Evolution of quantum systems driven by a Hamiltonian written in terms of the SU(1,1) group generators

G. Dattoli, A. Torre,* and R. Caloi†
Dipartimento Tecnologie Intersettoriali di Base, Divisione Fisica Applicata, Comitato Nazionale per la Ricerca e per lo Sviluppo dell'Energia Nucleare e delle Energie Alternative (ENEA), Centro Ricerche Energia Frascati, Casella Postale 65, I-00044 Frascati, Roma, Italy
(Received 1 August 1985)

In this paper we exploit the Wei-Norman algebraic disentangling procedure to study the evolution of quantum states driven by a time-dependent Hamiltonian linear combination of the generators of the SU(1,1) group.

The use of rigorous algebraic procedures[1,2] to deal with time-ordering problems has proved a useful tool to treat the evolution of quantum states driven by a large class of Hamiltonians often encountered in quantum optics.[3] Even if these techniques have been developed more than two decades ago and, in many cases, are definitively more powerful than the Feynman-Dyson expansion,[4] they are not widely used as perhaps they should be. The most interesting aspects of these methods are their simplicity and generality. They work indeed for many types of Hamiltonians without any recourse to specific assumptions.

In this Brief Report we go a step further along the line developed in Ref. 3, discussing the time-ordering problems arising in the analysis of the evolution of states driven by a Hamiltonian written in terms of SU(1,1) generators.[5] Hamiltonians of this kind have attracted noticeable interest, within the framework of the evolution of coherent states of the Perelemov type.[6] Furthermore, it has been recently shown that the Hamiltonian of the degenerate parametric oscillator of nonlinear optics preserves such states under time evolution.[7]

This problem, however, deserves interest also from the practical point of view. In fact, Hamiltonians expressed in terms of SU(1,1) generators are encountered in the analysis of the so-called squeezed states of the electromagnetic field.[8]

In Ref. 3 the following Hamiltonian has been considered:

$$\hat{H}=\frac{\omega(t)}{2}\hat{H}_1+\Omega^*(t)\hat{H}_2-\Omega(t)\hat{H}_3+\beta(t)\ (\hbar=1)\ , \tag{1}$$

where $\omega(t)$ and $\beta(t)$ are real time-dependent functions, and $\Omega(t)$ is a complex function. Furthermore, the $\hat{H}$ operators are the generators of the real, split, three-dimensional Lie algebra, with commutation relations

$$\begin{aligned}&[\hat{H}_1,\hat{H}_2]=2\lambda\hat{H}_2,\quad [\hat{H}_1,\hat{H}_3]=-2\lambda\hat{H}_3\ ,\\ &[\hat{H}_2,\hat{H}_3]=-\delta H_1\ ,\end{aligned} \tag{2}$$

where λ and δ are numbers, depending on the explicit form of the $\hat{H}$ operators.

When $\lambda=\delta=1$, we can recognize the commutation relation of the angular-momentum algebra;[3] when $\lambda=-\delta=1$, the algebraic structure is that of the SU(1,1) group, as is immediately realized by means of the following identification:[7]

$$\hat{H}_1=2\hat{K}_0,\quad \hat{H}_2=\hat{K}_+,\quad \hat{H}_3=-\hat{K}_-\ . \tag{3}$$

The evolution of a quantum system described by the Hamiltonian (1) written in terms of the generators (3) can be therefore studied by directly applying the techniques discussed in Ref. 3.

As first step we write

$$\begin{aligned}&\hat{H}(t)=\hat{H}_0(t)+\hat{V}(t)\ ,\\ &\hat{H}_0(t)=\omega(t)\hat{K}_0+\beta(t)\ ,\\ &\hat{V}(t)=\Omega^*(t)\hat{K}_++\Omega(t)\hat{K}_-\ .\end{aligned} \tag{4}$$

From the above expression we get an interaction Hamiltonian of the type

$$\begin{aligned}\hat{H}_I&=\exp\left[+i\int_0^t\hat{H}_0(t')\,dt'\right]\hat{V}(t)\exp\left[-i\int_0^t\hat{H}_0(t')\,dt'\right]\\ &=\overline{\Omega}^*(t)\hat{K}_++\overline{\Omega}(t)\hat{K}_-\ ,\\ \overline{\Omega}(t)&=\Omega(t)\exp\left[-i\int_0^t\omega(t')\,dt'\right]\ .\end{aligned} \tag{5}$$

In this connection the time displacement operator may be written

$$\hat{U}(t)=\hat{U}^{(0)}(t)\hat{U}_I(t)\ , \tag{6}$$

where

$$\begin{aligned}&i\frac{d}{dt}\hat{U}^{(0)}(t)=\hat{H}_0(t)\hat{U}^{(0)}(t),\quad \hat{U}^{(0)}(0)=\hat{I}\ ,\\ &i\frac{d}{dt}\hat{U}_I(t)=\hat{H}_I(t)\hat{U}_I(t),\quad \hat{U}_I(0)=\hat{I}\ .\end{aligned} \tag{7}$$

The first integration is trivial; the second requires more care. However, according to the Wei-Norman decoupling theorem (see Refs. 2 and 3), one can immediately write

$$\hat{U}_I(t)=\exp[2h(t)\hat{K}_0]\exp[g(t)\hat{K}_+]\exp[-f(t)\hat{K}_-]\hat{I}\ , \tag{8}$$

where the functions (h,g,f) are specified by the following system of linear differential equations:

$$\begin{aligned}&\dot{h}(t)=-i\overline{\Omega}(t)g(t)e^{2h(t)}\ ,\\ &\dot{g}(t)=-i\overline{\Omega}^*(t)e^{-2h(t)}-g(t)\dot{h}(t)\ ,\\ &\dot{f}(t)=i\overline{\Omega}(t)e^{2h(t)},\quad h(0)=f(0)=g(0)=0\ .\end{aligned} \tag{9}$$

It can be easily shown that the solution of system (9) depends on the solution of the single Riccati equation[2]

$u(t)=\dot{h}(t)$:

$$\dot{u}(t)-[u(t)]^2+p(t)u(t)+q(t)=0,\quad u(0)=0\ ,$$
$$p(t)=-\frac{\dot{\Omega}(t)}{\Omega(t)}+i\omega(t),\quad q(t)=|\Omega(t)|^2\ . \tag{10}$$

We can now discuss a more specific physical situation, namely, the case of the degenerate parametric oscillator discussed in Ref. 7.

In this case we realize the generators of SU(1,1) as[7]

$$\hat{K}_+=\tfrac{1}{2}(\hat{a}^\dagger)^2\ ,$$
$$\hat{K}_-=\tfrac{1}{2}(\hat{a})^2\ , \tag{11}$$
$$\hat{K}_0=\tfrac{1}{4}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)\ ,$$

where a and $a^\dagger$ are annihilation and creation operators ($[a,a^\dagger]=1$) and expand the wave function of the system at the generic time t in terms of the $D^\dagger(\frac{1}{4})$ representation of SU(1,1),[7] that is,

$$|\psi(t)\rangle=\sum_{l=-n}^{\infty}C_l(t)|n+l,\tfrac{1}{4}\rangle\ , \tag{12}$$

$n+l$ representing the Fock occupation number. To assume that $|\psi(0)\rangle=|n,\frac{1}{4}\rangle$ implies for the coefficients $C_l(t)$ of the expansion (12) the initial conditions $C_l(0)=\delta_{l,0}$. According to what has been discussed so far, the C_l coefficients can be immediately evaluated from the following matrix element:

$$C_l(t)=\langle n+l,\tfrac{1}{4}|\hat{U}(t)|n,\tfrac{1}{4}\rangle\ . \tag{13}$$

Using (6), (8), and (11)–(13), after some algebra we get

$$|\psi(t)\rangle=\sum_{l=-n}^{\infty}\left[\frac{g(t)}{2}\right]^{l/2}\exp\left[\tfrac{1}{4}[1+2(n+l)]\left[2h(t)-i\int_0^t\omega(t')\,dt'\right]\right]\sqrt{n!(n+l)!}\,H_n^l(g(t)f(t))\,|n+l,\tfrac{1}{4}\rangle\ , \tag{14}$$

where $H_n^l(x)$ is a polynomial defined as follows:

$$H_n^l(x)=\sum_{m=0}^{[n/2]}\frac{(-1)^m(x/2)^m}{2^m m!(n-2m)!(l/2+m)!}\ . \tag{15}$$

The symbol $[v]$ denotes the largest integer $\leq v$.

An analytical solution of (10) exists in a limited number of cases only, as, e.g., for the problem discussed in Ref. 7, where

$$\omega(t)=2\omega_0\ ,$$
$$\Omega(t)=2\Omega_0 e^{2i\omega_0 t}\ , \tag{16}$$

with ω_0 and Ω_0 constants.

In this case Riccati's equation is straightforwardly solved and therefore the system of differential equations (9) too. [The explicit expression of the Riccati equation now reads

$$\dot{u}-u^2+4\Omega_0^2=0,\quad u(0)=0\ ,$$

whose solution is $u=-2\Omega_0\tanh(2\Omega_0 t)$.] We must, however, stress that exact solutions can be found for a larger class of potential, as will be shown elsewhere. In this connection, assuming that the state $|\psi(t)\rangle$ evolves from the vacuum ($n=0$), the general expression (14) reduces to

$$|\psi(t)\rangle=\sum_{l=0}^{\infty}(-i/2)^{l/2}\exp[-\tfrac{1}{2}(1+2l)\omega_0 t]\frac{(\tanh 2\Omega_0 t)^{l/2}}{\sqrt{\cosh 2\Omega_0 t}}\frac{\sqrt{l!}}{(l/2)!}|l,\tfrac{1}{4}\rangle\ . \tag{17}$$

The results found in this paper coincide with those of Refs. 7 and 9, where the evolution of coherent SU(1,1) states has been studied.

In this Brief Report we have followed a different technique based on the Wei-Norman algebraic procedure,[2] which has allowed us to treat the problem from a rather general point of view.

Let us finally point out that the above developed analysis can be usefully applied to the solution of SU(1,1) Raman-Nath-type[3,10] equations, namely,

$$iC_l'=\omega(t)(n+l)C_l+[\Omega(t)\sqrt{(n+l+1)(n+l+2)}C_{l+2}+\Omega^*(t)\sqrt{(n+l)(n+l-1)}C_{l-2}]\ ,$$
$$C_l(0)=\delta_{l,0}\ . \tag{18}$$

This last point, the asymptotic limits for large n and the connection with earlier work, will be discussed in a forthcoming paper.

---

*Present address: Department of Physics and Astronomy, Dartmouth College, Hanover, NH 03755.

†Present address: Dipartimento di Fisica, Università degli Studi di Roma "La Sapienza," piazzale Aldo Moro 2, I-00185 Roma, Italy.

1W. Magnus, Commun. Pure Appl. Math. 7, 649 (1954).
2J. Wei and E. Norman, J. Math. Phys. A 4, 575 (1963).
3G. Dattoli, J. Gallardo, and A. Torre, J. Math. Phys. (to be published).
4R. P. Feynman, *Quantum Electrodynamics* (Benjamin, New York, 1961).
5G. Wybourne, *Classical Groups for Physicist* (Wiley, New York, 1974).
6A. M. Perelemov, Commun. Math. Phys. 26, 222 (1972).
7C. C. Gerry, Phys. Rev. A 31, 2721 (1985).
8H. P. Juen, Phys. Rev. A 13, 2226 (1976).
9C. C. Gerry, Phys. Lett. 119B, 381 (1982).
10See, e.g., F. Ciocci, G. Dattoli, and M. Richetta, J. Phys. A 17, 1333 (1984); G. Dattoli, M. Richetta, and I. Pinto, Nuovo Cimento 4D, 293 (1984).