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Intrinsic linewidth of a free-electron laser

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The intrinsic linewidth of a cw free-electron laser is calculated, and the result compared to the ordinary atomic laser. A classical (i.e., nonquantum) linewidth is obtained for a free-electron laser operating in the classical regime.

Free-electron lasers are more and more being established as a source of highly coherent light alternative to conventional atomic or molecular lasers. At first glance it would not be surprising if the two had very different features: While all conventional lasers vitally rely on the fact that the gain medium (i.e., the atoms) is a quantum-mechanical system, this is not so in the case of free-electron lasers (FEL's). All existing and almost all proposed FEL's allow for a classical treatment of the electrons for all purposes relevant to their operation. In spite of this fundamental difference the characteristics of the light emitted by conventional lasers and FEL's (viz., monochromaticity, photon statistics below and above threshold) have been found to be very similar. However, there is one problem that does not appear to have been addressed in the literature yet: This is the question of the ultimate linewidth achievable with an FEL. For the conventional laser, this limit is determined by quantum mechanics. For $\hbar \rightarrow 0$, no ultimate limit would exist. The question arises as to whether the FEL being a classical system might not be subject to such a limit or, in any event, what that limit might be.

This question may have appeared academic thus far. All FEL's to date have operated on a pulsed mode, so that the width of their frequency spectrum was determined by the finite duration of the laser pulses (typically of the order of picoseconds), and was therefore so large that "fundamental" limits were not really relevant. In the last year, however, an FEL has been operated at University of California, Santa Barbara (Ref. 1) making use of an electrostatic accelerator which produces electron pulses lasting several microseconds, which leads to laser pulses about six orders of magnitude longer than the typical ones mentioned above. Such an FEL may have a very narrow bandwidth (less than 1 MHz), and it is not ruled out that one could eventually operate a cw FEL with an electrostatic accelerator. Under these circumstances, the question of just how narrow the linewidth of an FEL can ultimately be is no longer academic. It is this question that we want to address in this Rapid Communication.

The laser linewidth can be calculated from the decay rate of the ensemble average of the laser electric field amplitude. A decay law of the form

$$\langle E(t)\rangle = \langle E(0)\rangle e^{-Dt/2} \tag{1}$$

(arising from randomization of the phase) gives a Lorentzian spectrum of full width at half maximum equal to D. This decay rate may be calculated in a quantized field picture from the decay rate of the off-diagonal elements of the field density matrix (see, e.g., Ref. 2). One has, in general,

$$\langle E(t)\rangle = \left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2} \sum_n \sqrt{n+1}\,\rho_{n,n+1}(t) \ , \tag{2}$$

and so the ansatz[2]

$$\rho_{n,n+1}(t) = e^{-Dt/2}\rho_{n,n+1}(0) \ , \tag{3}$$

gives Eq. (1). Accordingly, we will derive the intrinsic linewidth of a free-electron laser by solving for D in Eq. (3), using the evolution equations for the density matrix of the FEL. We will also use the fact, known from ordinary laser theory, that the decay rate D of the off-diagonal elements of ρ, at steady state, may be obtained from just the linear theory, provided that the number of photons at steady state is assumed to be known.

In Ref. 3 the change in the density matrix for the system of the field and a single electron was written in the form

$$\delta\rho_{ef}(T/2) = S\rho_{ef}(-T/2)S^\dagger - \rho_{ef}(-T/2) \ , \tag{4}$$

where S is the time evolution operator (this is the change over the time T that an electron interacts with the field). We shall use here, for simplicity, the single-particle formalism,[3,4] which is probably best known; the many-particle theory[5] is found to give the same results to zeroth order in the quantum recoil [see below, Eq. (8)], although one expects many-particle corrections (related, for instance, to amplified spontaneous emission) to arise in higher orders. Also to facilitate comparison with Refs. 2 and 4 we shall use the notation appropriate to the moving (Bambini-Renieri) frame. It has been shown in Ref. 6 how essentially the same description may be obtained in the laboratory frame; we shall go to this frame at the end of the calculation.

The operator S may be expanded[3] with respect to the quantum recoil, viz.,

$$S = S_0 + S_1 + \cdots \ , \tag{5}$$

where S_0 is independent, and S_1 is of first order in the quantum recoil. (Note that S_0 already contains multiphoton transitions of arbitrary order.) By the term "quantum recoil" we mean the parameter

$$\epsilon = \frac{2\hbar k^2 T}{m} \tag{6a}$$

$$= 2\pi\frac{\hbar\nu}{\gamma_0 mc^2}\left(\frac{L}{\lambda_q}\right) \ , \tag{6b}$$

where Eq. (6a) is written in terms of moving-frame quantities, and (6b) in terms of laboratory-frame quantities (see

Refs. 3 and 6, respectively; in Ref. 6, ϵ is written as qL). In Eq. (6a), k is the magnitude of the laser wave vector (equal to the wiggler wave vector in that frame), m the mass of the electron, and T the interaction time. In Eq. (6b), ν is the laser (angular) frequency, $\gamma_0 mc^2$ the energy of the electrons, L the wiggler length, and λ_q the wiggler spatial period. The size of ϵ determines the magnitude of quantum effects: When $\epsilon \ll 1$, one is in the classical regime, in which case the expansion in Eq. (5) is appropriate and gives the linear (small-gain) theory. We shall briefly comment on the quantum-regime results ($\epsilon \gtrsim 1$) at the end of this Rapid Communication.

In order to describe the oscillator FEL, losses have to be added to Eq. (4). This was done in Ref. 4. A "coarse-grained" time average allows one to write the differential equation

$$\frac{d\rho_{n,n+1}}{dt} = r\sum_N \langle \bar{p}+\hbar k(N-n),n|S|\bar{p},N\rangle \rho_{N,N+1}\langle \bar{p},N+1|S^\dagger|\bar{p}+\hbar k(N-n),n+1\rangle - \frac{\nu}{Q}(n+\tfrac{1}{2})\rho_{n,n+1} + \frac{\nu}{Q}[(n+1)(n+2)]^{1/2}\rho_{n+1,n+2} , \quad (7)$$

for the field part of the density matrix, i.e., $\rho = \mathrm{Tr}_e \rho_{ef}$, where the trace is over the electron variables. In Eq. (7) r is the rate at which electrons are injected in the FEL, $\bar{p}$ is their momentum, and ν/Q are the cavity losses. An important difference between this equation and the analogous one for the ordinary laser is that in the latter the term $\rho_{n,n+1}$ is coupled only to itself and to $\rho_{n-1,n}$ and $\rho_{n+1,n+2}$; this is due to the fact that in the atom only two levels are considered, and so the field can only change by absorption or emission of one photon, whereas in the free-electron laser the energy spectrum of the electron in the wiggler has an infinite number of levels, and multiphoton transitions couple in principle $\rho_{n,n+1}$ to all the other $\rho_{N,N+1}$; $N=0$ to ∞.

We now substitute the ansatz (3) into Eq. (7), and sum over n. We may exploit the fact that the photon-number distribution in steady state is sharply peaked at a value n_{ss} by approximating

$$\sum_n f(n)\rho_{n,n+1} \simeq f(n_{ss})\sum_n \rho_{n,n+1} .$$

We then have

$$-\tfrac{1}{2}D\sum_n \rho_{n,n+1} \simeq r\left[\sum_N \rho_{N,N+1}\right]\sum_n \langle n|S|n_{ss}\rangle\langle n_{ss}+1|S^\dagger|n+1\rangle - \frac{\nu}{Q}(n_{ss}+\tfrac{1}{2})\sum_n \rho_{n,n+1} + \frac{\nu}{Q}[n_{ss}(n_{ss}+1)]^{1/2}\sum_n \rho_{n,n+1} , \quad (8)$$

where we have dropped the reference to the electron momentum in the S-matrix element. The sum over the matrix elements of S may be evaluated using Eq. (5) and the explicit expressions given in Ref. 3.

In order to determine the linewidth we are interested in the real part of the quantity D [cp. Eq. (3)]. The result is, in the notation of Ref. 3,

$$\mathrm{Re}D = r\frac{j^2(T)}{2n_{ss}} + \frac{\nu}{Q}\frac{1}{4n_{ss}} + O\left[\frac{r}{n_{ss}}\epsilon j\frac{\partial j}{\partial(\beta T)}\right] + O\left[\frac{j^2}{n_{ss}}\right]^2 + O\left[\frac{j^2}{n_{ss}^2}\right] . \quad (9)$$

Here the first term is the ratio of the number n_{sp} of spontaneously emitted photons per electron [$j^2(T)=n_{sp}$; cp. Ref. 3] over the steady-state photon number, multiplied by the rate of injection. This term is classical. The second term is quantum mechanical ($n_{ss} \sim \hbar^{-1}$) and is much smaller (see the discussion below). The third term comes from S_1 and the remaining terms are higher-order contributions from S_0 which are all exceedingly small. Hence the dominant term is

$$\mathrm{Re}D = \frac{1}{2}\frac{r_{sp}}{n_{ss}} , \quad (10)$$

where $r_{sp} = n_{sp} r$ is the rate of spontaneous emission (photons/time). It is worth noting that in this form the result is identical to that obtained for the ordinary laser (see, e.g., Ref. 7). It therefore lends itself to the well-known intuitive explanation in terms of phase diffusion caused by the random emission of the spontaneous photons.[2,7]

We should here add a remark on our usage of the notion of "spontaneous emission." In analogy to the atomic laser terminology we refer by "spontaneous emission" to the radiation emitted when no photons of the respective modes are present. This does not imply that spontaneous emission be genuinely quantum mechanical. In fact, it is not in the case of the FEL. A more precise characterization might be as "bremsstrahlung (in the wiggler field)" or "spontaneously scattered radiation."

The crucial difference with the ordinary laser is that in the latter, spontaneous and net stimulated emission are simply proportional (the proportionality constant being n_{ss}). In the FEL in the classical regime, instead, one has different expressions for them. In laboratory-frame variables,[6] the gain per time α is

$$\alpha = r_{sp}2\pi\left[\frac{L}{\lambda_q}\right]\frac{\hbar\nu}{\gamma_0 mc^2}\left[-\frac{d}{dx}\ln\left[\frac{\sin^2 x}{x^2}\right]\right]_{x=\mu_0 L/2} \quad (11)$$

(where the detuning parameter μ_0 is defined in Ref. 6, for instance). Near the point of maximum linear gain ($\mu_0 L = 2.6$) the term in brackets in Eq. (11) is approximately equal to 1. If the laser is not too far above threshold, one expects the small-signal gain per unit time to be of the order of magnitude of ν/Q, i.e.,

$$\alpha \simeq \frac{\nu}{Q} \ll r_{sp} .$$

This is the reason of the dominance of the first term in Eq. (9). Using this, and introducing the laser output power $P = n_{ss}\hbar\nu(\nu/Q)$, the linewidth Eq. (10) may be written as

$$\mathrm{Re}D_{\mathrm{FEL}} \simeq \frac{1}{2}\frac{1}{2\pi}\left(\frac{\nu}{Q}\right)^2 \frac{\lambda_q}{L}\frac{\gamma_0 mc^2}{P} \quad , \tag{12}$$

to be compared to the result for the ordinary, atomic laser[2]

$$D_{\mathrm{atomic}} = \frac{1}{2}\left(\frac{\nu}{Q}\right)^2 \frac{\hbar\nu}{P} \quad . \tag{13}$$

We can see that Eq. (12) is classical (i.e., independent of $\hbar$) while Eq. (13) is not. Indeed, "spontaneous emission" in the FEL is an essentially classical phenomenon, arising from density fluctuations (shot noise) in the otherwise unbunched electron beam. (This may be easily derived from, e.g., the classical model in Ref. 8; the only essential feature is the discrete nature of the electrons.) It is important to notice that for fixed ν, Q, and P, Eq. (12) always gives a broader linewidth than Eq. (13), in the classical regime. This is easily seen from the fact that the ratio $D_{\mathrm{atomic}}/D_{\mathrm{FEL}}$ is precisely the quantum-recoil parameter ϵ of Eq. (6b), which is much smaller than 1 in the classical regime.

In the quantum regime (that is, when $\epsilon \gtrsim 1$), one cannot use the expansion (5), but the probability of multiphoton transitions is substantially smaller and straightforward perturbation theory may be used (at least in the single-particle theory). The result is a linewidth identical to the atomic laser, i.e., Eq. (13).

An interesting interpretation of (12) may be also obtained from the estimates in Ref. 6 of the order of magnitude of n_{ss}. One can see then that while $D_{\mathrm{atomic}} \sim (\nu/Q)/n_{ss}$, $D_{\mathrm{FEL}} \sim (\nu/Q)/N_e$, where N_e is the number of electrons inside the cavity.

Finally, as a numerical example, let $\nu/Q = 10^6$ s^{-1}, $L/\lambda_q = 100$ periods, $\gamma_0 mc^2 = 10$ MeV; with an output (continuous) power of 1 W, the linewidth turns out to be $D = 2.5\times10^{-3}$ Hz. This is an extremely small value, which shows that potentially a cw FEL could be a coherent source of great monochromaticity.

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