Canonical treatment of harmonic oscillator with variable mass

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By the use of a canonical transformation the problem of the harmonic oscillator with a time-dependent mass has been transformed to that of an oscillator with a time-dependent frequency. Pseudostationary and quasicoherent states are discussed.

I. INTRODUCTION

The problem of the time-dependent harmonic oscillator has been the object of renewed interest over the past decade. From the physical point of view, the most relevant applications arise in the quantum-mechanical treatment of the damped oscillator. This leads a variation of the mass parameter according to

$$M(t)=m\exp(2\delta t)\ . \tag{1.1}$$

We single out work by Kanai,[1] Hasse,[2,3] Tartglia,[4] Dekker,[5] and Caldirola.[6,7] Dodonov and Man'Ko[8] have considered the mass

$$M(t)=m\exp[2\Gamma(t)] \tag{1.2}$$

in order to treat the case of variable damping. Abdalla and Colegrave[9] give an exact solution for a particular $\Gamma(t)$. Remaud and Hernandez[10] point out that when energy is supplied to an oscillating system in a periodic cycle, the resulting dynamics may be described by

$$H(q,p,t)=\frac{p^2}{2M(t)}+\tfrac{1}{2}M(t)\omega_0^2q^2 \tag{1.3}$$

with a periodic $M(t)$.

This case could be considered to correspond to an imaginary damping coefficient δ in (1.1), but then the Hamiltonian would cease to be real (in classical mechanics) or self-adjoint (in quantum mechanics). This difficulty has been overcome by Colegrave and Abdalla,[11,12] who treat the strongly pulsating case with

$$M(t)=m\cos^2(\delta t+\epsilon)\ . \tag{1.4}$$

These authors give an exact solution which has been rederived using an alternative method by Leach.[13] Colegrave and Abdalla[14,15] have described the pulsating oscillator system as simulating externally imposed changes in the field and discussed the fluctuation which could result from the interaction of the cavity field with an atomic reservoir as, for instance, in laser production. However, the main purpose of the present paper is to exhibit, in a very simple way, a new treatment of the general problem of a variable-mass oscillator (1.3). As discussed in Ref. 11 we have employed canonical transformation theory to transform the variable-mass oscillator to one with a time-dependent frequency and to reduce the Hamiltonian (1.3) to the standard form

$$H(t)=\hbar\Omega(t)(A^\dagger A+\tfrac{1}{2})\ , \tag{1.5}$$

where A and $A^\dagger$ are the usual Dirac operators and $\Omega(t)$ is the effective frequency. It is one of our intentions in the present note to obtain a solution for the Schrödinger equation

$$H(t)\,|\,\psi(t)\rangle=i\hbar\frac{\partial}{\partial t}\,|\,\psi(t)\rangle\ , \tag{1.6}$$

where H is given by (1.3). The method we have used produces an exact solution for the Schrödinger equation (1.6) for a general time-varying mass function $M(t)$. A quantal treatment of this problem is based on the Hamiltonian (1.3) together with (1.2), and the order of presentation is as follows. After introducing the canonical transformation in Sec. II we consider in Secs. III and IV the dynamics of the mass oscillator first in the wave picture; then, we calculate the expectation values of dynamical quantities. In Sec. V we introduce the solution in the Heisenberg picture and in Sec. VI we discuss the relation between the coherent states and pseudostationary states. Finally, we present some examples to illustrate the validity of our method when the mass is taken as a function of the time.

II. THE REDUCTION TO A VARIABLE-FREQUENCY OSCILLATOR

As considered in Ref. 16 we transform the variable-mass Hamiltonian (1.3) via the canonical transformation

$$\begin{aligned} q_0(t)&=[M(t)/m]^{1/2}q(t)\ ,\\ p_0(t)&=[m/M(t)]^{1/2}p(t) \end{aligned} \tag{2.1}$$

to a form in which the time dependence is concentrated in the fluctuation function $\gamma(t)$ defined by

$$\gamma(t)=\frac{1}{2}\frac{d}{dt}[\ln M(t)]\ . \tag{2.2}$$

The new Hamiltonian is

$$H_0=\frac{p_0^2}{2m}+\tfrac{1}{2}m\omega_0^2q_0^2+\frac{\gamma(t)}{2}(p_0q_0+q_0p_0)\ . \tag{2.3}$$

Let us introduce an explicitly time-dependent Dirac operator

$$A_1(t)=[2m\hbar\Omega(t)]^{-1/2}\times\left\{m\left[\Omega(t)+i\left[\gamma(t)+\frac{\dot\Omega(t)}{2\Omega(t)}\right]\right]q_0+ip_0\right\} \quad (2.4)$$

where $\dot\Omega(t)$ indicates the derivative of $\Omega(t)$ with respect to t. Obviously $A_1(t)$ and its adjoint satisfy the canonical relation

$$[A_1(t),A_1^\dagger(t)]=1 . \quad (2.5)$$

From Eq. (2.4) and its adjoint we have

$$q_0(t)=[\hbar/2m\,\Omega(t)]^{1/2}[A_1(t)+A_1^\dagger(t)] \quad (2.6)$$

and

$$ip_0(t)=[m\,\Omega(t)\hbar/2]^{1/2}[A_1(t)-A_1^\dagger(t)]-i\left[\gamma(t)+\frac{\dot\Omega(t)}{2\Omega(t)}\right]\left[\frac{m\hbar}{2\Omega(t)}\right]^{1/2}[A_1^\dagger(t)+A_1(t)] . \quad (2.7)$$

Thus from Eqs. (2.6) and (2.7) substituted into Eq. (2.3) the new canonical Hamiltonian is

$$H_1(t)=-\frac{\Omega(t)}{4}\hbar[A_1(t)-A_1^\dagger(t)]^2+\frac{h}{4\Omega(t)}\left[\omega_0^2-\gamma^2+\left[\frac{\dot\Omega(t)}{2\Omega(t)}\right]^2\right][A_1(t)+A_1^\dagger(t)]^2$$
$$+\frac{i\hbar}{2}\frac{\dot\Omega(t)}{2\Omega(t)}[A_1^2(t)-A_1^{\dagger 2}(t)]+\frac{\partial F_2}{\partial t} , \quad (2.8)$$

where F_2 is the generating function given as in Ref. 17 by

$$F_2(q_0,p_1,t)=-\frac{1}{2\omega_0}\left[\gamma(t)+\frac{\dot\Omega(t)}{2\Omega(t)}\right]q_0^2+\frac{1}{2}\left[\frac{\Omega(t)}{\omega_0}\right]^{1/2}(q_0p_1+p_1q_0) , \quad (2.9)$$

where p_1 is a new momentum defined by

$$p_1=\frac{1}{[\Omega(t)\omega_0]^{1/2}}\left[\gamma(t)+\frac{\dot\Omega(t)}{2\Omega(t)}\right]q_0+[\omega_0/\Omega(t)]^{1/2}p_0 . \quad (2.10)$$

From Eq. (2.9) the last term in Eq. (2.8) is

$$\frac{\partial F_2}{\partial t}=-\frac{\hbar}{4\Omega(t)}\frac{d}{dt}\left[\gamma(t)+\frac{\dot\Omega(t)}{2\Omega(t)}\right](A_1^\dagger+A_1)^2-\frac{i}{2}\hbar\frac{\dot\Omega(t)}{2\Omega(t)}(A_1^2-A_1^{\dagger 2}) . \quad (2.11)$$

From Eqs. (2.8) and (2.11) we see that the original Hamiltonian (1.3) reduces to the remarkably simple result

$$H(t)\to H_1(t)=\hbar\Omega(t)(A_1^\dagger A_1+\tfrac{1}{2}) , \quad (2.12)$$

where $\Omega(t)$ is given by the equation

$$\frac{1}{2}\frac{d}{dt}\frac{\dot\Omega(t)}{\Omega(t)}-\frac{1}{4}\left[\frac{\dot\Omega(t)}{\Omega(t)}\right]^2+\Omega^2(t)=\omega_0^2-\dot\gamma(t)-\gamma^2(t) . \quad (2.13)$$

The general solution is (cf. Ref. 18)

$$\Omega(t)=(ax_1^2+bx_2^2+2cx_1x_2)^{-1} , \quad (2.14)$$

where $x_1(t)$ and $x_2(t)$ are two independent solutions of

$$\ddot x+(\omega_0^2-\dot\gamma-\gamma^2)x=0 \quad (2.15)$$

and the constants a, b, and c are related according to

$$ab-c^2=w^{-2},\quad w=x_1^2\frac{d}{dt}(x_2/x_1) . \quad (2.16)$$

Thus the problem of variable mass (1.3) is solved by applying the definition of the time-dependent Dirac operator (2.4).

III. SOLUTION OF THE SCHRÖDINGER EQUATION

We shall turn our attention to the task of using the Schrödinger picture to find the pseudostationary states. The Schrödinger equation corresponding to the Hamiltonian (2.3) is

$$\frac{\partial^2\psi}{\partial q_0^2}-\frac{m^2\omega_0^2}{\hbar^2}q_0^2\psi+\frac{2im}{\hbar}\gamma(t)q_0\frac{\partial\psi}{\partial q_0}=-\frac{2im}{\hbar}\frac{\partial\psi}{\partial t}-\frac{im}{\hbar}\gamma(t)\psi . \quad (3.1)$$

Let us define

$$y=\sqrt{\Omega(t)}q_0,\quad \phi(y)\equiv\psi(q_0) . \quad (3.2)$$

Then Eq. (3.1) becomes

$$\frac{\partial^2\phi}{\partial y^2}-\left[\frac{m\omega_0}{\hbar\Omega(t)}\right]^2y^2\phi+\frac{2im}{\hbar}\frac{1}{\Omega(t)}\left[\gamma(t)+\frac{\dot\Omega(t)}{2\Omega(t)}\right]y\frac{\partial\phi}{\partial y}=-\frac{2im}{\hbar\Omega(t)}\frac{\partial\phi}{\partial t}-\frac{im\gamma(t)}{\hbar\Omega(t)}\phi . \quad (3.3)$$

We seek a separated solution of the form

$$\phi(y,t)=Y(y)T(t)\times\exp\left[-\frac{im}{2\hbar\Omega(t)}\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]y^2\right] . \tag{3.4}$$

With aid of Eq. (2.13) and after straightforward work we find, with λ a constant of separation,

$$\frac{d^2Y}{dy^2}+\left[\lambda-\frac{m^2y^2}{\hbar^2}\right]Y=0 , \tag{3.5}$$

$$\frac{dT}{dt}+\frac{i\Omega(t)\hbar}{2m}\left[\lambda+\frac{im\dot{\Omega}(t)}{2\hbar\Omega^2(t)}\right]T=0 . \tag{3.6}$$

Equation (3.5) is the usual equation for the harmonic oscillator and requires the quantization

$$\lambda=(2n+1)m/\hbar \quad (n=0,1,2,\dots) . \tag{3.7}$$

The solutions of Eqs. (3.5) and (3.6) may be written as

$$Y_n(y)=H_n\left[\frac{m}{\hbar}y\right]\exp\left[-\frac{m}{2\hbar}y^2\right] , \tag{3.8}$$

$$T_n(t)=\Omega^{1/4}(t)\exp\left[-i(n+\tfrac{1}{2})\int_0^t\Omega(\tau)d\tau\right] . \tag{3.9}$$

Thus the corresponding solution of the time-dependent Schrödinger Eq. (1.6) including a normalization constant (with respect to q_0) is

$$\psi_n(q_0,t)=\left[\frac{m\Omega(t)}{\hbar\pi}\right]^{1/4}2^{-n/2}(n!)^{-1/2}H_n(\sqrt{m\Omega(t)/\hbar}q_0)\times\exp\left\{-\frac{m}{2\hbar}\left[\Omega(t)+i\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]\right]q_0^2\right\}\exp\left[-i(n+\tfrac{1}{2})\int_0^t\Omega(t')dt'\right] . \tag{3.10}$$

From Eq. (3.10) it is easy to realize that the Schrödinger wave function is not much more complicated in form than that for the time-independent oscillator.

IV. EXPECTATION VALUES AND MATRIX ELEMENTS

To calculate the matrix elements of V, T, and H_0 with respect to the states (3.10), we can show that

$$\langle n|V|n'\rangle=\frac{1}{4}\hbar\frac{\omega_0^2}{\Omega(t)}\{(2n+1)\delta_{n'n}+[(n+1)(n+2)]^{1/2}\delta_{n,n'-2}+[(n'+1)(n'+2)]^{1/2}\delta_{n-2,n'}\} , \tag{4.1}$$

$$\begin{aligned}\langle n|T|n'\rangle=&\frac{\hbar\Omega(t)}{2}(n+\tfrac{1}{2})\{1+[f(t)/\Omega(t)]^2\}\delta_{nn'}\\&-\frac{\hbar\Omega(t)}{4}[(n+1)(n+2)]^{1/2}\left[1-2i[f(t)/\Omega(t)]-\left[\frac{f(t)}{\Omega(t)}\right]^2\right]\delta_{n,n'-2}\\&-\frac{\hbar\Omega(t)}{4}[(n'+1)(n'+2)]^{1/2}\left[1+2i[f(t)/\Omega(t)]-\left[\frac{f(t)}{\Omega(t)}\right]^2\right]\delta_{n-2,n'} ,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\langle n|H_0|n'\rangle=&\hbar\Omega(t)\left[1+\frac{f(t)}{2\Omega^2(t)}\right](n+\tfrac{1}{2})\delta_{n'n}+\frac{\hbar}{4\Omega}[(n+1)(n+2)]^{1/2}[\dot{f}(t)+i\dot{\Omega}(t)]\delta_{n,n'-2}\\&+\frac{\hbar}{4\Omega}[(n'+1)(n'+2)]^{1/2}[\dot{f}(t)-i\dot{\Omega}(t)]\delta_{n-2,n'} .\end{aligned} \tag{4.3}$$

We notice that $\langle n|V|n'\rangle$ is time dependent and equal to $\omega_0/\Omega(t)$ times the value for an oscillator with constant mass

$$\langle n|V|n'\rangle=[\omega_0/\Omega(t)]\langle n|V|n'\rangle_{M(t)=m} . \tag{4.4}$$

The expectation values of V, T, H_0, and $q_0p_0+p_0q_0$ in the state ψ_n are

$$\langle n|V|n\rangle=[\omega_0/\Omega(t)]\langle n|V|n\rangle_{M(t)=m} , \tag{4.5}$$

$$\langle n|T|n\rangle=[\Omega(t)/\omega_0]\{1+[f(t)/\Omega(t)]^2\}\times\langle n|T|n\rangle_{M(t)=m} , \tag{4.6}$$

$$\langle n|H_0|n\rangle=[\Omega(t)/\omega_0][1+\dot{f}(t)/2\Omega^2(t)]\times\langle n|H_0|n\rangle_{M(t)=m} , \tag{4.7}$$

and

$$\langle n|q_0p_0+p_0q_0|n\rangle=-\frac{2\hbar}{\Omega(t)}f(t)(n+\tfrac{1}{2}) , \tag{4.8}$$

where

$$f(t)=\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right] . \tag{4.9}$$

The kinetic energy T and the potential energy V may be explicitly expressed as functions of the time. This will be done in the following section.

V. THE EQUATIONS OF MOTION

In this section we shall derive the equations of motion in the Heisenberg picture of quantum mechanics. The canonical Hamiltonian (2.3) gives

$$\frac{dq_0}{dt}-\gamma(t)q_0=p_0/m\ , \qquad \frac{dp_0}{dt}+\gamma(t)p_0=-m\omega_0^2 q_0\ . \tag{5.1}$$

The general solution is given by

$$q_0(t)=\left[\left[\frac{\Omega(0)}{\Omega(t)}\right]^{1/2}\cos I(t)+\frac{f(0)}{\sqrt{\Omega(t)\Omega(0)}}\sin I(t)\right]q_0(0)+\frac{p_0(0)}{m\sqrt{\Omega(t)\Omega(0)}}\sin I(t) \tag{5.2a}$$

and

$$p_0(t)=\left[\left[\frac{\Omega(t)}{\Omega(0)}\right]^{1/2}\cos I(t)-\frac{f(t)}{\sqrt{\Omega(t)\Omega(0)}}\sin I(t)\right]p_0(0) +mq_0(0)\left[\frac{f(0)\Omega(t)-f(t)\Omega(0)}{\sqrt{\Omega(t)\Omega(0)}}\cos I(t)-\sqrt{\Omega(t)\Omega(0)}\left[1+\frac{f(0)f(t)}{\Omega(0)\Omega(t)}\right]\sin I(t)\right]\ , \tag{5.2b}$$

where

$$I(t)=\int_0^t \Omega(t')dt' \tag{5.3}$$

and $f(t)$ is defined by Eq. (4.9). We may easily check that

$$[q_0(t),p_0(t)]=[q_0(0),p_0(0)]=i\hbar\ . \tag{5.4}$$

From Eqs. (5.2), we find the following expressions for $V(t)$ and $T(t)$:

$$V(t)=\Omega(t)^{-1}\left[\Omega(0)\cos^2 I(t)+\frac{f^2(0)}{\Omega(0)}\sin^2 I(t)+f(0)\sin[2I(t)]\right]V(0) +[\omega_0^2/\Omega(t)\Omega(0)]\sin^2[I(t)]T(0)+\frac{1}{4}\frac{\omega_0^2}{\Omega(t)}\left[\sin[2I(t)]+2\frac{f(0)}{\Omega(0)}\sin^2 I(t)\right][q_0,p_0]_+\ , \tag{5.5}$$

$$\begin{aligned}T(t)=\Omega^{-1}(0)&\left[\Omega(t)\cos^2 I(t)+\frac{f^2(t)}{\Omega(t)}\sin^2 I(t)-f(t)\sin[2I(t)]\right]T(0)\\ &+\tfrac{1}{2}[\Omega(0)\Omega(t)]^{-1}([\Omega^2(t)+f^2(t)][\Omega^2(0)+f^2(0)]\\ &\qquad+\{[f^2(0)-\Omega^2(0)][\Omega^2(t)-f^2(t)]-4\Omega(0)f(0)\Omega(t)f(t)\}\cos[2I(t)]\\ &\qquad-[f(0)\Omega(t)-f(t)\Omega(0)][\Omega(0)\Omega(t)+f(0)f(t)]\sin^2 I(t))V(0)\\ &-\frac{1}{4}\left[\frac{\Omega^2(t)-f^2(t)}{\Omega(t)}+2\frac{f(0)f(t)}{\Omega(0)}\right]\sin[2I(t)][q_0,p_0]_+\ .\end{aligned} \tag{5.6}$$

In the final section, we shall compare these results with those obtained by Colegrave and Abdalla in Refs. 11 and 16.

VI. QUASICOHERENT STATES

As is to be expected from our reduction of the problem to Eq. (2.12), it is clear that we may use the number states of the operator $A_1^\dagger A_1$ to construct coherent states in the form

$$|\alpha(t)\rangle=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n(t)\rangle \tag{6.1}$$

since we have ensured that $[A,A^\dagger]=1$ at all times, it follows that

$$A_1|\alpha(t)\rangle=\alpha(t)|\alpha(t)\rangle\ , \tag{6.2}$$

$$A_1^\dagger A_1|n(t)\rangle=n|n(t)\rangle,\quad n=0,1,2,\ldots\ . \tag{6.3}$$

From Eqs. (2.4) and (5.2) we can deduce that

$$A_1(t)=A_1(0)\exp\left[-i\int_0^t \Omega(t')dt'\right] . \tag{6.4}$$

It follows from Eqs. (6.2) and (6.4) that

$$\alpha(t)=\alpha(0)\exp\left[-i\int_0^t \Omega(t')dt'\right] . \tag{6.5}$$

As discussed in Ref. 8 the "best" coherent states are the eigenstates of the operator (2.4). Therefore, the expectation values given by

$$\langle\alpha\,|\,q_0\,|\,\alpha\rangle=\left[\frac{\hbar}{2m\Omega(t)}\right]^{1/2}[\alpha(t)+\alpha^*(t)] , \tag{6.6a}$$

$$\langle\alpha\,|\,p_0\,|\,\alpha\rangle=i\left[\frac{m\Omega\hbar}{2}\right]^{1/2}[\alpha^*(t)-\alpha(t)]-\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]\left[\frac{mh}{2\Omega(t)}\right]^{1/2}[\alpha^*(t)+\alpha(t)] \tag{6.6b}$$

and the product of uncertainties is

$$\Delta q_0^2\Delta p_0^2=\frac{\hbar^2}{4}\left[1+\frac{1}{\Omega^2(t)}\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]^2\right] . \tag{6.6c}$$

In the Schrödinger representation the quasicoherent state (6.2) is

$$\frac{\partial\psi_\alpha(q_0,t)}{\partial q_0}=\left\{-\frac{m}{\hbar}\left[\Omega(t)+i\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]\right]q_0+\alpha\left[\frac{2m\Omega(t)}{\hbar}\right]^{1/2}\right\}\psi_\alpha(q_0,t) \tag{6.7}$$

with solution

$$\psi_\alpha(q_0,t)=N_\alpha(t)\exp\left\{-\frac{m}{2\hbar}\left[\Omega(t)+i\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]\right]q_0^2+\alpha\left[\frac{2m\Omega(t)}{\hbar}\right]^{1/2}q_0\right\} , \tag{6.8}$$

where

$$N_\alpha(t)=\left[\frac{\Omega(t)}{\pi\hbar}\right]^{1/4}\exp[-\tfrac{1}{4}(\alpha+\alpha^*)^2] .$$

Alternatively, we may write, as in Ref. 19,

$$\psi_\alpha(q_0,t)=\left[\frac{m\Omega(t)}{\pi\hbar}\right]^{1/4}\exp\left\{-\frac{m}{\hbar}\left[\Omega(t)+i\left[\gamma(t)+\frac{\dot{\Omega}(t)}{2\Omega(t)}\right]\right]q_0^2\right\}$$
$$\times\exp(-\tfrac{1}{2}\,|\,\alpha\,|^2)\sum_{n=0}^{\infty}H_n(\sqrt{m\Omega(t)/\hbar}q_0)2^{-n/2}(n!)^{-1}\alpha^n(t) , \tag{6.9}$$

where $\alpha(t)$ is given by Eq. (6.5). Finally, comparing Eqs. (3.10) and (6.9), we find that

$$\psi_\alpha(q_0,t)=\exp(-\tfrac{1}{2}\,|\,\alpha\,|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\psi_n(q_0,t) \tag{6.10}$$

which is in agreement with Eq. (6.1).

VII. EXAMPLES

Finally, we shall introduce some examples to establish the validity of our method to recover all the results of Colegrave and Abdalla, with some more examples since our method depends on $\gamma(t)$ which is easy to obtain. Therefore, the task is to determine $\Omega(t)$, which can be done if one manages to have a solution for the differential equation (2.15). Thus we shall concentrate on $\Omega(t)$.

Let us consider the following cases for $\omega_0>\delta>0$:

(i) $M(t)=m\exp(-2\delta t)$,

(ii) $M(t)=m\cos^2(\delta t)$,

(iii) $M(t)=m(\mu_1 e^{-\delta t}+\mu_2 e^{\delta t})^2$,

(iv) $M(t)=me^{2\delta t}/(1+\mu e^{2\delta t})^2$,

(v) $M(t)=m\cos^n(\delta t)$,

(vi) $M(t)=m(1+\delta t)^{2n}$.

In order to calculate $\Omega(t)$ we need to find x_1 and x_2 of Eq. (2.15). We find for the cases from (i)—(iii) that

$$x_1=\cos(\omega t),\quad x_2=\sin(\omega t)\ , \tag{7.1a}$$

where $\omega=(\omega_0^2-\delta^2)^{1/2}$, $(\omega_0^2+\delta^2)^{1/2}$, $(\omega_0^2-\delta^2)^{1/2}$, respectively, and from Eqs. (2.14) and (2.16) we find

$$\Omega\rightarrow\omega=\text{const}\ . \tag{7.1b}$$

In case (iv), as discussed in Ref. 9, we have

$$x_1=\cos(\omega t)+\frac{\delta}{\omega}r(t)\sin(\omega t)\ , \tag{7.2a}$$

$$x_2=[1+\delta^2/\omega^2 r(t)r(0)]\sin(\omega t)-\frac{\delta}{\omega}[r(t)-r(0)]\cos(\omega t)\ , \tag{7.2b}$$

where

$$r(t)=(1-\mu e^{2\delta t})/(1+\mu e^{2\delta t})\ . \tag{7.2c}$$

Thus the effective frequency reduces to

$$\Omega=(\omega_0^2-\delta^2)^{1/2}=\omega\ . \tag{7.2d}$$

To continue our discussion we shall consider case (v), which corresponds to a strongly harmonic oscillator to power n; in this case the solution of Eq. (2.15) is

$$x_1=\cos^{n/2}(\delta t)\{[1-\sin(\delta t)]/2\}^{-\alpha}{}_2F_1\left[\alpha,\frac{2\alpha-n+1}{2},1\pm\frac{1}{\delta}(\delta^2n^2+4\omega_0^2)^{1/2},\frac{2}{1-\sin(\delta t)}\right]\ , \tag{7.3a}$$

$$x_2=\cos^{n/2}(\delta t)\{[1-\sin(\delta t)]/2\}^{-\beta}{}_2F_1\left[\beta,\frac{2\beta-n+1}{2},1\mp\frac{1}{\delta}(\delta^2n^2+4\omega_0^2)^{1/2},\frac{2}{1-\sin(\delta t)}\right]\ , \tag{7.3b}$$

where ${}_2F_1$ is well known as the hypergeometric function,[20]

$$\beta=\frac{n}{2}\{1\mp[1+(2\omega_0/\delta n)^2]^{1/2}\}\ , \tag{7.3c}$$

and

$$\alpha=\frac{n}{2}\{1\pm[1+(2\omega_0/\delta n)^2]^{1/2}\}\ . \tag{7.3d}$$

As a special case, if $n=1$ we find

$$\begin{aligned}x_1&=\cos^{1/2}(\delta t)P_s(\sin(\delta t))\ ,\\ x_2&=\cos^{1/2}(\delta t)Q_s(\sin(\delta t))\ ,\end{aligned} \tag{7.4a}$$

where P_s is the Legendre function of order s, Q_s is the second solution of Legendre's equation, and s is given by

$$s=(2\omega-\delta)/2\delta,\quad \omega=(\omega_0^2+\delta^2/4)^{1/2}\ . \tag{7.4b}$$

Finally it is interesting to discuss a slow linear change in mass to power n given by (see, for example, Refs. 13 and 14)

$$M(t)=m(1+\delta t)^{2n}\ . \tag{7.5}$$

In this case the solution of Eq. (2.15) is expressed in terms of Bessel functions; thus

$$x_1=\sqrt{1+\delta t}\,J_r((\omega_0/\delta)(1+\delta t))\ , \tag{7.6a}$$

$$x_2=\sqrt{1+\delta t}\,J_{-r}((\omega_0/\delta)(1+\delta t))\ , \tag{7.6b}$$

where $r=n-\frac{1}{2}$.

By taking $n=1$ Eqs. (7.6a) and (7.6b) are reduced to

$$x_1=\cos(\omega_0 t),\quad x_2=\sin(\omega_0 t)\ . \tag{7.7}$$

Since the method adopted in the present paper deals with all variable-mass parameters, this gives us advantage in dealing with a mass varying periodically without zeros.[21] For example the pulsating oscillator mass as in Ref. 22 is defined by

$$M(t)=m\exp[2\mu\sin(\nu t)]\ . \tag{7.8}$$

Thus, the fluctuating function $\gamma(t)$ defined by Eq. (2.2) is

$$\gamma(t)=\mu\nu\cos(\nu t)\ . \tag{7.9}$$

Substituting into Eq. (2.15) we find

$$\ddot{x}+[\omega_0^2+\mu\nu^2\sin(\nu t)-\mu^2\nu^2\cos^2(\nu t)]x=0\ . \tag{7.10}$$

There are two possibilities to deal with this problem, either to consider $\mu\ll 1$, and then Eq. (7.10) is of the form of Hill's equation for which a standard treatment exists, or to use the rotating-wave approximation (RWA) which gives an asymptotic solution for all values of μ. However, one can also apply the perturbation method; in this case Eq. (7.10) gives

$$x_1=\cos(\omega_0 t)+\lambda\{2\omega_0[1+\cos(\nu t)]\sin(\omega_0 t)-\nu\sin(\nu t)\cos(\omega_0 t)\}\ , \tag{7.11a}$$

$$x_2=\sin(\omega_0 t)+\lambda\{2\omega_0[1-\cos(\nu t)]\cos(\omega_0 t)-\nu\sin(\nu t)\sin(\omega_0 t)\}\ , \tag{7.11b}$$

where

$$\lambda=\mu\nu/(4\omega_0^2-\nu^2)\ . \tag{7.11c}$$

Then, the augmented frequency $\Omega(t)$ is

$$\Omega(t)=\omega_0\{1+2\lambda[\nu\sin(\nu t)-2\omega_0\sin(2\omega_0 t)]\}\ . \tag{7.11d}$$

If we use the rotating-wave approximation, we shall find at exact resonance when $\omega_0=\nu/2$ that

$$x_1=\cos(\omega_0 t)\exp(\mu\omega_0 t) \tag{7.12a}$$

and

$$x_2 = \sin(\omega_0 t)\exp(-\mu\omega_0 t) \; . \tag{7.12b}$$

VIII. CONCLUSION

The problem of the motion of a particle with a time-dependent mass $M(t)=m\exp[2\Gamma(t)]$ in a uniform nonstationary gravitational field can be described exactly by means of the Schrödinger equation with Hamiltonian (1.3). This is a real physical problem, and as we remarked in the Introduction the harmonic oscillator with variable mass has extensive applications in physics, especially in quantum optics, see Colegrave and Abdalla,[14,15,23] in plasma physics[24,25] and in studying the early stages of the evolution of the universe; see, e.g., Ref. 26. Besides, unstable systems can be also described with the aid of the concept of the time-dependent mass.[27] However, the most significant effect of the variable-mass parameter in a harmonic oscillator is to change the natural frequency ω_0 to an effective frequency $\Omega(t)$ given by Eq. (2.14). This effective frequency allows us to make a separation of variables in solving the Schrödinger equation (1.6) and also to overcome the difficulty in obtaining a solution of this problem in the Heisenberg picture. We believe that all the results reported in the present paper enable us to analyze the quantal motion of a variable-mass harmonic oscillator in a way that is straightforward and reliable, depending as it does on well-established procedures for time-dependent Hamiltonians, and can be easily compared with those obtained by other authors. For example, the wave function given in Sec. III, with the replacement $\gamma(t)\to-\gamma$ and $\Omega(t)\to\omega=(\omega_0^2-\gamma^2)^{1/2}$, leads to the result of Hasse,[2] Tartglia,[4] and of Colegrave and Abdalla.[16] In Sec. V, by making the same replacement, we obtain for the equations of motion the results of the above authors and those given by Dodonov and Man'Ko[8] and in Ref. 28, while the kinetic and potential energies are in agreement with the results in Ref. 16. Note also that the replacement $\gamma(t)\to-\nu\tan(\nu t)$ and $\Omega(t)\to\omega=(\omega_0^2+\gamma^2)^{1/2}$ throughout Secs. II–VI covers the results in Ref. 11. Finally, if we let $\gamma(t)\to\delta/(1+\delta t)$ and $\Omega(t)\to\omega_0$ and substitute into Eqs. (4.5) and (4.6) we are left with Eqs. (5.17) and (5.18) of Ref. 12. We feel that the present paper has important implications for dealing with problems in quantum optics and plasma physics and perhaps with other branches of physics including solid-state physics and quantum field theory.

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