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Collective variable description of a free-electron laser

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We give a collective variable description of high-gain free-electron-laser amplifiers which reproduces the solution of microscopic equations quite satisfactorily both in the linear and in the nonlinear regime.

Free-electron lasers (FEL's) are increasingly the object of intense theoretical investigations and experimental efforts due to their relevance as sources of powerful, coherent radiation virtually in the whole electromagnetic spectrum. The FEL process can be described as a stimulated magnetic bremsstrahlung from a beam of relativistic electrons injected in a suitable magnetic array (undulator or wiggler), where the particles can efficiently couple to a radiation field transferring a fraction of their kinetic energy to the photons.[1]

The existence of an FEL high-gain regime has been demonstrated in recent years.[2] For example, Orzeckowski *et al.*[2] obtained a peak power as high as 80 MW for radiation in the millimeter range ($\lambda \simeq 9$ mm) in a single-pass amplifier experiment; that is, the simplest FEL mode of operation, which unlike the oscillator mode does not require mirrors, or different stages in the electron-radiation interaction as in the optical klystron mode. The radiated intensity exhibits an impressive exponential growth before saturation effects set in. This behavior occurs even starting with a negligible input power; the physical basis for this regime of self-amplified spontaneous emission (SASE) is a collective instability for the system.[3-6]

A simple description of the SASE instability can be given in terms of only three (complex) collective variables,[4] which has allowed both a classical[3-6] and a quantum[7] analysis of the linear stage of exponential amplification. On the other hand, the saturation regime, described in terms of trapping of the electrons in the ponderomotive potential of the combined (radiation+undulator) fields,[8] requires the numerical integration of the $2N+2$ coupled evolution equations for the position and momentum (or phase and energy) variables of the $N \gg 1$ particles and the complex field amplitude.[3,8-11]

We point out that a long-time (i.e., long undulator) analysis of the FEL equations in the high-gain regime shows "well behaved," nearly periodic undamped oscillations of the radiated intensity, though of irregular amplitude, at least for a sufficiently high number of electrons, despite the underlying chaotic behavior of the particles.[4] This has further stimulated us to develop a description of the FEL by means of few relevant collective variables which holds for the whole dynamics, i.e., also well beyond the linear stage. It would be the FEL analog, though in purely classical terms, of the description of conventional lasers usually given in terms of few relevant macroscopic variables, that is the atomic polarization and population inversion and the field amplitude.[12] However, this analogy is relevant only as regards our attitude to the problem. Actually, conventional lasers are the prototypes of open systems which undergo a dissipative dynamics, whereas FEL dynamics is basically Hamiltonian. Furthermore, the familiar laser instability leads to a stationary (lasing) state, whereas in high-gain FEL amplifiers we are interested in the transient evolution due to the SASE instability.

In this Rapid Communication we formulate a collective variable description of the FEL which reproduces the behavior predicted by the $2N+2$ evolution equations quite satisfactorily both in the linear and in the nonlinaear regime. In particular, we reproduce the same cubic characteristic equation of the linear analysis which rules the instability and the high-gain behavior of the microscopic description, and obtain the values of the build-up time of radiated intensity, that is the time at which the first peak is emitted, and the height of this peak within a few percent with respect to the values obtained from the full set of microscopic equations.

The microscopic Hamiltonian model of a single-pass FEL is given by the following well-known set of equations:[4]

$$d\sigma_J/d\tau = p_J, \quad (J=1,\ldots,N) \ , \tag{1a}$$

$$dp_J/d\tau = -[A\exp(i\sigma_J)+\text{c.c.}], \quad (J=1,\ldots,N) \ , \tag{1b}$$

$$dA/d\tau = N^{-1}\sum_{J=1}^{N}\exp(-i\sigma_J)+i\delta A \ , \tag{1c}$$

where, using the same notations of Ref. 3 to which we refer for more details,

$$\begin{aligned}
&\sigma_J = (k+k_0)z_J - \left(\frac{\omega}{2\omega_0}+\delta\right)\tau \ ,\\
&p_J = \frac{1}{\rho}\frac{\gamma_J-\langle\gamma\rangle_0}{\langle\gamma\rangle_0}; \quad |A|^2 = \frac{|E_0|^2 V/4\pi}{\rho N\langle\gamma\rangle_0 m_0c^2} \ ,\\
&\tau = 2\omega_0\rho t; \quad \delta = \frac{1}{\rho}\frac{\langle\gamma\rangle_0-\gamma_R}{\langle\gamma\rangle_0}; \quad \rho = \left(\frac{\mathscr{K}}{4}\frac{\Omega_p}{\omega_0}\right)^{2/3} \ ,\\
&\mathscr{K} = \frac{eB_0}{m_0c\omega_0}; \quad \Omega_p = \left(\frac{4\pi e^2 N}{m_0 V\langle\gamma\rangle_0^3}\right)^{1/2} \ .
\end{aligned} \tag{2}$$

The meaning of the dimensionless variables and parameters in Eqs. (1a)–(1c) is the following: σ_j and p_j are the phase (position) variable and the energy variation or "momentum" variable, respectively, of the jth electron; A is the scaled complex field amplitude of the radiation field, τ the scaled time, and δ the detuning parameter. In Eqs.

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(2) these quantities are expressed in terms of the amplitude E_0 and the frequency $\omega = ck$ of the radiation field, the magnetostatic amplitude B_0 and frequency $\omega_0 = ck_0$ associated with the undulator periodicity, the energy γ_j of the jth electron in rest mass units, the initial mean energy $\langle\gamma\rangle_0 = N^{-1}\sum_j \gamma_j(0)$, the resonance energy γ_R, and the generalized Pierce parameter ρ which contains the undulator parameter $\mathcal{K}$ and the relativistic plasma frequency Ω_p. The FEL resonance relation is $\omega = 2\omega_0\gamma_R/(1+\mathcal{K}^2)$. Note that if the undulator length is $L_0 = 2\pi c N_0/\omega_0$, N_0 being the number of periods, the scaled time is defined in the interval $0 \le \tau \le 4\pi\rho N_0$ corresponding with the time interval $0 \le t \le L_0/c$ in which the FEL process takes place. The set of Eqs. (1), which depend only on the detuning parameter δ, is the same studied in Refs. 4 and 6 and can be obtained from Eqs. (13)–(15) of Ref. 3 in the approximation $\rho p_j << 1$, which has been shown to be valid for $\rho << 1$.

Let us define the mean value of B as $\langle B\rangle = N^{-1}\sum_j B_j$, where B_j is any dynamical variable referring to the jth electron. In this way we define the average momentum

$$\langle p\rangle = N^{-1}\sum_j p_j \ , \tag{3}$$

the average kinetic energy

$$\langle p^2\rangle = N^{-1}\sum_j p_j^2 \ , \tag{4}$$

and the bunching parameter

$$b = \langle \exp(-i\sigma)\rangle = N^{-1}\sum_j \exp(-i\sigma_j) \ , \tag{5}$$

such that $0 \le |b| \le 1$. This parameter approaches one when the electrons bunch in an optical wavelength, which allows for an efficient energy transfer from the particles to the radiation field.

Equations (1) admit two constants of motion,[6] which according to definitions (3)–(5) can be written as

$$\langle p\rangle + |A|^2 = |A|_0^2 \ , \tag{6}$$

$$\frac{\langle p^2\rangle}{2} + i(A^*b - Ab^*) - \delta|A|^2 = \frac{\langle p^2\rangle_0}{2} + i(A^*b - Ab^*)_0 - \delta|A|_0^2 \ . \tag{7}$$

In Eq. (6) we have taken into account that $\langle p\rangle_0 = 0$ by definition [Eqs. (2) and (3)]; hence in Eq. (7) $\langle p\rangle_0^2 = (\Delta p)_0^2$ is the initial energy spread. The first constant of motion, Eq. (6), is the total momentum conservation; the second one, Eq. (7), is associated with the existence of a Hamiltonian for Eqs. (1), as discussed in Ref. 4. Anyway, Eqs. (6) and (7) can be easily verified by differentiating with respect to time and using Eqs. (1).

The field Eq. (1c) can be written as (overdot $= d/d\tau$)

$$\dot{A} = b + i\delta A \ , \tag{8}$$

where we have just used the definition (5) of the bunching parameter $b(t)$. By this defintion and Eq. (1a) one finds the evolution equation for $b(t)$

$$\dot{b} = -i\mathcal{P} \ , \tag{9}$$

where we have defined the phase-momentum average

$$\mathcal{P} = \langle \exp(-i\sigma)p\rangle = N^{-1}\sum_j \exp(-i\sigma_j)p_j \ . \tag{10}$$

From definition (10) and Eqs. (1) one gets the following equation of motion for $\mathcal{P}(t)$:

$$\dot{\mathcal{P}} = -A - i\langle \exp(-i\sigma)p^2\rangle - A^*\langle \exp(-2i\sigma)\rangle \ , \tag{11}$$

where, according to the definition of the mean value,

$$\langle \exp(-i\sigma)p^2\rangle = N^{-1}\sum_j \exp(-i\sigma_j)p_j^2 \ , \tag{12a}$$

$$\langle \exp(-2i\sigma)\rangle = N^{-1}\sum_j \exp(-2i\sigma_j) \ . \tag{12b}$$

Thus, Eq. (9) has implied the introduction of a new quantity, $\mathcal{P} = \langle \exp(-i\sigma)p\rangle$; in turn, the evolution equation for $\mathcal{P}$, Eq. (11), involves two further mean values; clearly, this process generates a hierarchy of equations which contain higher and higher moments, as usual in nonlinear dynamics. We try to truncate this hierarchy by writing a closed system of evolution equations for only three complex macroscopic variables, namely, the field amplitude A, the electron bunching parameter b, and phase-momentum average $\mathcal{P}$.

To this end we truncate the hierarchy by assuming that

$$\langle (p - \langle p\rangle)^2 \exp(-i\sigma)\rangle = \langle (p-\langle p\rangle)^2\rangle \langle \exp(-i\sigma)\rangle \ . \tag{13}$$

This factorization ansatz can be related to the underlying chaotic behavior of the N-electron system demonstrated in Ref. 4. The validity of approximation (13) can be numerically checked by solving the exact set of Eqs. (1). The result is shown in Fig. 1, where we compare the time evolution of the real and imaginary parts of the (unfactorized) left-hand side and the (factorized) right-hand side of Eq. (13). By Eq. (13) the term (12a) can be expressed as

$$\langle \exp(-i\sigma)p^2\rangle = \langle p^2\rangle b + 2\langle p\rangle(\mathcal{P} - \langle p\rangle b) \ . \tag{14}$$

The quantity $\langle p^2\rangle$ in Eq. (14) can be easily evaluated with the help of the constant of motion relation (7), obtaining

$$\langle \exp(-i\sigma)p^2\rangle = 2[ib(Ab^* - A^*b) + \langle p\rangle(\mathcal{P} - \langle p\rangle b) + \delta b(|A|^2 - |A|_0^2)] + b[\langle p^2\rangle_0 - 2i(Ab^* - A^*b)_0] \ . \tag{15}$$

As concerns the term $A^*\langle \exp(-2i\sigma)\rangle$, we drop it in Eq. (11) since the numerical solution of Eqs. (1) shows that its effect is negligible with respect to that of the term (12a). Now by inserting Eq. (15) into (11) we obtain a closed system of equations for the three complex collective variables A, b, and $\mathcal{P}$ which read

$$\dot{A} = b + i\delta A \ , \tag{16a}$$

$$\dot{b} = -i\mathcal{P} \ , \tag{16b}$$

$$\dot{\mathcal{P}} = -A + 2b(Ab^* - A^*b) + 2i(|A|^2 - |A|_0^2)[\mathcal{P} + (|A|^2 - |A|_0^2 - \delta)]b - b[2(Ab^* - A^*b)_0 + i\langle p^2\rangle_0] \ . \tag{16c}$$

In Eq. (16c) the constant of motion relation (6) has been used to eliminate $\langle p\rangle$.

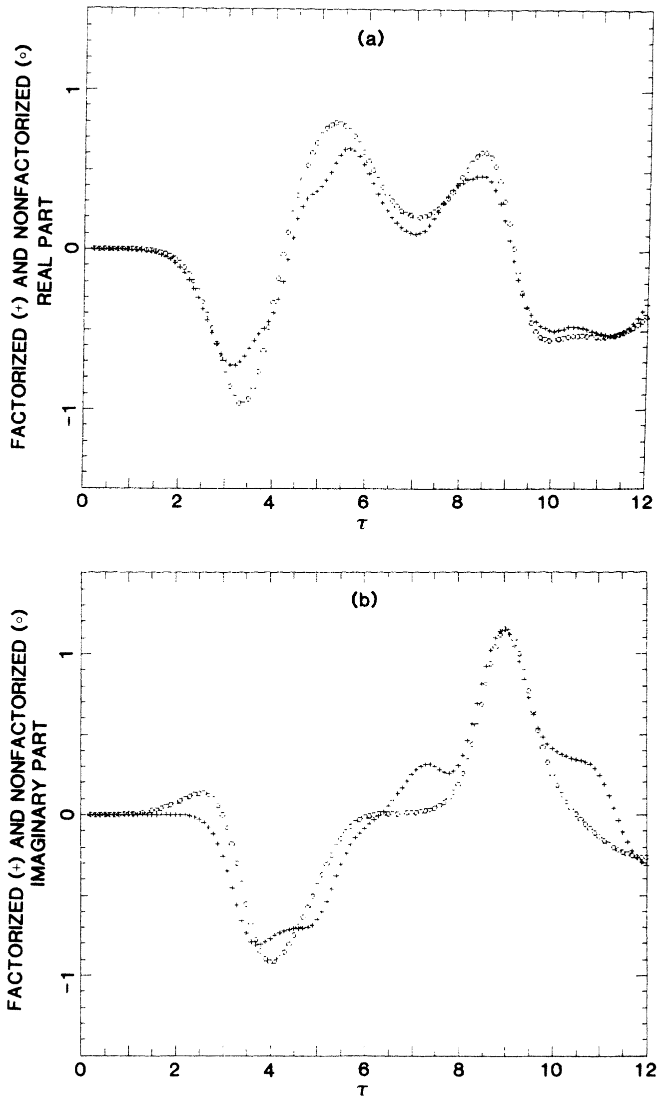

FIG. 1. Check of the factorization ansatz (13). (a) real part of $\langle (p-\langle p\rangle)^2 \exp(-i\sigma)\rangle$ (crosses) and real part of $\langle (p-\langle p\rangle)^2\rangle \times \langle \exp(-i\sigma)\rangle$ (open circles) vs dimensionless time τ, from Eqs. (1) for $N=100$, $\delta=0$, and the initial condition $(\mathrm{Re}A)_0=(\mathrm{Im}A)_0=0.03$, $(\sigma_j)_0$ randomly distributed ($|b|_0=0.15$), $(p_j)_0$ distributed according to a Gaussian with $\langle p\rangle_0=0$, and $(\Delta p)_0=0.05$. (b) imaginary part of $\langle (p-\langle p\rangle)^2 \exp(-i\sigma)\rangle$ (crosses) and imaginary part of $\langle (p-\langle p\rangle)^2\rangle \langle \exp(-i\sigma)\rangle$ (open circles) vs dimensionless time τ, from Eqs. (1) as in case (a).

To check the validity of Eqs. (16) we first linearize around the initial condition, which is an equilibrium condition,

$$A_0=b_0=\mathcal{P}_0=0 \ . \tag{17}$$

The linearized system is

$$\dot{A}=b+i\delta A; \quad \dot{b}=-i\mathcal{P}; \quad \dot{\mathcal{P}}=-A \ . \tag{18}$$

This system is equivalent to the linear system for collective variables of Refs. 3 and 4 obtained via linearization of the microscopic Eqs. (1). In fact, looking for exponential solutions as $\exp(i\lambda\tau)$, one gets the well-known cubic characteristic equation

$$\lambda^3-\delta\lambda^2+1=0 \ , \tag{19}$$

which gives the same instability analysis and exponential behavior as in Refs. 3 and 4. Hence the collective variable description of Eqs. (16) gives the same short-time behavior as the microscopic description of Eqs. (1). In particular, the initial equilibrium condition (17) is unstable for $\delta \le (3/2)^{2/3}$, leading to a strong self-bunching of the electrons and exponential growth of the radiated intensity. As concerns the long-time behavior, in Fig. 2 we compare the radiated intensity calculated from the numerical integration

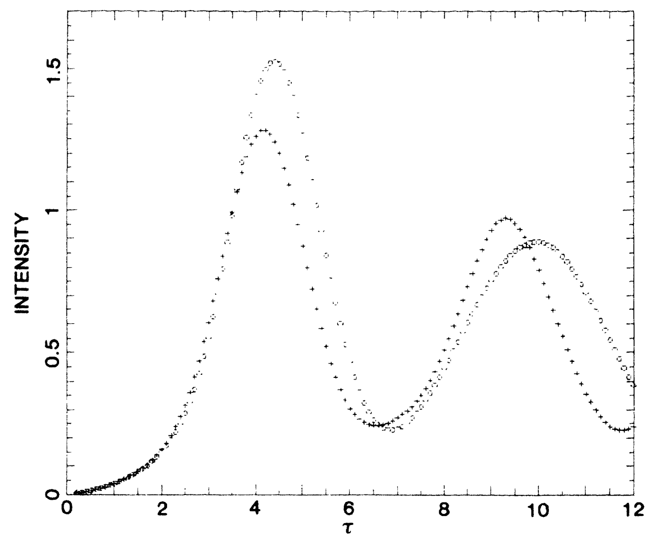

FIG. 2. Dimensionless radiated intensity $|A|^2$ vs dimensionless time τ. Crosses: from Eqs. (1) as in Fig. 1; open circles: from Eqs. (16) for $\delta=0$ and with the initial condition $(\mathrm{Re}A)_0=(\mathrm{Im}A)_0=0.03$, $(\mathrm{Re}b)_0 \simeq 3.8\times 10^{-2}$, $(\mathrm{Im}b)_0 \simeq 0.15$, $(\mathrm{Re}\mathcal{P})_0 \simeq -4.5\times 10^{-3}$, $(\mathrm{Im}\,\mathcal{P})_0 \simeq -5.2\times 10^{-3}$, $(\Delta p)_0^2 \simeq 2\times 10^{-3}$.

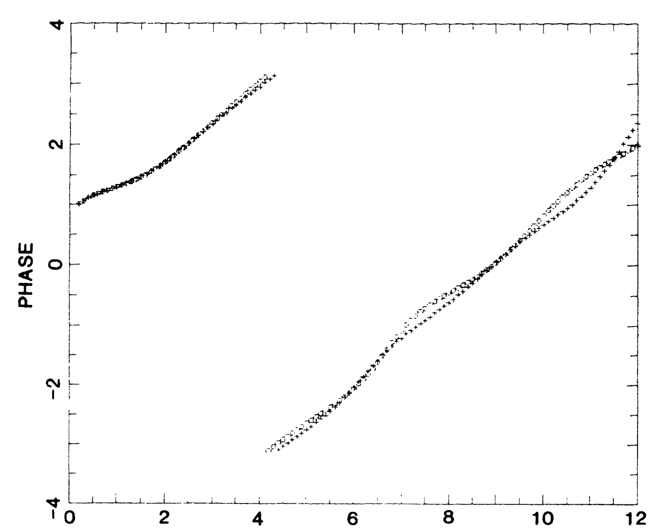

FIG. 3. Time evolution of the radiation field phase. Crosses: from Eqs. (1) in the same case considered in Figs. 1 and 2; open circles: from Eqs. (16) in the same case considered in Fig. 1.

of the microscopic Eqs. (1) for $N=100$ electrons, and the same quantity obtained from Eqs. (16); the comparison has been performed for operation on resonance, i.e., $\delta=0$, when the growth rate is maximum. Equations (1) were integrated with an initial condition with randomly distributed electron phases ($|b|_0 \simeq 0.15$) and a small field ($|A|_0 \simeq 0.04$) to simulate electron and field noise, and with electron momenta distributed according to a Gaussian with zero mean value and uncertainty $(\Delta p)_0^2 \simeq 0.05$ to simulate energy spread. The initial values for the real and imaginary parts of the field amplitude A used in Eqs. (1) and those calculated from Eqs. (1) for the real and imaginary parts of the electron collective variables b and $\mathscr{P}$, ($|\mathscr{P}|_0 \simeq 7\times 10^{-3}$) were taken as initial values for system (16). The agreement appears to be qualitatively and even quantitatively remarkable. In particular, the build-up time is approximated within 5% and the peak height within 15%. This agreement is also evident from Fig. 3, where we compare the time evolution of the field phase in the two cases.

In conclusion, we think that Eqs. (16) provide a good collective variable description of the essential features of the dynamic of high-gain FEL amplifiers.

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