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Stochastic and regular motion in a four-particle system

J. Reichl and H. Büttner
Physikalisches Institut, Universität Bayreuth, D-8580 Bayreuth, Federal Republic of Germany
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Two anharmonic (harmonic) oscillators were coupled by an exponential nonlinearity. The temporal development of the phase-space distance indicates alternating regular and stochastic motion as a function of the coupling parameter and the energy.

Recently, there has been much interest in the onset of chaotic motion of nonlinear Hamiltonian systems with a few degrees of freedom.[1,2] Since it was also found that in many-particle systems nonlinear interactions may support heat currents as a sign of thermodynamiclike behavior[3,4] it seems to be important to understand the transition from regular to chaotic behavior in Hamiltonian systems. This question has a long history[5,6] and quite recently it was found for two coupled oscillators that stochastic and regular motion may alternate as a function of an internal coupling parameter.[7] In this Communication we present results for a four-particle system, namely, two anharmonically coupled nonlinear Toda oscillators, each of which represents an integrable two-particle system. For comparison we have studied two harmonically bound oscillators coupled by the same nonlinearity. The general question is how this coupling changes the integrability of the constituents.

For the classical calculation we start with the Hamiltonian

$$H_1=(p_1^2+p_2^2+p_3^2+p_4^2)/2m+A\{\cosh[b(q_1-q_2)]+\cosh[b(q_3-q_4)]+C\cosh[b(q_2-q_3)]-2-C\}\ , \tag{1}$$

which reads after a transformation to dimensionless variables:

$$\tilde{H}_1=(p_x^2+p_y^2+p_u^2-p_xp_u-p_yp_u)/2+\cosh x+\cosh y+C\cosh u-2-C\ . \tag{2}$$

The transformation is accomplished by introducing

$$\begin{aligned}&x=b(q_1-q_2),\ \ y=b(q_3-q_4),\ \ u=b(q_2-q_3),\ \ Q=m(q_1+q_2+q_3+q_4)\ ,\\&ap_x=p_1-mP,\ \ ap_y=p_1+p_2+p_3-3mP,\ \ ap_u=p_1+p_2-2mP,\ \ 4mP=p_1+p_2+p_3+p_4\ ,\end{aligned} \tag{3}$$

where the following abbreviations were used:

$$a=(Am/2)^{1/2},\ \ t'=bt(2A/m)^{1/2}\ . \tag{4}$$

This transformation also means that we may neglect the center-of-mass motion and only describe the remaining degrees of freedom. The following equations of motion are easily found:

$$\begin{aligned}\ddot{x}&=-\sinh x+C\sinh u/2\ ,\\\ddot{y}&=-\sinh y+C\sinh u/2\ ,\\\ddot{u}&=-C\sinh u+(\sinh x+\sinh y)/2\ .\end{aligned} \tag{5}$$

The dot indicates a time derivative and the parameter C describes the coupling between the two identical oscillators with relative displacement x and y, respectively. The corresponding dynamics for harmonic oscillators are given by

$$\begin{aligned}\ddot{x}&=-x+C\sinh u/2\ ,\\\ddot{y}&=-y+C\sinh u/2\ ,\\\ddot{u}&=-C\sinh u+(x+y)/2\ .\end{aligned} \tag{6}$$

It is interesting to note that in this case another "normal" coordinate is easily found in the difference coordinate $x-y$, while this is not possible for the Toda oscillator. Both systems are nonintegrable but the harmonic one has one more constant of motion: They have different degrees of nonintegrability.

Of special interest is the dynamical behavior as a function of the coupling parameter C and also of the total energy E in the system. The dynamical behavior is studied by looking at the temporal development of the phase-space distance of two nearby initial points in phase space. In a first investigation the energy E is held fixed and the coupling constant C is varied over several decades, making sure that the same initial condition is always used. The results for the Toda system are indicated in Table I. It shows those representative values of C where we found linear or exponential temporal behavior of the phase-space distance, which means that in the surroundings of these values the indicated behavior can be observed. (We do not refer to those parameter values where there is an onset of linear or exponential behavior, since these are hard to define by an integration over a finite time interval.) The comparison with the calculations of the harmonic oscillators shows some interesting differences for the higher-energy case, where this system seems to be more "integrable." The main result is the alternating behavior between stochastic and regular motion. This result was also found quite recently in work on coupled nonlinear oscillators with two degrees of freedom by Deng and Hioe,[7] although here the energy was not kept fixed while the coupling parameter was changed.

Even more interesting is the fact that for a constant coupling strength C there is also an alternating stochastic and

TABLE I. The increase of the phase-space distance with time is linear (L) or exponential (X) for various representative coupling constants C and two different energies E. The initial conditions for the momenta and coordinates are $p_x = p_y = (2E/3)^{1/2}$, $p_u = 0$, $y = u = 0$, $\cosh x = 1 + E/3$ (for the Toda oscillators, upper part), and $x = (2E/3)^{1/2}$ (for the harmonic oscillators, lower part).

C	$E=4$	$E=8$
10.000	L	X
1.000	L	L
0.700	L	X
0.600	X	X
0.500	L	X
0.300	X	X
0.195	L	X
0.100	X	X
0.085	L	X
0.080	X	X
0.077	L	X
0.074	X	X
0.070	L	X
0.030	L	X
0.010	L	L
10.000	L	L
1.000	L	L
0.500	L	L
0.200	L	L
0.113	X	L
0.108	L	L
0.105	X	L
0.102	L	L
0.100	X	L
0.090	X	L
0.070	L	L
0.050	L	X
0.040	L	X
0.010	L	L
0.001	L	L

TABLE II. Temporal development of the phase-space distance as in Table I for three different coupling constants as a function of the energy E.

E	$C=0.080$	0.084	0.090
1.00	L	L	L
2.00	X	L	L
2.25	L	X	L
2.50	X	X	X
2.75	L	L	X
3.00	X	X	X
3.25	L	X	L
3.50	L	L	X
3.75	X	L	X
4.00	X	L	X
5.00	X	X	X
6.00	X	X	X
8.00	X	X	X
10.00	X	X	X
1.0	L	L	L
2.0	L	L	L
3.0	L	L	L
4.0	L	L	X
4.5	X	X	X
5.0	X	X	X
5.5	X	X	X
6.0	X	X	L
7.0	L	L	L
8.0	L	L	L
10.0	L	L	L
15.0	L	L	L
17.0	L	L	X
20.0	X	X	X

regular behavior with increasing energy E. In Table II the results are shown for a definite initial condition. For various parameters C at least three different energy regions were found with a transition from a stochastic to a regular motion and back to a stochastic one. The same initial condition was chosen for the harmonic case and again the typical alternating behavior was found. Note, however, that there are broader energy regions with a definite characteristic motion. These results were confirmed by calculations with other initial conditions. It seems, therefore, a general fact that in systems with two or more degrees of freedom the dynamics may change from regular to stochastic and back to regular motion as a function of total energy and coupling parameter C.

Although the phase-space distance is best suited to discriminate between regular and stochastic behavior, it does not seem to be very sensitive to the differences between systems of various degrees of nonintegrability. The two coupled harmonic oscillators behave similarly to the two nonlinear Toda oscillators in this respect. Further discussions on this problem will be published soon.

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