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Quantum measurements and the standard quantum limit

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We discuss quantum measurements and find a repeated-measurement scheme to monitor the free-mass position to an arbitrary accuracy.

Sensitivity demands on gravitational-wave detection experiments lead naturally to the question of whether there are fundamental limits in the detection schemes. In the early and mid 1970's Braginsky and his colleagues[1–6] became increasingly concerned about the impact of quantum-mechanical limits on the efforts to detect gravitational waves. Quantum limits for certain measurement schemes were formulated. For example, standard quantum limits for amplitude and phase measurements of a mechanical oscillator or ultimate sensitivity limit of a resonant gravitational-wave antenna using a linear motion detector[1,7,8] and the standard quantum limit for repeated measurements of free-mass position[3,9] were studied and formulated. The last case is especially important for the laser-interferometric gravitational-wave detection experiments.[10–15] Together with the increasing sensitivity of laser interferometers, there have been considerable deliberations about the scope of validity of the standard quantum limit for repeated monitoring of free-mass position.[16–23]

In studying the effect of the quantum-mechanical properties of test objects on the accuracy of measurement of forces, Braginsky and Vorontsov[3] argued for a limit on the uncertainty Δx of the second measurement of position of a free mass m after a time τ to be

$$\Delta x \geq (\hbar\tau/m)^{1/2} . \tag{1}$$

Equation (1) is called the *standard quantum limit* (SQL) for monitoring the position of a free mass.[9] For a free mass, the evolution of position operator x in Heisenberg picture is

$$x(t)=x(0)+p(0)t/m , \tag{2}$$

where p is the momentum operator. Braginsky and Vorontsov's argument can be epitomized as follows. By the time τ of the second measurement, the variance of x (squared uncertainty) increases to

$$(\Delta x)^2(\tau)=(\Delta x)^2(0)+[(\Delta p)^2(0)/m^2]\tau^2 \geq 2\Delta x(0)\Delta p(0)\tau/m \geq \hbar\tau/m . \tag{3}$$

The last inequality is due to the uncertainty principle $\Delta x(0)\Delta p(0) \geq \frac{1}{2}\hbar$.

Yuen[16] has pointed out a flaw in this argument. From (2), the variance of x at time τ is not given by (3), but by

$$(\Delta x)^2(\tau)=(\Delta x)^2(0)+(\Delta p)^2(0)(\tau^2/m^2) +[\langle x(0)p(0)+p(0)x(0)\rangle -2\langle x(0)\rangle\langle p(0)\rangle]\tau/m . \tag{4}$$

The argument (3) implicitly assumes the correlation term [last term in (4)] vanishes. Yuen[16,19] indicated that there are Hamiltonian-realizable measurement processes that leave the free mass in a *contractive state* for which the correlation term is negative and hence, for a while, the variance of x decreases with time. By a second measurement at time τ of different apparatus setup, the uncertainty of position measurement can be smaller than $(\hbar\tau/m)^{1/2}$.

Equations (1), (3), and (4) are about the intrinsic uncertainties of the free-mass wave function. In a measurement process, measurement uncertainty is also crucial. This point is emphasized by various authors[20–24] after the appearance of Yuen's paper.[16] Believing that the flaw discovered by Yuen[16] lies in the argument, not in the SQL, Caves[21] sharpened the statement of the SQL, gave a new, heuristic argument incorporating measurement error for the SQL, and analyzed a linear measurement model that supports the heuristic argument. He considered Yuen's first measurement as a preparation procedure for the second and the two should be regarded as a single measurement. Caves emphasized on *repeated identical measurement* and sharpened the statement of the SQL as follows. Let a free mass m undergo unitary evolution during the time τ between two measurements of its position x, made with identical measuring apparatuses; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$.

One important question implicitly and/or explicitly in these deliberations is what is the definition of a measurement of a certain variable A. Liberally, *any process that can give us some information about the variable A can be considered a measurement of A*.[23] After the discussions of measurements and uncertainties by Heisenberg[25] and Bohr,[26] von Neumann[27] formulated a system of ideal quantum measurements systematically. These ideal (or von Neumann) measurements are tied directly to the probabilistic interpretation of quantum mechanics. After an ideal measurement of A, the system with original wave function $|\psi\rangle$ will be left in an eigenstate $|A\rangle$ of A and the probability of achieving this is $|\langle\psi|A\rangle|^2$ or

$|\psi(A,\ldots,)|^2$. Hence the limits set by uncertainty principles have a direct bearing on von Neumann measurements. Although von Neumann measurements are usually the only measurements treated in the discussion of measurements in a usual quantum-mechanics textbook or a usual quantum-mechanical course, many people have felt that they are too restricted. In fact, besides a few examples such as the Stern-Gerlach experiments, one seldom finds von Neumann measurements directly on an observable studied. Most informations in atomic, nuclear, particle, and solid state physics are not obtained by von Neumann measurements in the first few stages of the settings. The last stages are still usually described by von Neumann measurements. Systematic works beyond von Neumann measurements are very much needed in precision measurements and measurements of weak forces or signals.

In 1965, Arthurs and Kelly[28] formulated an important model of a simultaneous measurement of a pair of conjugate observables. In this model, a pair of conjugate variables p and q are coupled to meter variables P_x and P_y linearly with the following interaction Hamiltonian:

$$H_{\text{int}}=K(qP_x+pP_y)\,, \tag{5}$$

where p,q are the position and momentum to be measured and P_x,P_y are the momenta of the two single degrees of freedom of the meter.[29] The meter is initially prepared in the appropriate Gaussian states $M(x),N(y)$ where

$$\begin{aligned} M(x)&=(2/\pi b)^{1/4}\exp(-x^2/b)\,,\\ N(y)&=(2b/\pi)^{1/4}\exp(-by^2)\,, \end{aligned} \tag{6}$$

and b is the "balance." Arthurs and Kelly considered the case of real b; but it can be readily generalized[22,30] to complex b with $b=\bar{b}/1+i\epsilon$ (b,ϵ real). Here we consider this generalized Arthurs-Kelly scheme. The coupling constant K is supposed to be large enough so that an impulse approximation can be made. At time $t=1/K$ after the interaction, the meter positions x and y are measured in a von Neumann way to have the values x_m and y_m. Arthurs and Kelly showed that the expected value of x is equal to the expected value of q before interaction and that the expected value of y is equal to the expected value of p before the interaction. The variances of x and y are related to the variances of q and p before the interaction by

$$(\Delta x)^2=(\Delta q)^2+\bar{b}/2,\ \ (\Delta y)^2=(\Delta p)^2+(1/2|b|)\,. \tag{7}$$

From (7) and the uncertainty principle $\Delta q\,\Delta p\geq\frac{1}{2}$, one can readily derive

$$\Delta x\Delta y\geq 1 \tag{8}$$

as the proper uncertainty relation for this kind of joint measurement. Moreover, after the measurement, the state of the system is given by

$$\psi(t)=(1/\pi b)^{1/4}\exp[-(1/2b)(q-x_m)^2+iqy_m]\,. \tag{9}$$

Therefore this measurement is complete, in that the state of the system after the measurement is dependent only on the meter readings and not otherwise on the state of the system before measurement. The Arthurs-Kelly model can be generalized to more than one pair of conjugate variables.[9] Thus, by delaying the von Neumann measurement to the second or further stage, more general measurements can be included. Yuen and colleagues[30] have generalized ideal measurements to include an overcomplete set of states as a result of the measurements. This generalization includes the Arthurs-Kelly model as a special case.

In all the above generalizations, the state of the system after measurement is in an eigenstate or approximate eigenstate of the variable to be measured. Gordon and Louisell[31] considered the case that after measurement, the system is in a different state. In their formalism, a quantum measurement is described by a set of operators $|\psi^S\rangle\langle\psi^M|$ such that $\langle\psi^M|\rho|\psi^M\rangle$ gives the measurement probability in the state described by the density operator ρ while $|\psi^S\rangle$ is the state after the measurement. The ordinary position measurement is then described by $|x\rangle\langle x|$. Although after most measurements in microphysics, the state of the system is left in a different state, it is not clear that, in general, the Gordon-Louisell measurement can be realized by an interaction Hamiltonian. Yuen[16] has used the contractive-state Gordon-Louisell measurement to construct specific examples which violate the limit (1) in the second measurement.[32] Whether these measurements are Hamiltonian-realizable remains to be solved.

After the above discussions of quantum measurements and the recent deliberations of SQL, we construct a repeated measurement scheme to monitor the free-mass position to an arbitrary accuracy. Consider the system (x,p) to be measured, coupled linearly to a meter of four degrees of freedom with the following interaction Hamiltonian:

$$H_{\text{int}}(t)=K_1(t)(xP_1+pP_2)+K_2(t)(xP_3+pP_4)\,, \tag{10}$$

where P_1, P_2, P_3, P_4 are the meter momenta with corresponding position variables X_1, X_2, X_3, X_4. At $t=0$ we turn the meter coupling on. At $t=t_1\ll\tau$ we turn the meter coupling off. We assume $K_1(t)$, $K_2(t)$ are large enough in certain intervals of $(0,t_1)$ and t_1 is small enough so we can use the impulse approximation. Solving the variables in the Heisenberg picture between $t=0$ and $t=t_1$, we have

$$\begin{aligned} x(t)&=x(0)+C_1(t)P_2(0)+C_2(t)P_4(0)\,,\\ p(t)&=p(0)-C_1(t)P_1(0)-C_2(t)P_3(0)\,,\\ X_1(t)&=X_1(0)+C_1(t)x(0)+d_{11}(t)P_2(0)+d_{12}(t)P_4(0)\,, \end{aligned} \tag{11}$$

with similar formulas for $X_2(t)$, $X_3(t)$, $X_4(t)$. Here

$$\begin{aligned} C_1(t)&=\int_0^t K_1(t')dt'\,,\\ C_2(t)&=\int_0^t K_2(t')dt'\,,\\ d_{11}(t)&=\int_0^t dt''\int_0^{t''}dt'K_1(t'')K_1(t')\,,\\ d_{12}(t)&=\int_0^t dt''\int_0^{t''}dt'K_1(t'')K_2(t')\,. \end{aligned} \tag{12}$$

Suppose at $t=0$, the x,X_1X_2,X_3,X_4 degrees of freedom are uncorrelated. Then at $t=t_1$, the expectation values and variances of X's can be calculated readily

$$\begin{aligned}\langle X_1(t_1)\rangle &= C_1(t_1)\langle x(0)\rangle + \langle X_1(0)\rangle \\ &\quad + d_{11}(t_1)\langle P_2(0)\rangle + d_{12}(t_1)\langle P_4(0)\rangle , \\ \Delta X_1^2(t_1) &= C_1^2(t_1)\Delta x^2(0) + \Delta X_1^2(0) + d_{11}^2(t_1)\Delta P_2^2(0) \\ &\quad + d_{12}^2(t_1)\Delta P_4^2(0)\end{aligned} \tag{13}$$

with similar formulas for X_2,X_3,X_4.

Now we specialized to the case

$$K_1(t) = \begin{cases} 2/t_1, & 0 \le t < t_1/2 \\ 0, & t_1/2 < t \le t_1, \end{cases} \qquad K_2(t) = \begin{cases} 0, & 0 \le t < t_1/2 \\ 2/t_1, & t_1/2 < t \le t_1 \end{cases} \tag{14}$$

and prepare the meter in a Gaussian state so that at $t=0$, the total wave function is

$$\phi(x,X_1,X_2,X_3,X_4,t=0) = F(x)(2/\pi)(1+\epsilon^2)^{-1/4} e^{-X_1^2/b_1} e^{-b_1 X_2^2} e^{-X_3^2/b_2} e^{-b_2 X_4^2} , \tag{15}$$

where $b_1 > 0$ and $b_2 = b/(1+i\epsilon)$ with $b>0$. For $\epsilon > 0$, X_3 degree of freedom is in a contractive state at $t=0$. Between $t=0$ and $t=t_1/2$, X_3,X_4,P_3,P_4 are not coupled to either x,p or X_1,X_2,P_1,P_2, and we are in an Arthurs-Kelly scheme for x,p and X_1,X_2,P_1,P_2. The total wave function at $t=t_1/2$ is[28]

$$\phi(x,X_1,X_2,X_3,X_4,t=t_1/2) = \left[\frac{1}{\pi b_1}\right]^{1/4} \exp\left[-\frac{1}{2b_1}(x-x_1)^2 + ixX_2\right] F_1(X_1,X_2)\left[\frac{2}{\pi}\right]^{1/2}(1+\epsilon^2)^{-1/4} e^{-X_3^2/b_2} e^{-b_2 X_4^2} \tag{16}$$

where

$$F_1(X_1,X_2) = \left[\frac{1}{4\pi^3 b_1}\right]^{1/4} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2b_1}(X_1-u)^2\right] e^{-iuX_2} F(u)\,du . \tag{17}$$

Between $t=t_1/2$ and $t=t_1$, we are in a generalized Arthurs-Kelly scheme for x,p and X_3,X_4,P_3,P_4. The total wave function at $t=t_1$ can be solved as

$$\phi(x,X_1,X_2,X_3,X_4,t=t_1) = \frac{1}{(1+\epsilon^2)^{1/4}}\left[\frac{1}{\pi b_2}\right]^{1/4} \exp\left[-\frac{1}{2b_2}(x-X_3)^2 + ixX_4\right] F_1(X_1,X_2) F_2(X_1,X_2,X_3,X_4) , \tag{18}$$

where

$$\begin{aligned} F_2(X_1,X_2,X_3,X_4) = {} & \left[\frac{1}{4\pi^3 b_2}\right]^{1/4} \left[\frac{1}{1+\epsilon^2}\right]^{1/4} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2b_2}(X_3-u)^2\right] e^{-iuX_4} \\ & \times \left[\frac{1}{\pi b_1}\right]^{1/4} \exp\left[-\frac{1}{2b_1}(u-X_1)^2 + iuX_2\right] du . \end{aligned} \tag{19}$$

At $t=t_1$, we make a von Neumann measurement of X_1,X_2,X_3,X_4. The outcome is $X_{1m},X_{2m},X_{3m},X_{4m}$. The system wave function is reduced to

$$\bar{F}(x,t=t_1) = \frac{1}{(1+\epsilon^2)^{1/4}}\left[\frac{1}{\pi b_2}\right]^{1/4} \times \exp\left[-\frac{1}{2b_2}(x-X_{3m}) + ixX_{4m}\right] . \tag{20}$$

The expectation values and variances of X_1,X_2,X_3,X_4 are obtained from (11)–(14) as

$$\begin{aligned} \langle X_1(t_1)\rangle &= \langle X_3(t_1)\rangle = \langle x(0)\rangle , \\ \langle X_2(t_1)\rangle &= \langle X_4(t_1)\rangle = \langle p(0)\rangle , \\ \Delta X_1^2 &= \Delta x^2(0) + \frac{b_1}{2} , \\ \Delta X_2^2 &= \Delta p^2(0) + \frac{1}{2b_1} , \\ \Delta X_3^2 &= \Delta x^2(0) + \frac{b}{2} + b_1 , \\ \Delta X_4^2 &= \Delta p^2(0) + \frac{1}{2b} + \frac{1}{b_1} . \end{aligned} \tag{21}$$

Thus we see both X_{1m} and X_{3m} measure $x(0)$; X_{1m} with less uncertainty and X_{3m} with more uncertainty. The resulting wave function (20) of the system depends only on the readings of X_{3m} and X_{4m} and can be left in a contractive state.

Given b_1, choose $\epsilon=\tau/mb_1$ and $b=2b_1(1+\tau^2/m^2b_1^2)$. At $t=t_1$, the system wave function is given by (15) with F replaced by $\bar{F}$. Between t_1 and $\tau(t_1 \ll \tau)$, the wave function evolves freely. At $t=\tau$, the system becomes a minimum-uncertainty state with $\Delta x^2(\tau)=b/4(1+\epsilon^2)=b_1/2$. Thus from Eq. (21), for the second and subsequent measurements $\Delta X_1^2=b_1$. Thus we can measure the position before each measurement to an arbitrary precision b.

Whether this measurement is a measurement of position in the sharpened SQL statement,[21] we have to leave it to Caves to decipher. What I want to stress here is that the deliberations of SQL do lead us to more of an understanding of the structures of quantum measurements, and that we should consider and investigate more general quantum measurements more explicitly.

Discussions. Since we anticipate evolution between the first and the second measurement, the above scheme leave the free mass in a contractive state [mostly by the interaction with (X_3,P_3)] so that just before the second measurement, Δx is small and X_{1m} will give a precise value of x. We can also use the interaction $H_{\rm int}=p^2X_0$ to *evolve back* the free mass with X_0 having a large classical value $\langle X_0\rangle=-\tau/mt_1$ in the first half of the measurement. During the second half of the measurement we can use the Arthurs-Kelly method to measure x as precisely as we want.

Either the method (10) with (15) or the method mentioned in the last paragraph needs more degrees of freedom of the meter than methods considered before. For von Neumann measurements, the largeness of the apparatus puts a precision limit on the measurement of an operator which does not commute with the conserved quantities.[33–35] This and the above examples lead to the question of whether the largeness of the apparatus plays any limiting role to the precision of a single or repeated measurements of position or other operators in the above class.

If there is a classical force acting on the free mass between two measurements, it is only to change the mean position and mean momentum of this free mass, not the dispersive properties of it. Hence only the mean value of X_1,X_2,X_3,X_4 is going to be changed. This force is readily detectable if it affects the free mass more than the resolution of X_1. Therefore any small force can be monitored as closely as one wishes by the above schemes. At present, a squeezed state of light is close to being produced.[36] If, somehow, a scheme of using squeezed states of light to swing mirrors to a contractive state in an interferometric configuration and to measure the position difference of the mirrors to arbitrary precision can be found, it would be very interesting.

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