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Entrainment regions for periodically forced oscillators

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A natural oscillator subjected to periodic forcing with adjustable frequency and amplitude may behave aperiodically or be entrained to oscillate with an integer multiple of the forcing period. Entrainments occur in subsets of the frequency-amplitude plane and are well understood for very small forcing amplitude. We investigate the mechanisms by which entrainments are terminated at moderate amplitudes and support our theory by numerical experiments on a model chemical oscillator.

Models involving forced oscillators arise in many fields of science and engineering. Here we study a periodically forced, autonomously oscillating chemical reactor. The dynamics of such systems are known to be very rich in phenomena, but realistic models are often far too complex to analyze either analytically or numerically. To understand and classify some of the basic phenomena it is therefore useful to examine simplified models (perhaps even caricatures) which are designed to isolate some particular aspect of the behavior. In this spirit, Kai and Tomita[1] (KT) studied entrainment in a periodically forced Brusselator described by the equations

$$\dot{X} = A + X^2 Y - BX - X + a \cos\omega t, \quad \dot{Y} = BX - X^2 Y \ , \tag{1}$$

where a and ω are parameters. The constants are taken to be $A = 0.4$ and $B = 1.2$ so that the unforced Brusselator ($a = 0$) has a limit cycle with natural frequency $\omega_0 = 0.375\,037\,5$. This limit cycle is an asymptotically stable periodic orbit enclosing a source.

For $a > 0$ it is natural to investigate the behavior of (1) by studying the *stroboscopic map* induced by sampling solutions of (1) at integer multiples of the period $2\pi/\omega$ of the forcing term. A periodic point for the stroboscopic map corresponds to an *entrainment* of the underlying oscillator by the forcing term. KT computed the entrainment regions in the $(\omega/\omega_0, a)$ parameter plane for periods ≤ 4 and their results are shown in Fig. 1. For $a = 0$ the stroboscopic map restricted to the limit cycle is conjugate to a rigid rotation of a circle and it follows from the standard theory described below that there is an entrainment region, also called an Arnol'd horn or a resonance region, entering the first quadrant from every rational point on the ω/ω_0 axis.

A rather curious feature of the KT calculations is the apparent smooth closing of the two- , three- , and four-period entrainment regions. We investigate this phenomenon both numerically and mathematically. Our computations indicate that the three-period resonance regions do indeed close off smoothly, but that they each contain in their interior a point of Hopf bifurcation. Moreover, our computations indicate that at least some of the four- and five-period regions do not close off smoothly, but seem to terminate in a cusp at which a Hopf bifurcation occurs. We give an explanation of these phenomena based on mathematical arguments and state some conjectures inspired by the computations and the arguments.

The stroboscopic maps form a two-parameter family of diffeomorphisms $x \to f_\mu(x)$, where $x \in \mathbb{R}^2$, $\mu = (\sigma, a) \in \mathbb{R}^2$, and f_μ is a C^∞ function of x and μ. For $a = 0$, $f_{(\sigma,0)}$ has a hyperbolic attracting C^∞ invariant circle $S_{(\sigma,0)}$ on which the map is C^∞ conjugate to a rigid rotation with rotation number $\rho = 1/\sigma$. We choose the reciprocal of the rotation number as a parameter to conform with the notation of KT. Standard perturbation theory for invariant manifolds[2] implies that, for small a, the map has a hyperbolic attracting invariant circle S_μ which is smooth but not necessarily C^∞.

The map f_μ restricted to S_μ yields a two-parameter family of circle diffeomorphisms. We let σ vary in a neighborhood of a rational number $\sigma_0 = q/p$ and seek the *Arnol'd horn* corresponding to rotation number $\rho_0 = p/q$, i.e., we seek the set of points $\mu = (\sigma, a)$ for which there are periodic points on S_μ with period q and rotation number ρ_0. Under generically satisfied hypotheses on a finite number of Fourier coefficients of $f_{(\sigma,0)}$,[3] in the neighborhood of the σ axis the ρ_0-Arnol'd horn is a wedge-shaped region emanating from $(\sigma_0, 0)$, as illustrated in Figs. 1, 2(a), and 3. Under the same generic hypotheses and for μ in the interior of the horn with a sufficiently small,[4] S_μ contains exactly two periodic orbits with period q, one consisting of sinks which we label $s_1, s_2, \ldots, s_q$ and the other consisting of saddles la-

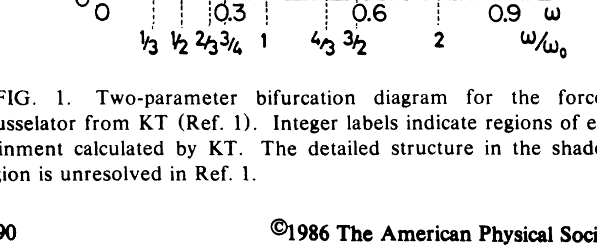

FIG. 1. Two-parameter bifurcation diagram for the forced Brusselator from KT (Ref. 1). Integer labels indicate regions of entrainment calculated by KT. The detailed structure in the shaded region is unresolved in Ref. 1.

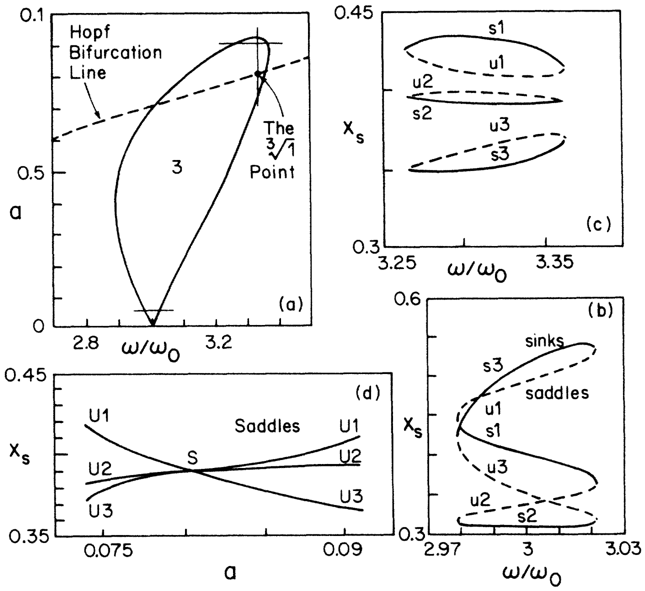

FIG. 2. (a) The Arnol'd horn for $\omega/\omega_0=\frac{3}{1}$. The Hopf bifurcation line is indicated by — — —. (b) One-parameter bifurcation diagram across the horn at $a=0.005$. (c) One-parameter bifurcation diagram across the horn at $a=0.09$. (d) One-parameter bifurcation diagram for a vertical line through the bifurcation locus at the $1^{1/3}$ bifurcation point S. At S the period-3 saddles collapse on a fixed point.

beled $u_1, u_2, \ldots, u_q$. Furthermore, these sinks and saddles come together along the boundaries of the horn to form *saddle nodes*, i.e., fixed or periodic points with exactly one eigenvalue equal to 1. A generic condition on the second-order term in the normal form together with a condition on the dependence of the map on the parameters ensures that the saddle nodes occur along curves in the parameter plane. In particular, for each sufficiently small $a_0>0$ the intersection of the ρ_0-Arnol'd horn with the line $a=a_0$ is a σ interval which we denote by I_0.

The saddle-node bifurcations which occur at the end points of I_0 induce pairings of the saddles and sinks. Suppose, for example, that we have labeled the saddles and sinks so that the bifurcation at the left-hand end point of I_0 involves the pairs $(s_1,u_1), (s_2,u_2), \ldots, (s_q,u_q)$. Generically, the pairing induced by the bifurcation at the right-hand end point of I_0 must be different.[4] In particular, the labels can always be chosen so that the pairing at the right-hand end point is $(s_1,u_2), (s_2,u_3), \ldots, (s_{q-1},u_q), (s_q,u_1)$. The fact that the saddle-node bifurcations occur between different pairs of saddles and sinks along the two different sides of the Arnol'd horn is of critical importance in the arguments given below. The computed structure of the saddles and sinks and their interactions with the saddle nodes is shown in Fig. 2(b) for the KT model with $q=3$ and $a_0=0.005$.

We computed[5] (by implementing contraction mappings for boundary value problems and not through initial value problem simulation) the resonance horns for $\omega/\omega_0=\frac{3}{1}, \frac{3}{2}$, and $\frac{3}{4}$ in the KT model. These regions are shown in Figs. 2(a) and 3. Except for the tip (i.e., the vertex on the ω/ω_0 axis) each of the horns appears to be bounded by smooth curves corresponding to elementary saddle-node bifurcations. Moreover, we found no evidence of any saddle-node bifurcations involving the period-three orbits occurring in the interior of the horns.

The relationships between the saddle nodes for the Arnol'd horn corresponding to $\omega/\omega_0=\frac{3}{1}$ are shown in Fig. 2(b) for $a=0.005$ and in Fig. 2(c) for $a=0.09$. The observed changes in the connections can occur if there is a point of Hopf bifurcation with a third-order resonance located in the interior of the horn. Numerically we found a curve of Hopf bifurcation running through each 3 horn as shown in Figs. 2(a) and 3. On each of these curves there is a point where the eigenvalues of the corresponding fixed point are cube roots of unity.

The structure of the Hopf bifurcation for the third-order resonance is given by Arnol'd.[4] For all parameter values in a punctured neighborhood of the bifurcation point there are saddle points forming a period-three orbit. These saddle points coalesce at the bifurcation point. The structure is illustrated in Fig. 2(d).

Consider the set

$$\Gamma=\{(x,\mu)\in\mathbb{R}^4: f_\mu^q(x)=x\} \ .$$

Let A_0 denote the Arnol'd horn emanating from $(\sigma_0,0)$

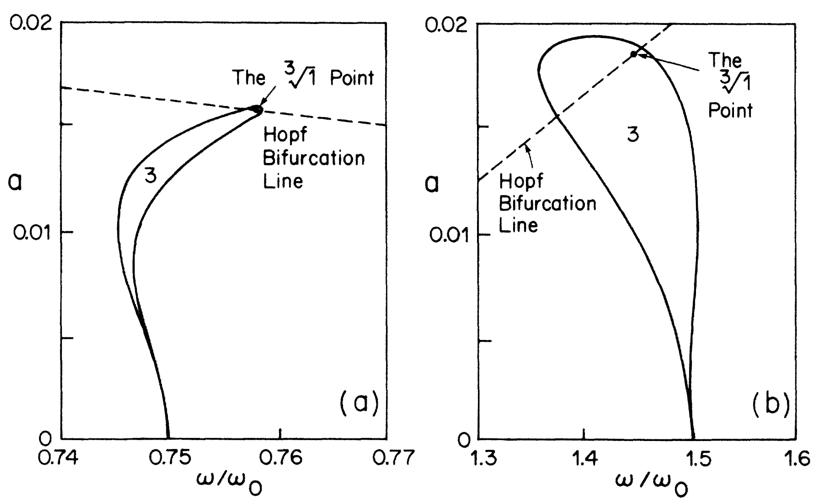

FIG. 3. The Arnol'd horns for (a) $\omega/\omega_0=\frac{3}{4}$ and (b) $\omega/\omega_0=\frac{3}{2}$.

and let Γ_0 denote the component of Γ containing the periodic orbit corresponding to A_0. Note that A_0 is just the projection of Γ_0 onto the parameter plane, i.e.,

$$A_0 = \{\mu \in \mathbb{R}^2 : (x,\mu) \in \Gamma_0\} \ .$$

An Arnol'd horn A_0 is said to be a *simple disklike resonance region* if (i) A_0 is bounded and topologically a closed disk, (ii) except for the tip, the boundary of A_0 consists only of elementary saddle-node bifurcation points, and (iii) there are no other saddle-node bifurcation points in Γ_0. As we observed above, the three-period Arnol'd horns in the KT model appear to be simple disklike resonance regions.

Our numerical observations can be explained by the following conjecture which could be promoted to a theorem if the hypotheses were clearly stated and if the details of the proof were provided. We shall return to this matter on another occasion.

Conjecture 1. Let A_0 be an Arnol'd horn for a generic two-parameter family. If A_0 is bounded and topologically a closed disk, and if, except for the tip, the boundary of A_0 consists only of saddle-node bifurcation points, then there must be a point $(x,\mu) \in \Gamma_0$ with μ in the interior of A_0 such that 1 is an eigenvalue of $D^q f_\mu(x)$.

We outline an indirect proof. Suppose for contradiction that, for every $(x,\mu) \in \Gamma_0$ with μ in the interior of the resonance horn A_0, the spectrum of $Df_\mu^q(x)$ does not contain 1. Then the implicit function theorem implies that the projection of Γ_0 onto the interior of A_0 is a covering map. Since by hypothesis the interior of A_0 is an open disk, this covering must consist of a family of open disks, each projecting onto the interior of A_0. Recall that, for parameter points in the interior of the horn near its tip, Γ_0 consists of exactly q sinks, which we can label $s_1, s_2, \ldots, s_q$, and q saddles, which we can label $u_1, u_2, \ldots, u_q$. Therefore, the covering must consist of exactly $2q$ disks, each of which represents a continuation of one of these saddles or sinks.

Now consider a parameter point on the left-hand boundary of A_0 near the tip. As we observed above, the saddle-node bifurcation induces a pairing of the saddles and sinks at such a parameter point and we can assume without loss of generality that it is $(s_1,u_1), (s_2,u_2), \ldots, (s_q,u_q)$. As we move the parameter point away from the tip and along the boundary, we travel all the way around the horn and return to a neighborhood of the tip on the right-hand side. Since, by hypothesis, we have encountered only saddle nodes along this path, the pairing we started with cannot change. Therefore the pairing along the right-hand boundary near the tip is also $(s_1,u_1), (s_2,u_2), \ldots, (s_q,u_q)$. However, this contradicts the fact that the pairing on the right-hand boundary must be different from the pairing on the left and completes the proof.

The argument we have sketched shows that, for simple disklike resonance regions there must be a point in the interior of the region where an eigenvalue of the periodic point is equal to 1. Since there are no points of saddle-node bifurcations in the interior of A_0, we can eliminate saddle nodes as well as some other points with co-dimension two. Namely, neither a simple cusp[6] nor a Bogdanov point[7] can occur in the region, since these both have saddle nodes occurring in every neighborhood.

For generic two-parameter families, that leaves only the possibility that $D^q f_\mu(x)$ is the identity and x is a fixed point of f_μ^k for some k which divides q. For $q=3$, the only possibility for a generic family is that x is a fixed point of f_μ and $Df_\mu(x)$ has eigenvalues which are primitive cube roots of unity. Arnol'd[4] shows that such a fixed point can occur generically in the interior of A_0 but cannot occur generically on the boundary. Thus we are led to the following conjecture which states roughly that the structure we observed in the KT model is the generic one.

Conjecture 2. Let A_0 be a three-period resonance horn for a generic two-parameter family. If A_0 is a simple disklike resonance region then there must be an (x,μ) in Γ_0 with μ in the interior of A_0, such that $f_\mu(x)=x$ and the eigenvalues of $Df_\mu(x)$ are primitive cube roots of unity.

For $q \geq 3$ the situation is not so clear. Arnol'd[4] shows that, for $q \geq 5$ the resonance region in a neighborhood of the Hopf bifurcation is generically a cusp and therefore a Hopf bifurcation cannot occur in the interior of a resonance region. However, since we want to conclude that $k=1$, the argument above generalizes only for prime q. Thus we are led to conjecture 3.

Conjecture 3. If A_0 is a q-period Arnol'd horn for a generic two-parameter family, where $q \geq 5$ is prime, then A_0 cannot be a simple disklike resonance region.

Actually it is quite reasonable to speculate that conjecture 3 holds for arbitrary $q \geq 5$. For $q=4$, Arnol'd's analysis is not definitive and it appears that a Hopf bifurcation can occur generically either as a cusp on the boundary or as an interior point of its resonance region.[4] Clearly, much more work needs to be done on these questions.

For the KT model we computed the four-period resonance regions for $\omega/\omega_0 = \frac{4}{3}$ and $\frac{4}{1}$ and the five-period region for $\omega/\omega_0 = \frac{5}{1}$. To the limit of our numerical accuracy, we found that they are not simple disklike resonance regions, but instead terminate in cusps at Hopf bifurcation points. Thus there is some question about the accuracy of the region shown in Fig. 1 for $\omega/\omega_0 = \frac{4}{3}$.

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