by M. M. Popovic (Springer, Berlin, 1985); L. Rosenberg, Adv. At. Mol. Phys. **18**, 1 (1982); M. H. Mittleman, Comments At. Mol. Phys. **11**, 91 (1982).
[2] A. Weingartshofer, J. K. Holmes, J. Sabbagh, and S. L. Chin, J. Phys. B **16**, 1805 (1983).
[3] N. M. Kroll and K. M. Watson, Phys. Rev. A **8**, 804 (1973).
[4] R. Shakeshaft, Phys. Rev. A **28**, 667 (1983); **29**, 383 (1984).
[5] B. J. Choudbury, Phys. Rev. A **11**, 2194 (1975).
[6] J. Bergou and S. Varró, J. Phys. A **14**, 1469 (1981).
[7] M. H. Mittleman, Phys. Rev. A **19**, 134 (1979); **20**, 1965 (1979).
[8] C. Leone, P. Cavaliere, and G. Ferrante, J. Phys. B **17**, 1027 (1984).
[9] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw Hill, New York, 1953), Vol. II.
[10] L. Rosenberg, Phys. Rev. A **21**, 157 (1980).
[11] E. J. Kelsey and L. Rosenberg, Phys. Rev. A **19**, 756 (1979).
[12] L. Rosenberg, Phys. Rev. A **20**, 275 (1979); **22**, 2485 (1980).