Effect of additive and multiplicative noise on the first bifurcations of the logistic model

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The statistical dynamics of the response of the logistic map, $x_{n+1}=rx_n(1-x_n)$, towards additively and multiplicatively coupled fluctuating forces is studied analytically and numerically in quantitative detail in the range of control parameter r where the unforced system shows the first transcritical and the first pitchfork bifurcation.

I. INTRODUCTION

Nonlinear dissipative systems that undergo a transition under quasistatic variation of the driving stress can show peculiar response behavior when also time-dependent forces are applied. Hydrodynamic and laser instabilities, bifurcations in "simple" nonlinear (model) systems, and discrete maps that are perturbed by time-dependent forces coupling additively or multiplicatively to a state variable of the system in question are examples.

Studies[1–6] of the effect of "time"-dependent forcing on systems with discrete dynamics seem to have been concentrated on uncorrelated forces coupled to the logistic map[7] (see, however, Refs. 8–11 for other forced discrete systems and Ref. 12 for time periodic forcing of the logistic map). This research is mostly aimed at understanding within qualitative and sometimes semiquantitative terms the influence of noise on the period-doubling-bifurcation sequence[13] in the control parameter range beyond the first simple bifurcations.

The effects of additive or multiplicative random forcing on the very first bifurcations of the logistic map, on the other hand, have not yet been investigated properly. Since an incorrect statement[1] about the equivalence of additive and multiplicative noise might have misled people into believing that the response at small control parameters to noise is well understood and uninteresting, we undertook the investigation reported here.

We shall present detailed quantitative, analytical, and numerical results on the statistical dynamics of the logistic map's response to additively or multiplicatively coupled time-dependent forces that fluctuate according to various prescribed statistical properties.

In Sec. II we give the basic formulas for a statistical description. Section III contains analytical and numerical results on the behavior of the map under additive and multiplicative forcing in the control parameter range $r<3$ of the first transcritical bifurcation. In Sec. IV we investigate analytically and numerically the first noisy pitchfork bifurcation. In Sec. V we summarize our main results.

II. THE SYSTEM

We shall investigate the statistical dynamics of the response of the logistic map

$$x_{n+1}=rx_n(1-x_n) \tag{2.1}$$

towards statistically stationary stochastic forces ξ_n with vanishing mean

$$\langle \xi_n \rangle = 0 \tag{2.2a}$$

and unit covariance

$$\langle \xi_n^2 \rangle = 1 \tag{2.2b}$$

in a range of the control parameter r where the first bifurcations of the unperturbed system are located as shown in Fig. 1. The forces couple to the variable x either multiplicatively,

$$x_{n+1}=r_n x_n(1-x_n)\ , \tag{2.3a}$$

via a fluctuating control parameter

$$r_n = r(1+\Delta\xi_n)\ , \tag{2.3b}$$

or additively,

$$x_{n+1}=rx_n(1-x_n)+\Delta\xi_n\ . \tag{2.4}$$

Here Δ measures the noise intensity or, alternatively, the coupling strength between ξ and x.

A. Averages

The statistical properties of the noise are prescribed by the path probability density $\mathscr{P}[\xi]$ for a particular se-

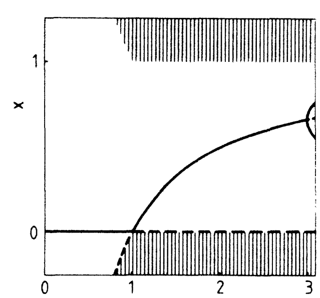

FIG. 1. Bifurcation diagram of the logistic map (2.1). Hatched area shows the basin of attraction of the fixed point at $-\infty$.

quence $[\xi]=\ldots,\xi_{n-1},\xi_n,\ldots$ of forces. Thus we use ensemble averages over noise realizations $[\xi]$ weighted by $\mathcal{P}[\xi]$:

$$\langle F[\xi]\rangle=\prod_n \int_{-\infty}^{\infty} d\xi_n F[\xi]\mathcal{P}[\xi] \;. \tag{2.5}$$

An example of a function $F[\xi]$ depending on the noise is x_{n+1} which is determined by the history of the forcing up to "time" n and by the initial value x_{n_0} at time n_0. Therefore, noise averages (2.5) of functions of x still depend on the starting value x_{n_0}. However, if n_0 lies in the distant "past," $n_0 \to -\infty$, and if the noise amplitudes are sufficiently large to overcome the "potential barriers" that restrict the dynamics of the unforced system to a subset of its phase space (e.g., to a period-2 cycle), then the forced system will have "forgotten" its initial condition $x_{-\infty}$ at any finite time n. We shall discuss these points further later on where we compare our analytical results for ensemble averages with numerical results obtained as time averages and, in addition, for some cases, as ensemble averages.

B. Some general statistical properties

Before we investigate the statistical dynamics of the forced map in detail we want to mention some general properties. First of all, any average evaluated for system (2.3) or (2.4) is an even function of Δ if, as in our case, the forces are distributed symmetrically around zero. This result holds for arbitrary statistical dynamics of the forcing since only the combination $\Delta\xi_n$ enters the equations of motion (2.3) and (2.4). This property also implies that any average evaluated for small noise amplitudes to second order in Δ depends at most on two-point correlations of the random forces. Higher cumulants of ξ enter only via higher orders Δ^{2k} for $k>1$.

Thus the statistical dynamics of the response of the logistic map (2.1) towards small-amplitude noise is up to order Δ^2 universal in the sense that it is independent of the detailed statistics of the forcing (i.e., higher-order cumulants) and depends only on its two-point correlation

$$D(\,|n-m|\,)=\langle \xi_n \xi_m\rangle \;. \tag{2.6}$$

This dependence, however, is in general different for multiplicative and additive forcing.

Other general results are relations between moments and correlation functions of x with itself or with ξ that may be derived from the equations of motion and that are particularly simple as long as time translational invariance is not broken. An example is the relation

$$\langle x_n^2\rangle=\frac{r-1}{r}\langle x_n\rangle+\begin{cases}\Delta\langle \xi_n x_n(1-x_n)\rangle & \text{(multiplicative forcing)} & \text{(2.7a)}\\ 0 & \text{(additive forcing)} & \text{(2.7b)}\end{cases}$$

which follows from averaging the equation of motion with multiplicative forcing (2.7a) or with additive forcing (2.7b) if time translational invariance, $\langle x_{n+1}\rangle=\langle x_n\rangle$, holds. (That, by the way, is violated if x_n always diverges for large enough n as in the case of additive forcing at $r=1$.)

From the above relation (2.7) one can draw some interesting conclusions for $\langle x_n\rangle$ and $\langle(\delta x_n)^2\rangle$, the mean square of the fluctuation

$$\delta x_n = x_n-\langle x_n\rangle \;. \tag{2.8}$$

In order to derive them simultaneously for both forcing types let us assume for the moment that the ξ_n in the multiplicatively forced system are uncorrelated (the additive forces need not be specified for the following discussion). Then the last term in (2.7) vanishes also in this case, $\langle \xi_n x_n(1-x_n)\rangle=0$, since ξ_n and x_n are uncorrelated—x_n depends only on ξ_j with $j<n$—and one obtains in both cases

$$\langle(\delta x_n)^2\rangle=\langle x_n\rangle\left[\frac{r-1}{r}-\langle x_n\rangle\right] . \tag{2.9}$$

Thus if $\langle x_n\rangle=0$, $\langle(\delta x_n)^2\rangle=0$ also. Furthermore, $\langle(\delta x_n)^2\rangle$ being positive semidefinite enforces at $r=1$ that $\langle x_n\rangle=0=\langle(\delta x_n)^2\rangle$ (*provided* that time translational invariance holds). For $r>1$ one infers $0\le\langle x_n\rangle\le 1-1/r$ so that forcing suppresses the mean of x_n with respect to the fixed point $x^*=1-1/r$ of the unforced system. Finally, for $r<1$ one finds $1-1/r\le\langle x_n\rangle\le 0$ so that $\langle x_n\rangle$ is either zero or negative.

III. BEHAVIOR NEAR THE FIRST INSTABILITY

In the absence of noise, $\Delta=0$, the system shows a transcritical bifurcation at the threshold value $r=r_c(\Delta=0)=1$ of the control parameter. The fixed point $x^*=0$ ($x^*=1-1/r$) being stable (unstable) below threshold becomes unstable (stable) above.[7]

A. Additive versus multiplicative noise

From the equations of motion (2.3) and (2.4) for the perturbed systems it is clear that $x^*=0$ remains a fixed point only for multiplicative forcing. Only in this case will there be as a function of control parameter r a sharp bifurcation from $x^*=0$ to a time-dependent orbit with nonvanishing amplitudes x_n.

For additive forcing one might expect at first sight the order parameter $(\langle x_n^2\rangle)^{1/2}$ to show a rounded, imperfect bifurcation with a gradual growth as r increases towards $r=1$. However, the system's response to additive perturbations near the static, $\Delta=0$, instability at $r=1$ is more dramatic (cf. Sec. III D). Right at $r=1$, for example, any additive noise, $\Delta\xi_n$, however small Δ may be, drives the system towards $-\infty$, independent of its initial condition. The reason, shown in Fig. 1, is that the basin of attraction of $-\infty$ (marked by the hatched area) touches the fixed point shown by the solid line just at $r=1$. Furthermore, additive noise will eventually kick the system into the basin of attraction of $-\infty$ also for $r\neq 1$ unless the size of the forcing is specifically bounded from above and below with the restrictions becoming more and more severe as $r\rightarrow 1$. Thus additive forcing changes the static, $\Delta=0$, bifurcation structure of the system completely while multiplicative noise does not change the bifurcation behavior of the system near $r=1$ qualitatively.

B. Stability threshold of $x^*=0$ for parametric forcing

The fixed point $x^*=0$ of the parametrically forced system (2.3) is stable against perturbations $[\xi]$ if an infinitesimal initial deviation x_0 decays to zero in the long time limit, i.e., if

$$1 \geq \lim_{N\rightarrow\infty} \left| \frac{\partial x_N}{\partial x_0} \right|_{x_0=0} = \lim_{N\rightarrow\infty} \prod_{n=0}^{N-1} r\,|1+\Delta\xi_n| \ . \quad (3.1)$$

At the stability threshold $r_c(\Delta)$ defined by the equality in (3.1) one has

$$1=\lim_{N\rightarrow\infty} \left[\prod_{n=0}^{N-1} r_c(\Delta)\,|1+\Delta\xi_n| \right]^{1/N} \quad (3.2)$$

or, equivalently,

$$r_c(\Delta)=\exp(-\langle \ln|1+\Delta\xi| \rangle) \ . \quad (3.3)$$

The average in (3.3) comes up as a time average, but since the forcing is ergodic we have

$$\lim_{N\rightarrow\infty} \left[N^{-1} \sum_{n=0}^{N-1} \ln|1+\Delta\xi_n| \right] = \langle \ln|1+\Delta\xi| \rangle \ ,$$

and statistical stationarity allows one to drop the time index on the right side.

Some remarks are in order to elucidate the richness of the formula (3.3) for the stability threshold.

(1) It holds for arbitrary statistical dynamics of the forcing [with the appropriate interpretation of the average, formula (3.3) also applies to periodic forcing[12,14]].

(2) The threshold $r_c(\Delta)$ does not depend on the correlation properties of the forcing. To evaluate (3.3) one needs to know only the stationary distribution $P(\xi)$ of the amplitudes of $[\xi]$. Thus parametric modulation with periodic or quasiperiodic forces having the same distribution of amplitudes as a stochastic series of forces entails the same stability threshold $r_c(\Delta)$ of $x^*=0$. It is amusing to check this result on a computer, e.g., with uncorrelated sequences of $\xi=\pm1$ (dichotomous noise) versus deterministic sequences of various periodicity lengths.

(3) *Any* forcing whatsoever with small amplitudes, $\Delta \ll 1$, causes stabilization of the trivial state $x^*=0$ since

$$r_c(\Delta)=1+\tfrac{1}{2}\Delta^2+O(\Delta^4) \quad (3.4)$$

is bigger than the critical value, $r_c(\Delta=0)=1$, in the absence of forcing. Moreover, the threshold shift, $r_c(\Delta)-r_c(\Delta=0)$, of the stability boundary of $x^*=0$ induced by parametric modulation grows proportional to the mean square of the forcing amplitudes. The same universal stabilization of the basic state was found[15] for the parametrically modulated, damped Duffing oscillator, i.e., a continuous system showing a pitchfork bifurcation for static restoring forces. Also, there the shift of the stability threshold induced by modulation with small amplitudes grows proportional to the mean square of the forces.[15,16]

(4) Large forcing amplitudes can cause stabilization $[r_c(\Delta)>1]$ or destabilization $[r_c(\Delta)<1]$ of the fixed point $x^*=0$ depending on Δ and the form of the stationary distribution $P(\xi)$ of the amplitudes. Consider as two representative examples modulation with forces that take on only two values,

$$P(\xi)=\tfrac{1}{2}[\delta(\xi-1)+\delta(\xi+1)] \ , \quad (3.5a)$$

or which are distributed with equal weight over a finite interval,

$$P(\xi)=\frac{1}{\sqrt{12}} \begin{cases} 1 & \text{if } |\xi|<\sqrt{3} \\ 0 & \text{otherwise.} \end{cases} \quad (3.5b)$$

The resulting stability thresholds

$$r_c(\Delta)=|1-\Delta^2|^{-1/2} \quad (3.6a)$$

and

$$r_c(\Delta)=\frac{e}{|1-\tilde{\Delta}^2|^{1/2}} \left| \frac{1-\tilde{\Delta}}{1+\tilde{\Delta}} \right|^{1/2\tilde{\Delta}} , \quad \tilde{\Delta}=\Delta\sqrt{3} \quad (3.6b)$$

shown in Fig. 2 by curves a and b, respectively, lie partly above and partly below $r_c(\Delta=0)$.

The stability threshold (3.6a) diverges at $|\Delta|=1$. With $\xi_n=\pm1$ the probability that $r_n=r(1+\Delta\xi_n)$ approaches zero for $|\Delta|\rightarrow 1$ is just $\frac{1}{2}$, and so x is (eventually) forced for any value of r towards the attractor $x^*=0$. That holds, by the way, for any starting value x_0. So for $\Delta\sim1$ this kind of forcing destroys any dynamics whatsoever of the logistic map. For forces that are distributed according to (3.5b), on the other hand, the probability of finding ξ in an (infinitesimal) interval $d\xi$ around $\pm1/\Delta$ such that $r(1+\Delta\xi)$ is (infinitesimally) close to zero is too small [being only $\sim d\xi/(r\Delta)$] to force x towards $x^*=0$

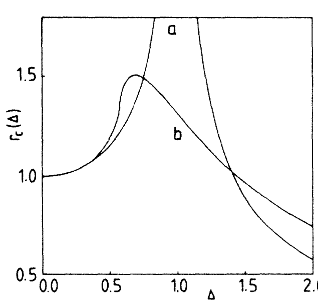

FIG. 2. Stability boundaries (3.6a) and (3.6b) of $x^*=0$ in the presence of multiplicative forcing. Curve a is for forces that take only values ±1. For curve b the forces are distributed according to (3.5b) with equal weight over a finite interval. The fixed point $x^*=0$ is stable for all r,Δ below the respective curves.

for large values of r. Thus the stability threshold (3.6b) does not diverge.

We should like to mention that the stability curve of the $x=0$ state of a Duffing oscillator subjected to a (sinusoidal) parametric modulation is qualitatively similar[16] to curve a.

(5) The stability threshold $r_c(\Delta)$ of $x^*=0$ for the generalized version, $x_{n+1}=x_n g(x_n;r_n)$, of the map (2.1) and (2.3) is given by the solution of $\langle \ln|g(0;\hat{r})|\rangle=0$ with $\hat{r}=r_c(\Delta)(1+\Delta\xi)$.

(6) The "critical slowing down" behavior of the relaxation towards the stable fixed point $x^*=0$ can be obtained for $r=r_c(\Delta)(1-\epsilon)$ shortly below the stability threshold $r_c(\Delta)$ from the linearized version of the map (2.3) together with (3.2). One finds that the time $\tilde{n}$ during which an initial (infinitesimal) deviation decreases by a factor e^{-1} does not depend on the forcing,

$$\tilde{n}=[-\ln(1-\epsilon)]^{-1}\simeq\epsilon^{-1} . \tag{3.7}$$

C. Statistical dynamics for small multiplicative noise above threshold

In this subsection we discuss the statistical behavior of the logistic map with random parametric forcing above threshold, i.e., for combinations of the parameters r and Δ such that $r>r_c(\Delta)$. Then the fixed point $x^*=0$ is no longer stable according to (3.1)–(3.3).

1. Bifurcation behavior

From some (unsystematic) numerical tests we found that x_n tends to diverge quite rapidly when Δ is large even when r is supercritical by only a small margin. Also, when the ξ's are bounded, $|\xi_n|<\xi_{\max}$, as in (3.5) the trajectory escapes to $-\infty$ whenever the supercritical control parameter exceeds $4/(1+\Delta\xi_{\max})$. Then the peak height $(1+\Delta\xi_n)r/4$ of the parabola (2.3a) exceeds 1 whenever ξ_n is close to $\xi_{\max}$. Thus there is a good chance that $x_{n+1}>1$ with a subsequent attraction towards $-\infty$.

For small Δ, on the other hand, and not too large a distance $r-r_c(\Delta)$ above threshold, the system responds to the forcing with finite, statistically stationary fluctuations. We are interested in the statistical properties of the latter. As an example of such a property we show with thick solid lines in Fig. 3 the numerically determined behavior of the order parameter $(\langle x_n^2\rangle)^{1/2}$ and of the mean $\langle x_n\rangle$. The stationary distribution of the noise had the box shape (3.5b) with $\Delta=0.5$ and the two-point correlation spectrum was white, $\langle\xi_n\xi_m\rangle=\delta_{n,m}$. The stability threshold (3.6b) of the $x^*=0$ fixed point is for this noise $r_c(\Delta=0.5)=1.188$ while the small-Δ expansion up to second order (3.4) gives 1.125. Figure 3 is quite representative since other types of noise generate for small Δ close to threshold very similar behavior of the first moments $\langle x_n\rangle$ and $\langle x_n^2\rangle$.

2. Small-Δ perturbation theory

To elucidate this behavior analytically we shall use in the following a small-Δ expansion. Inserting the decomposition of the fluctuating orbit above threshold,

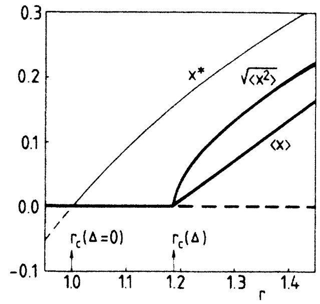

FIG. 3. Bifurcation diagram. Thick solid lines are for parametric noise with a white two-point correlation spectrum and the box-shaped stationary distribution (3.5b) with $\Delta=0.5$. Thin solid lines denote the unforced system, $\Delta=0$. The stability thresholds $r_c(\Delta=0.5)=1.188$ (3.6b) and $r_c(\Delta=0)=1$ are marked by arrows.

$$x_n(\Delta)=x_n^{(0)}+\Delta x_n^{(1)}+\Delta^2 x_n^{(2)}+\cdots , \tag{3.8}$$

into the equation of motion (2.3) one obtains a sequence of linear, inhomogeneous difference equations. Their solutions are

$$x_n^{(0)}=x^*=1-1/r , \tag{3.9a}$$

$$x_n^{(1)}=x^*\sum_{i=0}^{n-1}(2-r)^i\xi_{n-1-i} , \tag{3.9b}$$

$$x_n^{(2)}=\sum_{i=0}^{n-1}(2-r)^{n-1-i}x_i^{(1)}[(2-r)\xi_i-rx_i^{(1)}] . \tag{3.9c}$$

Here we have used the fixed point $x^*=1-1/r$ of the unperturbed system as the initial value, i.e., $x_0=x_0^{(0)}=x^*$, so that $x_0^{(k)}=0$ for $k>0$. This choice is not necessary but analytically convenient since it eliminates those transients that would otherwise appear in (3.9) and that decay anyway in the long time limit. The factor $(2-r)$ in (3.9) is the derivative $\partial x_{n+1}^{(0)}/\partial x_n^{(0)}$ of the unperturbed map at x^*.

3. Stationary mean $\langle x(\Delta)\rangle$

From (3.9) one finds that $\langle x_n^{(1)}\rangle=0$ in accordance with the symmetry properties discussed in Sec. II B and that $\langle x_n(\Delta)\rangle$ is determined up to second order in Δ by sums containing only the two-point correlation (2.6) of the noise. For an arbitrary spectrum of this correlation function the formulas for most averages are rather involved. So for the sake of greater transparency we restrict the following discussion to forces with a Brownian two-point correlation spectrum,[17] i.e.,

$$\langle\xi_n\xi_m\rangle=\delta_{n,m}+(1-\delta_{n,m})e^{-\gamma|n-m|} \tag{3.10}$$

with the white-noise case being included in the limit $\gamma\to\infty$. Then one finds for $n\to\infty$ the stationary mean

$$\langle x(\Delta)\rangle=x^*-\frac{\Delta^2}{r(3-r)}[1-(r-1)M]+O(\Delta^4) \tag{3.11a}$$

with

$$M=\frac{(r-2)e^{-\gamma}}{1+(r-2)e^{-\gamma}} \tag{3.11b}$$

being determined by the system's "memory" of previously applied forces. M vanishes for white-noise forcing, $\gamma \to \infty$, where the position x_n at time n and the forces ξ_{n-j} at earlier times $n-j$ are uncorrelated. The same holds true with finite γ for $r=2$ where M (3.11b) changes its sign. This interesting point will be investigated further below. The significance of the factor $(3-r)$ appearing in the denominator of (3.11a), which thus restricts the radius of convergence of the expansion (3.8) to values $r<3$, will be discussed in Sec. IV.

The mean (3.11) of the fluctuating trajectory above threshold lies for all $r<3$ below the fixed point of the unforced map as discussed in Sec. II B (see also Fig. 3). As an aside we mention that $\langle x(\Delta)\rangle$ is bigger (smaller) for Brownian noise than for white noise if $r>2$ ($r<2$).

The small-Δ expression (3.11) vanishes at $r=r_c(\Delta) =1+\Delta^2/2$, i.e., it is consistent with the stability threshold of the $x^*=0$ fixed point derived in Section III B. The initial slope $s_1(\Delta)$ with which $\langle x(\Delta)\rangle$ grows above r_c,

$$s_1(\Delta)=\left.\frac{\partial\langle x(\Delta)\rangle}{\partial r}\right|_{r=r_c(\Delta)}$$

$$=1-\Delta^2\tfrac{3}{4}[1-\tfrac{2}{3}M(r=1)]+O(\Delta^4)\ , \qquad (3.12)$$

is smaller than the slope $[s_1(\Delta=0)=1]$ of the fixed point of the unforced system at $r_c(\Delta=0)$. That can also be seen in Fig. 3. So in the immediate vicinity of $r_c(\Delta)$ one has

$$\langle x(\Delta)\rangle=[r-r_c(\Delta)]s_1(\Delta)+O((r-r_c)^2)\ . \qquad (3.13)$$

4. *Order parameter and mean square of fluctuations*

For the stationary square of the order parameter one finds

$$\langle x^2(\Delta)\rangle=(x^*)^2-\Delta^2\frac{r-1}{r^2(3-r)}[1+2(2-r)M]+O(\Delta^4) \qquad (3.14)$$

and thus the mean-squared fluctuation is given by

$$\langle[\delta x(\Delta)]^2\rangle=\Delta^2\frac{r-1}{r^2(3-r)}[1-2M]+O(\Delta^4)\ . \qquad (3.15)$$

The square brackets in (3.14) and (3.15) are positive for $1<r<3$ and so the order parameter $(\langle x^2\rangle)^{1/2}$ is always below the fixed point x^* of the unforced map (see also Fig. 3). The memory term M enhances the fluctuations (3.15) for $r<2$ and depresses them for $r>2$ in comparison with white noise.

To determine the growth behavior of the order parameter and of the mean fluctuation shortly above threshold we shall restrict ourselves to white noise. In that case the relation (2.7), $\langle x^2(\Delta)\rangle=\langle x(\Delta)\rangle(r-1)/r$, is most conveniently used together with the form (3.13) to obtain

$$\langle x^2(\Delta)\rangle=[r-r_c(\Delta)]a_1(\Delta)$$
$$+[r-r_c(\Delta)]^2a_2(\Delta)+O((r-r_c)^3) \qquad (3.16)$$

with the growth coefficients

$$a_1(\Delta)=\tfrac{1}{2}\Delta^2+O(\Delta^4)\ , \qquad (3.17a)$$

$$a_2(\Delta)=1-\tfrac{9}{4}\Delta^2+O(\Delta^4)\ . \qquad (3.17b)$$

To derive (3.16) we had to evaluate the coefficient

$$s_2(\Delta)=\left.\frac{1}{2}\frac{\partial^2\langle x(\Delta)\rangle}{\partial r^2}\right|_{r=r_c(\Delta)}$$

$$=-1+\tfrac{9}{8}\Delta^2+O(\Delta^4) \qquad (3.18)$$

of the next term in (3.13). Note that with increasing noise the growth behavior of the order parameter $[\langle x^2(\Delta)\rangle]^{1/2}$ above threshold changes from $\sim(r-r_c)$ at $\Delta=0$ to $\sim(r-r_c)^{1/2}$ at finite Δ (cf. Fig. 3).

From the relation (2.9) one finds the supercritical growth of the mean-squared fluctuation to be

$$\langle[\delta x(\Delta)]^2\rangle=[r-r_c(\Delta)]b_1(\Delta)$$
$$+[r-r_c(\Delta)]^2b_2(\Delta)+O((r-r_c)^3)\ , \qquad (3.19)$$

$$b_1(\Delta)=\tfrac{1}{2}\Delta^2+O(\Delta^4)\ , \qquad (3.20a)$$

$$b_2(\Delta)=-\tfrac{3}{4}\Delta^2+O(\Delta^4)\ . \qquad (3.20b)$$

5. *Stationary distribution $W(x)$*

The stationary distribution of the fluctuating positions x_n is given by the average

$$W(x)=\lim_{n\to\infty}\langle\delta(x-x_n)\rangle\ . \qquad (3.21)$$

To evaluate it one can use a cumulant expansion. To second order in Δ the cumulants of x_n are determined by x^* and the statistical properties of $x_n^{(1)}$ (3.9b) and $x_n^{(2)}$ (3.9c). The latter are given by sums over n fluctuating quantities multiplied with weights $(2-r)^j$. Since for $|2-r|\simeq 1$ the above fluctuations enter the sums with roughly equal weight one can expect for $n\to\infty$ that $x_n^{(1)}$ as well as $x_n^{(2)}$ are roughly Gaussian distributed independently of the forcing statistics. Hence truncating the cumulant expansion after the second cumulant suggests itself as a reasonable approximation.

Doing that one obtains for $W(x)$ a Gaussian distribution centered at $\langle x\rangle$ (3.11) with a variance $\langle(\delta x)^2\rangle$ given by (3.15) and (3.19). Therefore, $W(x)$ becomes a δ function for $\Delta\to 0$ and, for finite Δ, when r approaches the threshold $r_c(\Delta)$ from above. For control parameters r such that $|r-2|\gtrsim 0.7$ the Gaussian approximation to $W(x)$ agrees very well with x histograms that we evaluated numerically for various stationary distributions $P(\xi)$ of the noise.

However, as $|r-2|$ decreases, the initial Gaussian distribution $W(x)$ changes if $P(\xi)$ is not Gaussian. The reason is obvious from (3.9). For example, right at $r=2$ the actual value of $x_n^{(1)}$ at time n no longer depends on the history of the previously applied forces (memory effects vanish as discussed already in Sec. III B 3) but only on ξ_{n-1}. Then the statistics of $x^{(1)}$ and ξ are the same since

$$x_n^{(1)}=\tfrac{1}{2}\xi_{n-1}\ \text{ for } r=2\ . \qquad (3.22)$$

Therefore, one expects for small Δ and $r=2$, where according to (3.9) $x_n \simeq (1+\Delta\xi_{n-1})/2$, the stationary distribution of x to be identical (up to scale factors) to that of ξ, i.e.,

$$W(x)=\frac{2}{\Delta}P\left[\frac{2}{\Delta}\left[x-\frac{1}{2}\right]\right] . \qquad (3.23)$$

A similar formula was found by Frazer *et al.*[9] at the first superstability point of the cubic map in the presence of additive white noise by using different methods. We have numerically confirmed the validity of (3.23) at $r=2$ for small Δ with various stationary force distributions.

To summarize, $W(x)$ is approximately Gaussian for any stationary distribution of small amplitude noise if $|2-r|$ is close to 1. For $r=2$, however, x is distributed like ξ. In between there ia a gradual change from one form to the other if $P(\xi)$ is non-Gaussian.

6. *Two-point correlations*

The stationary two-point correlation function of the fluctuations

$$C(n)=\langle \delta x_{k+n}\,\delta x_k\rangle \qquad (3.24)$$

is given up to second order in Δ by

$$C(n)=(2-r)^n C(0)+N(n) \qquad (3.25a)$$

with $n>0$ being implied. Here $C(0)$ denotes the equal-time correlation (3.15). The term

$$N(n)=\Delta^2(x^*)^2\frac{e^{-\gamma n}-(2-r)^n}{1+2(r-2)\cosh\gamma+(r-2)^2} \qquad (3.25b)$$

represents a "memory" effect of the forcing on the correlations of the fluctuations (see also the discussion in Secs. III C 3 and III C 4). If it is small, $\gamma \gg 1$, then the correlations decay towards zero monotonously for $r<2$ and in an oscillatory manner for $r>2$. This behavior is enforced by the topology of the map. It causes in the absence of parametric modulation these two different dynamics for the approach to the fixed point $x^*=\frac{1}{2}$ at the peak of the logistic parabola.

Correlations decay fastest for $r=2$. There the time dependence of $C(n)$ is identical to that of the noise correlation $D(n)$ (3.10):

$$C(n;r=2)=\delta_{n,0}C(0)+\frac{\Delta^2}{4}(1-\delta_{n,0})e^{-\gamma n} . \qquad (3.25c)$$

Thus at $r=2$ not only the stationary distribution $W(x)$ of the response x but also its (two-point) correlation dynamics $C(n)$ are the same as the respective quantities $P(\xi)$ and $D(n)$ of the forcing.

D. Small additive noise

Here we discuss briefly the statistical dynamics of the response of the map for control parameters $0<r<3$ In the presence of small additive noise.

1. *Small-Δ expansion*

In close analogy to a previous subsection we perform a small-Δ expansion (3.8) of the orbit $x_n(\Delta)$, this time around the stable fixed point

$$x^*=\begin{cases}0 & \text{for } 0<r<1\\ 1-1/r & \text{for } 1<r<3\end{cases} \qquad (3.26)$$

of the unforced map. Then the equation of motion (2.4) is decomposed up to second order in Δ into the system

$$x_{n+1}^{(1)}=qx_n^{(1)}+\xi_n , \qquad (3.27a)$$

$$x_{n+1}^{(2)}=qx_n^{(2)}-r(x_n^{(1)})^2 \qquad (3.27b)$$

with

$$q=r(1-2x^*)=\begin{cases}r & \text{for } 0<r<1\\ 2-r & \text{for } 1<r<3.\end{cases} \qquad (3.27c)$$

Its solution is

$$x_n^{(1)}=\sum_{i=0}^{n-1} q^i\xi_{n-1-i} , \qquad (3.28a)$$

$$x_n^{(2)}=-r\sum_{i=0}^{n-1} q^{n-1-i}(x_i^{(1)})^2 . \qquad (3.28b)$$

2. *Stationary averages*

For our representative noise with a Brownian two-point correlation spectrum one finds for the stationary mean

$$\langle x(\Delta)\rangle = x^*-\Delta^2\frac{r}{(1-q)^2(1+q)}\times\left[1+2\frac{qe^{-\gamma}}{1-qe^{-\gamma}}\right]+O(\Delta^4) . \qquad (3.29a)$$

The square of the order parameter

$$\langle x^2(\Delta)\rangle = \frac{r-1}{r}x^*+\Delta^2\frac{1-r}{(1-q)^2(1+q)}\times\left[1+2\frac{qe^{-\gamma}}{1-qe^{-\gamma}}\right]+O(\Delta^4) \qquad (3.29b)$$

follows immediately with relation (2.7).

According to (3.29a) the stationary mean is negative for $0<r<1$ with its absolute size increasing as r approaches 1. There the small-Δ-expansion formulas (3.29) lose their validity. Note, however, that (3.29) agrees very well with numerically determined averages as long as the latter exist, i.e., as long as x_n does not diverge. That in turn is the case whenever the distance $|r-1|$ is so large that the noise cannot kick the trajectory into the basin of attraction of the fixed point $x=-\infty$ of the system (2.4). As an aside we mention that a "smallest possible distance" cannot be defined properly except maybe as a mean quantity since the orbit depends on the forcing sequence $[\xi]$ realized in one particular numerical experiment.

In Fig. 4 we show with thick solid lines the numerically determined stationary average $\langle x(\Delta)\rangle$ and the order pa-

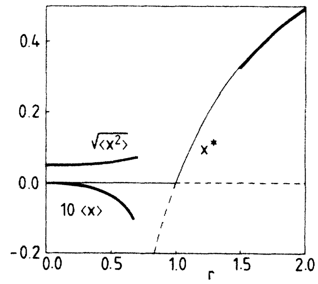

FIG. 4. Order parameter and mean orbit (thick solid lines) vs r for additive dichotomous noise of amplitude $\Delta=0.05$. Thin solid lines show the bifurcation for $\Delta=0$. For $0.69 \lesssim r \lesssim 1.45$ the orbits escaped in our numerical simulations to $-\infty$. $\langle x \rangle$ and $[\langle x^2 \rangle]^{1/2}$ are identical within a pencil's width for $r>1.45$.

rameter $[\langle x^2(\Delta)\rangle]^{1/2}$ as a function of r for dichotomous noise of amplitude $\Delta=0.05$. For r values within the gap of the solid curves the orbits escaped to $-\infty$. Except for this gap which increases with growing Δ the order parameter shows the typical characteristics of a rounded, imperfect bifurcation of a system that without the perturbation displays a perfect one (thin solid lines in Fig. 4). That the fixed point at $-\infty$ becomes so strongly attracting near $r=1$ is in a sense only an annoying complication of an otherwise simple bifurcation behavior. The agreement of the solid curves with the small-Δ result (3.29) for white noise, $\gamma=\infty$, is perfect.

3. Equivalence of additive and multiplicative noise

So far we have seen that the response of the map towards additive and multiplicative noise is totally different for small values of r, say, up to the stability threshold $r_c(\Delta)$ (3.3) of $x^*=0$ under parametric forcing. If, however, $r>r_c(\Delta)$ is sufficiently far above 1 so that the fixed point at $-\infty$ does not attract the additively perturbed orbits, then, for small Δ, multiplicative and additive noise causes similar response behavior: A comparison of (3.9b) with (3.28a) shows that the orbits $x_n(\Delta)=x^*+\Delta x_n^{(1)}$ up to first order in Δ are the same if one uses the same noise sequence $[\xi]$ and scales the noise intensity Δ for the two forcing types according to

$$\Delta_{\text{additive}} \leftrightarrow \frac{r-1}{r}\Delta_{\text{multiplicative}} . \tag{3.30}$$

We checked the validity of (3.30) for $r<3$ as well as for $r \geq 3$. We found that numerically determined stationary distributions $W(x)$ resulting from additive and multiplicative forces $[\xi]$ with the same statistics agreed much better with the scaling (3.30) than with the one derived by Crutchfield *et al.*[1] with rather *ad hoc* approximations.

As an interesting aside we mention that (3.30) can also be derived from the requirement that the moment generating functions $\langle \exp(ikx_{n+1})\rangle$ and $\langle \exp(ik\tilde{x}_{n+1})\rangle$ are the same, i.e.,

$$\begin{aligned}&\langle \exp[ikr(1+\Delta\xi_n)x_n(1-x_n)]\rangle \\ &\qquad = \langle \exp\{ik[r\tilde{x}_n(1-\tilde{x}_n)+\tilde{\Delta}\xi_n]\}\rangle .\end{aligned} \tag{3.31}$$

Here the tilde refers to additive forcing. For small amplitudes and uncorrelated forces (3.31) yields

$$\frac{r^2\langle x_n^2(1-x_n)^2 e^{ikrx_n(1-x_n)}\rangle}{\langle e^{ikr\tilde{x}_n(1-\tilde{x}_n)}\rangle} \simeq \frac{\tilde{\Delta}^2}{\Delta^2} . \tag{3.32}$$

Factorizing the numerator and approximating

$$r^2\langle x_n^2(1-x_n)^2\rangle \simeq \langle x_{n+1}^2\rangle \simeq (1-1/r)^2$$

leads then to (3.30).

Since most results presented in Sec. III C are strongly dominated by $x_n^{(1)}$, one merely has to use (3.30) to find approximately the corresponding small-Δ statistical properties of x_n in the presence of additive noise. For example, the discussion on the different shapes of the stationary distribution $W(x)$ for $r>r_c(\Delta)$ in Sec. III C 5 applies equally well to additive noise. Thus also for additive noise coupled to the system at $r=2$ the stationary distribution $W(x)$ has the same form as $P(\xi)$.

IV. THE EFFECT OF NOISE ON THE FIRST PERIOD-DOUBLING BIFURCATION

In this section we discuss the effect of small-amplitude noise on the first period-doubling pitchfork bifurcation of the unperturbed map at $r=3$. We shall concentrate on the additively forced map (2.4). The multiplicatively forced system (2.3) shows for small Δ in the vicinity of $r=3$ similar behavior that can be related quantitatively to the statistical dynamics of the response under additive forcing by scaling the noise according to (3.30).

A. Noisy pitchfork bifurcation

In the absence of forcing, $\Delta=0$, the fixed point $x^*=1-1/r$ of the system (2.1) loses its stability at $r=3$ by generating a period-2 limit cycle and thereby breaks time translational invariance. From the bifurcating limit cycle ($n\rightarrow\infty$) of

$$d_n = x_{n+1}-x_n \rightarrow (-1)^n d^* \tag{4.1}$$

one identifies the value of the order parameter above the threshold $r=3$ as

$$d^* = \pm\frac{1}{r}\sqrt{(r-3)(r+1)} . \tag{4.2}$$

Its magnitude is the separation of the pitchfork branches, which grows initially with the square root of the control parameter's distance above threshold. The phase of the order parameter depends on the initial condition x_0: at fixed time n ($\rightarrow\infty$) either $d_n=+|d^*|$ or $d_n=-|d^*|$.

In the presence of small noise the branches of the pitchfork bifurcation are broadened.[1,3] In Fig. 5(a) we show as a function of r all positions x_n between time $n=1000$ and 1150 for uncorrelated noise ($\Delta=0.01$) with the box-shaped stationary distribution (3.5b) of amplitudes.

Note that the pitchfork topology is still visible in Fig. 5(a). In fact, small noise perturbs the dynamics of d_n only slightly if r is sufficiently above threshold so that the overlap between the broadened pitchfork branches is small—$|d_n|$ is still roughly of size $|d^*|$—and, more importantly, the sequence of signs $d_n/|d_n|$ is still (mostly) alternating as for $\Delta=0$. This is the situation where the

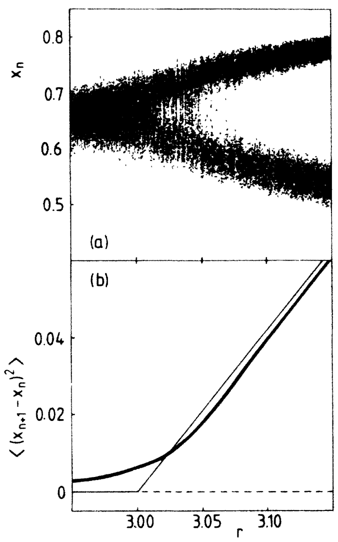

FIG. 5. The first pitchfork bifurcation in the presence of uncorrelated additive noise with a box-shaped stationary distribution and $\Delta=0.01$. (a) shows as a function of r the resulting positions x_{1000} up to x_{1150}. The thick solid line in (b) is the square of the stationary order parameter $d^2=\langle (x_{n+1}-x_n)^2 \rangle$ for the above noise. Thin lines show the bifurcation diagram of d^2 in the absence of noise, $\Delta=0$.

internally generated period-2 dynamics of the unforced map dominates the influence of the external noisy forces. With d_n and similarly x_n still being strongly correlated with the initial value d_0 or x_0, numerically determined ensemble averages over different noise realizations as well as time averages depend strongly on the initial value.

On the other hand, immediately above threshold where the smeared pitchfork branches still overlap, the internal period-2 dynamics is disrupted more effectively by the external forcing. (A mechanical analogue might consist of a double-well potential with the minima becoming shallow for $r \to 3$ so that external forces become more and more successful in disrupting an internally generated periodic motion between the wells. The acousto-optical bistable system studied by Vallée *et al.*[11] or the driven electrical circuit of Perez and Jeffries[18] seem to display these characteristics.)

The appropriate order parameter describing the time translational symmetry-breaking bifurcation in the presence of noise is[19]

$$d=(\langle d_n^2 \rangle)^{1/2}=[\langle (x_{n+1}-x_n)^2 \rangle]^{1/2} . \tag{4.3}$$

We have evaluated d by ensemble averaging over up to 30 000 realizations of noise histories as well as by time averaging for various initial conditions and control parameters. We found that the results were the same, that $\langle d_n^2 \rangle$ becomes rapidly stationary, and that it is everywhere practically independent of the initial condition. This, by the way, is not at all the case for $\langle d_n \rangle$, as discussed already if r is sufficiently far above threshold. [There, ensemble averages, e.g., yield $\langle d_{2n} \rangle \simeq |d^*|$ ($\simeq -|d^*|$) if x_0 is close to the lower (upper) branch of the unperturbed pitchfork where $d_0>0$ ($d_0<0$).]

In Fig. 5(b) we compare d^2 (thick solid line) for the noise that generated the fluctuating x_n shown in Fig. 5(a) with the square of the order parameter in the absence of noise (thin solid lines). The rounded imperfect bifurcation shows that in the presence of noise, time translational symmetry is broken everywhere with $d(r,\Delta)$ being finite for all r. For the large-r values in Fig. 5(b), noise restores partly time translational invariance in comparison with the unforced system, i.e., $d<|d^*|$.

At smaller r, on the other hand, noise enhances or even induces the symmetry breaking. For example, right at $r=3$ we found the order parameter $d(r=3,\Delta)$ to grow proportional to $\sqrt{|\Delta|}$ for small Δ. In particular, for uncorrelated noise with the box-shaped stationary distribution (3.5b), the rounding of the perfect $\Delta=0$ bifurcation grows as

$$d^2(r=3,\Delta)\simeq 0.62\,|\Delta| . \tag{4.4}$$

The theoretical problems associated with this nonanalytical small-Δ behavior are discussed in Sec. IV B.

B. Theoretical attempts

In Secs. III C and III D we found that the small-Δ expansion of $\langle x \rangle$ and $\langle (\delta x)^2 \rangle$ breaks down near the first period-doubling bifurcation of the unperturbed map at $r=3$. There the expansion coefficients of the lowest nontrivial order Δ^2 diverge $\sim (3-r)^{-1}$ for both moments and for both types of forcing, additive [cf. Eqs. (3.11a) and (3.15)] as well as multiplicative [cf. Eqs. (3.29)]. The reason for this divergence is the nonanalytical growth $\sim |\Delta|$ of these two moments at $r=3$. We did numerical simulations with our standard uncorrelated noise with a box-shaped stationary distribution and found in the amplitude range $5\times 10^{-4} \lesssim \Delta \lesssim 10^{-2}$ the following power laws for the first four stationary moments at $r=3$:

$$\langle x \rangle \simeq \tfrac{2}{3}-0.24\,|\Delta| , \tag{4.5a}$$

$$\langle (\delta x)^2 \rangle \simeq 0.16\,|\Delta| , \tag{4.5b}$$

$$\langle (\delta x)^3 \rangle \simeq -0.12\Delta^2 , \tag{4.5c}$$

$$\langle (\delta x)^4 \rangle \simeq 0.06\Delta^2 . \tag{4.5d}$$

They give rise to the linear growth (4.4) of

$$d^2=\langle (x_{n+1}-x_n)^2 \rangle = \langle (r-1-x_n)^2 x_n^2 \rangle + \Delta^2 \tag{4.6}$$

at $r=3$ and small Δ. With increasing distance $|3-r|$, however, there is a crossover of the Δ dependence of (4.4)—(4.6) to an analytical behavior.

A perturbation theory with a finite-order Taylor expansion in Δ cannot reproduce (4.4) and (4.5). Thus one is led to investigate as a theoretical alternative approximations to the infinite hierarchy of equations of motion coupling moments of increasing order with increasing complexity. We write down only the equations for the first three moments,

$$u_n := \langle x_n \rangle, \quad v_n := \langle (\delta x_n)^2 \rangle , \quad w_n := \langle (\delta x_n)^3 \rangle , \tag{4.7a}$$

for the map (2.4) with additive perturbations by uncorrelated forces:

$$u_{n+1}=r(u_n-u_n^2-v_n)\ , \tag{4.7b}$$

$$v_{n+1}=\Delta^2+r^2[(a_n^2-v_n)v_n-2a_nw_n+\langle(\delta x_n)^4\rangle]\ , \tag{4.7c}$$

$$w_{n+1}=r^3\{a_n(a_n^2-6v_n)w_n+3a_n^2[v_n^2-\langle(\delta x_n)^4\rangle] +3a_n\langle(\delta x_n)^5\rangle+3v_n\langle(\delta x_n)^4\rangle-2v_n^3 -\langle(\delta x_n)^6\rangle\}\ . \tag{4.7d}$$

Here we used the abbreviation

$$a_n=1-2u_n\ . \tag{4.7e}$$

In analogy to the "quasinormal" approximation in the statistical theory of turbulence,[20] the "closure"

$$\langle(\delta x_n)^4\rangle=3v_n^2,\ \langle(\delta x_n)^5\rangle=0,\quad \langle(\delta x_n)^6\rangle=15v_n^3 \tag{4.8}$$

suggests itself as a simple but *ad hoc* approximation. Comparing (4.5d) with (4.5b) one sees that the Gaussian approximation to $\langle(\delta x_n)^4\rangle$ is only off by about 30% at $r=3$ and small Δ. Moreover, it monotonously improves there with increasing Δ.

Note that in view of (4.5c) it is inconsistent, at least at the bifurcation threshold $r=3$, to neglect w_n in (4.7c) and to truncate the moment hierarchy at the v level. That has been done recently by Napiórkowski.[6] [This author investigated the system $y_{n+1}=1-\mu y_n+\tilde{\Delta}\tilde{\xi}_n$ which can be mapped onto (2.4) by $\mu=r(r-2)/4$, $\tilde{\Delta}=4\Delta/(r-2)$, $y=4(x-1/2)/(r-2)$.] But even his truncated version [$w=0$; $\langle(\delta x)^4\rangle-v^2=0$] of (4.7b) and (4.7c) yields at $r=3$ a fixed point $v\sim|\Delta|$ that is inconsistent with his assumption $v\sim\Delta^2$.

With the approximation (4.8) the infinite system of equations of motion for the moments is "closed" to a nonlinear discrete three-dimensional map in (u,v,w) space. However, it is very important to keep in mind that the dynamics generated by this map [(4.7) and (4.8)] is constrained by a restriction of the initial values (u_0,v_0,w_0) that is imposed by the basic equation of motion (2.4): If the forces $[\xi_n]$ start acting upon the system at time 1 with ξ_0, then x_0 is not a fluctuating quantity (unless one uses randomly chosen initial conditions which we shall not do) and thus $v_0=0=w_0$. Therefore, of the various fixed points, cycles, and so forth of the $3d$ map [(4.7) and (4.8)], one has to select just the relevant ones, i.e., those which can be reached from start values $(u_0,0,0)$ on the u axis. The others are irrelevant for the (approximate) description of the statistical dynamics of the original system (2.4).

However, even with the constraint the dynamical behavior of (4.7) and (4.8) is very rich, in particular for $r>3$. Since we feel that many of these partly unusual higher bifurcations of (4.7) and (4.8) are artifacts of the quasinormal truncation of the infinite moment hierarchy, we shall concentrate in the following on the small-Δ limit in the immediate vicinity of $r=3$.

But even in this narrow parameter range where we found numerically only fixed points, problems caused by the complexity of the map and by the constraint on the initial values did not allow a full analytical treatment: To evaluate all fixed points of (4.7) and (4.8) analytically requires finding the zeros of a polynomial of degree seven. However, we could determine analytically those fixed points that grow for finite but small Δ out of the relevant ones, (0,0,0) and $(1-1/r,0,0)$, at $\Delta=0$ (cf. below). But, having done so, we were unable to incorporate into their stability analysis the restriction imposed by the constraint to only those deviations from the fixed points that lie on trajectories starting from $(u_0,0,0)$. We therefore checked numerically to which of the analytically determined fixed points the initial values $(u_0,0,0)$ were attracted.

At $r=3$, e.g., one finds to lowest nontrivial order in $|\Delta|$ the fixed point

$$(u,v,w)^*\simeq\left[\frac{2}{3}-\frac{|\Delta|}{2\sqrt{6}},\frac{|\Delta|}{3\sqrt{6}},-\frac{\Delta^2}{6}\right] \tag{4.9}$$

which grows out from $(\frac{2}{3},0,0)$ to be stable in the above described sense. The other six fixed points are irrelevant and will not be discussed further. The coefficients of the Δ powers in (4.9) agree within about 25% with the values in (4.5a)–(4.5c) that were determined from a numerical simulation of (2.4). The stationary order parameter (4.6) corresponding to the fixed point (4.9) of the $3d$ map is given for small Δ by

$$d^2(r=3,\Delta)\simeq\frac{4}{3\sqrt{6}}|\Delta|\ . \tag{4.10}$$

Also, that compares rather well with the numerically determined linear growth (4.4) of the squared order parameter; the slope is off by about 10%.

These small-Δ results obtained at $r=3$ from the quasinormal approximation to the moment hierarchy are satisfactory. But we do not think that a pursuit of this approach, e.g., to higher r and/or larger Δ, is promising. The problems involved in (4.7) and (4.8) or in an extension thereof do not seem to be smaller than those arising in a more direct description of the original system (2.4).

V. CONCLUSION

We have investigated the statistical dynamics of the response of the logistic map towards additively or multiplicatively coupled time-dependent fluctuating forces $\Delta\xi_n$ for control parameters in the range of the first transcritical and the first pitchfork bifurcation. Our main results follow.

(1) Additive noise destroys the transcritical bifurcation at $r=1$. There, any additive forcing with zero mean drives the trajectory towards $-\infty$. For sufficiently large $|r-1|$ the response of the map to additive forcing is statistically stationary, and averages evaluated for small Δ via a perturbation expansion agree with numerical experiments.

(2) Multiplicative noise leaves the first bifurcation sharp with $x^*=0$ being a stationary state for any kind of parametric forcing. The simple exact formula for the stability boundary $r_c(\Delta)$ of this state does not depend on the correlation properties of the forcing—random and periodic perturbations entail the same stability threshold of $x^*=0$ if the stationary distributions $P(\xi)$ of their amplitudes are the same. Furthermore, *any* small-amplitude multiplicative forcing enhances the stability of the fixed point $x^*=0$ up to $r_c(\Delta)=1+\Delta^2/2+O(\Delta^4)$. Large noise either stabilizes or destabilizes the basic state depending

on Δ and the form of $P(\xi)$. These stability properties resemble those of the parametrically forced Duffing oscillator.

(3) For not too large distances $r-r_c$ above threshold the fluctuations of x generated by multiplicative noise are bounded and statistically stationary, provided that Δ is not too large. Moments and correlations evaluated with a small-Δ perturbation expansion agree quantitatively with numerical simulations. The supercritical growth of the order parameter close to threshold is changed by multiplicative noise from $\sim(r-r_c)$ for $\Delta=0$ to $(r-r_c)^{1/2}$ for finite Δ.

(4) The stationary distribution $W(x)$ of the map's response to any sort of small-amplitude, uncorrelated multiplicative noise is Gaussian if r is sufficiently far away from $r=2$ (where the first superstable fixed point of the unforced map is located). Right there, $W(x)$ has the same shape as $P(\xi)$. If the latter is non-Gaussian, $W(x)$ smoothly changes its shape to a Gaussian one as $|r-2|$ increases.

(5) We presented a careful investigation of the conditions under which additive and multiplicative noise cause similar statistical response behavior. For small Δ and r sufficiently beyond the above discussed threshold r_c, we derived an explicit r-dependent relation between equivalent multiplicative and additive force amplitudes that agrees with numerical experiments. In particular, for $r \lesssim 3$ our relation agrees with numerical experiments much better than the one derived previously by Crutchfield *et al.*[1]

(6) Close to $r=3$, where the first period doubling of the unforced system breaks time translational symmetry, the squared order parameter $\langle (x_{n+1}-x_n)^2\rangle$ shows a rounded imperfect bifurcation, averages vary nonanalytically with growing Δ, and the small-Δ expansion breaks down. A factorization approximation that truncates the infinite sequence of equations for increasing moments gives rise to a system of three nonlinear difference equations. Because of various problems arising in this approach we do not consider it to be satisfactory, although it reproduces the nonanalytic growth of moments and of the order parameter for small Δ quantitatively.

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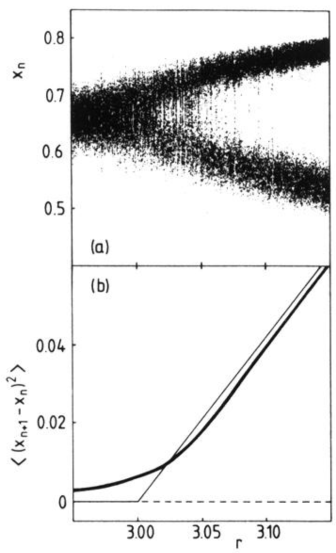

FIG. 5. The first pitchfork bifurcation in the presence of uncorrelated additive noise with a box-shaped stationary distribution and $\Delta = 0.01$. (a) shows as a function of r the resulting positions x_{1000} up to x_{1150}. The thick solid line in (b) is the square of the stationary order parameter $d^2 = \langle (x_{n+1} - x_n)^2 \rangle$ for the above noise. Thin lines show the bifurcation diagram of d^2 in the absence of noise, $\Delta = 0$.