PHYSICAL REVIEW A VOLUME 33, NUMBER 3 MARCH 1986

Logarithmic nonlinearities within the framework of stochastic mechanics

Antônio B. Nassar
Department of Physics, University of California, Los Angeles, Los Angeles, California 90024
(Received 23 July 1985)

We investigate special types of solutions of the hydrodynamical version of a generalized Schrödinger-Langevin equation (GNLSLE), derived in an earlier work, via stochastic mechanics. Within the same scheme of stochastic differential equations, we decompose the stochastic process associated with the GNLSLE into two independent processes: a classical Langevin-type process and a pure quantum Nelson-type process.

It has previously been demonstrated that logarithmic nonlinear Schrödinger equations (LNLSE's) possess many attractive and distinct features.[1–18] Two types of LNLSE's have been extensively investigated.

(1) For the description of conservative quantum systems, Bialynicki-Birula and Mycielski (BBM) constructed a nonlinear wave mechanics,[1] based on Schrödinger-type and Klein–Gordon-type equations with the nonlinear term $-b\ln|\psi|^2$ (BBM's nonlinearity).[2–5] In both cases the most remarkable feature is the existence of exact solitonlike solutions of Gaussian shape.[1–4] In their heuristic considerations, BBM dwelt upon the realm of the Copenhagen interpretation of quantum mechanics and proposed to apply it to atomic physics, describing phenomena in an intermediate region.[1] Subsequently, the physical reality of such a nonlinear Schrödinger equation (NLSE) has been questioned as a result of negative (neutron interferometer) experimental results.[6–8] Very recently, however, Hefter[9] has given physical grounds for the use of the BBM's NLSE by applying it to nuclear physics and obtaining qualitative and quantitative positive results (besides explaining the reason for the negative results of previous experimental tests). He argues, then, that the only consistent interpretation of such a NLSE is that of an equation for extended objects, such as nucleons and α particles, and not for point particles as originally suggested by BBM.

(2) For the description of nonconservative quantum systems, Kostin[10] formulated heuristically a nonlinear wave mechanics, based on a nonlinear Schrödinger-Langevin equation (NLSLE) with a nonlinear term $(\hbar\nu/2i)\times[\ln(\psi/\psi^*)]$. In this case, one finds that no solitonlike solution exists for the damped field-free or constant-field particle problem,[11] although a solitonlike solution has been found for the damped-harmonic-oscillator ground-state problem.[12] This NLSLE has subsequently been rederived by Skagerstam[13] and Yasue[14] within the stochastic reformulation of quantum mechanics, and has found extensive use in many applications, such as in the works of Weiner and Forman,[15] Yasue,[16] and Griffin and Kan.[12] In fact, very recently, Caldeira and Leggett[17] have given a possible justification for the use of nonlinear wave equations (such as Kostin's NLSLE) for the description of nonconservative systems, based on their conclusion that damping tends to destroy interference effects of two Gaussian wave packets in a harmonic potential.

Recently, within the stochastic reformulation of quantum mechanics, we were able for the first time to show that *both* aforementioned nonlinearities appear quite naturally.[18] Via a unified approach, we have derived a generalized nonlinear Schrödinger-Langevin equation (GNLSLE), for the description of nonconservative quantum systems, which encompasses both features described by BBM's and Kostin's NLSE's.[18]

In this paper, we investigate the possibility (or not) of finding solitonlike solutions of the hydrodynamical version of the GNLSLE for two cases: (1) with a stochastic external field $V=-xA(t)$ (temperature dependent) we show that no solitonlike solution exists; (2) with a nonstochastic (constant) external field $V=-gx$ (zero temperature) we show the possibility of a special solitonlike solution. We associate the former case with what we call a stochastic external field (temperature-dependent) Nelson-Langevin process, whereas the latter is associated with a nonstochastic external field (zero-temperature) Nelson-Langevin process. One more remark is worth emphasizing. There are two conceptually different origins of stochasticity here: one due to thermal classical fluctuations (of Langevin type) and another due to quantum fluctuations (of Nelson type).

Let us begin by reformulating the GNLSLE through Nelson's stochastic mechanics:[19–23] a stochastic formulation of quantum mechanics in terms of subquantum random fluctuations, resulting from the action of a stochastic invariant thermostat. The basic assumption here is that the system under consideration consists of a quantum extended particle in a viscous medium subject to an external potential V and that to each quantum state with the wave function

$$\psi(x,t)=[\rho(x,t)]^{1/2}\exp[iS(x,t)] \ , \tag{1}$$

there corresponds a stochastic process $q(t)$ satisfying the stochastic differential equation

$$\dot q(t)=v_+(q(t),t)+\eta(t) \ , \tag{2}$$

where $v_+(q(t),t)$ is the forward velocity field, and $\eta(t)$ is a Gaussian white noise with expectation values

$$\langle\eta(t)\rangle=0 \ ,$$
$$\langle\eta(t)\eta(t')\rangle=(\hbar/m)\delta(t-t') \ . \tag{3}$$

The dynamics of the stochastic process $q(t)$, corresponding to the GNLSLE, is determined through the hydrodynamical coupled set of equations[18]

$$\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}(vu)-\frac{\hbar}{2m}\frac{\partial^2 v}{\partial x^2} \ , \tag{4}$$

and

$$\frac{\partial v}{\partial t}=-\frac{1}{m}\frac{\partial V}{\partial x}-v\frac{\partial v}{\partial x}-\nu v+u\frac{\partial u}{\partial x}+\lambda u+\frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2} \ , \tag{5}$$

where $v=\frac{1}{2}(v_+ + v_-)=(\hbar/m)\partial S/\partial x$ is the current velocity, $u=\frac{1}{2}(v_+ - v_-)=(\hbar/2m)\partial \ln\rho/\partial x$ is the stochastic velocity, $v_+(v_-)$ is the forward (backward) velocity, and ρ is the probability density of the corresponding process. The term $-\nu v$ accounts for the external dissipative force (p.m.) corresponding to the nonlinear term $(\hbar\nu/2i)\ln(\psi/\psi^*)$ in Kostin's NLSE, whereas λu is the internal nondissipative stochastic force (p.m.) corresponding to the nonlinear term $-b\ln|\psi|^2$ in BBM's NLSE. Notice that in our interpretation of the BBM nonlinearity, b is an $\hbar$-dependent constant $(b=\hbar\lambda/2)$ such that in the Newtonian limit $\lambda u \to 0$.[18]

Inasmuch as v and u are gradients, we rewrite Eqs. (4) and (5) as $[V=xF(t)]$,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0 \ , \tag{6}$$

and

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\nu v-\frac{F(t)}{m}=-\frac{1}{m}\frac{\partial}{\partial x}(V_{qu}+V_{\rm BBM}) \ , \tag{7}$$

where $F(t)$ is a time-dependent arbitrary external force and $V_{qu}\equiv-(\hbar^2/2m)\rho^{-1/2}(\partial^2\rho^{1/2}/\partial x^2)$, $V_{\rm BBM}\equiv-(\hbar\lambda/2)\ln\rho$ are the Madelung-Bohm and the BBM quantum potentials, respectively. The corresponding GNLSLE of Eqs. (6) and (7) is

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\left[\frac{\hbar\nu}{2i}\ln\left(\frac{\psi}{\psi^*}\right)-\frac{\hbar\lambda}{2}\ln|\psi|^2-xF(t)\right]\psi.$$

The time development of the quantities ρ and v (the solution set of the problem) is uniquely determined from the above system of equations if initial conditions are imposed on them: $v(x,0)=v_0$ and $\rho(x,0)=\rho_0$.

Next, by following Hasse[24] and Roy and Singh,[25] we look for solitonlike solutions of the form

$$\rho=\rho(\epsilon) \ , \tag{8}$$

with $\epsilon=x-q_c(t)$, where $q_c(t)$ is the center of gravity of the soliton, which will appear to travel along the respective classical particle path in a viscous medium.

From (8), we have

$$\frac{\partial\rho}{\partial t}=-\dot q_c\rho' \text{ and } \frac{\partial\rho}{\partial x}=\frac{d\rho}{d\epsilon}\equiv\rho' \ . \tag{9}$$

Inserting Eq. (9) into Eq. (6), one obtains

$$v=\dot q_c \ , \tag{10}$$

where the constant of integration must be zero to ensure that v stays finite as $\rho\to 0$.

Substituting Eqs. (10) and (8) into Eq. (7), we arrive at

$$\ddot q_c+\nu\dot q_c-\frac{F(t)}{m}=-\frac{1}{m}\frac{d}{d\epsilon}(V_{qu}+V_{\rm BBM}) \ , \tag{11}$$

which can be split into[24,25]

$$\ddot q_c+\nu\dot q_c-F(t)/m=0 \ , \tag{12}$$

and

$$\frac{d}{d\epsilon}\left[\frac{\hbar^2}{2m^2}\frac{R''}{R}+\frac{\hbar\lambda}{m}\ln R\right]=0 \ , \tag{13a}$$

with

$$R\equiv\rho^{1/2} \ . \tag{13b}$$

A solution of Eq. (13) is given by[24]

$$\rho(x-q_c)=(m\lambda/\pi\hbar)^{1/2}\exp[-(x-q_c)^2/(\hbar/m\lambda)] \ , \tag{14}$$

which can easily be verified by substitution.

The associated stochastic process $q(t)$ is then, with the help of Eq. (2),

$$\dot q(t)=\{[p_c(t)/m]-\lambda[q(t)-q_c(t)]\}+\eta(t) \ . \tag{15}$$

Following Ruggiero and Zannetti,[26] we write the stochastic process $q(t)=q_c(t)+\epsilon(t)$ as the sum of two independent components: the classical (Langevin) solution $q_c(t)$ and the pure quantum (Nelson) fluctuation $\epsilon(t)$ owing to the zero-point motion. In terms of the three-dimensional representation process $[q_c(t),p_c(t),\epsilon(t)]$ we have

$$\dot q_c(t)=p_c(t)/m \ , \tag{16a}$$

$$\dot p_c(t)=-\nu p_c(t)+F(t) \ , \tag{16b}$$

$$\dot\epsilon(t)=-\lambda\epsilon(t)+\eta(t) \ . \tag{16c}$$

Viewed through Eq. (16b), the Kostin nonlinear term $[(\hbar\nu/2i)\ln(\psi/\psi^*)]$ is a precise quantum transcription of a phenomenologically classical dissipative and irreversible process. On the other hand, the BBM nonlinear term $[-(\hbar\lambda/2)\ln|\psi|^2]$ is an intrinsic quantum component (with no classical counterpart) and can be identified only through the stochastic differential equation for the pure quantum fluctuations [Eq. (16c)].

Case 1. For $F(t)=A(t)$, where $A(t)$ is a Gaussian white noise with expectations $\langle A(t)\rangle=0$ and $\langle A(t)A(t')\rangle=2\nu mk_BT\delta(t-t')$, we call the process $q(t)$ [as viewed through Eqs. (16)] a stochastic external field (temperature-dependent) Nelson-Langevin process. In order to obtain the probability density solution of this process $(\bar\rho)$, where quantum and thermal effects compete, we must take the convolution of (14) with $W(q_c,t)$ — the probability of some value of q_c at time t for given initial conditions: $q_c(0)=0$ and $\dot q_c(0)=v_0$.[27,28] It is found, with the help of lemma I of Chandrasekhar,[29] that

$$W(q_c,t)=(\pi\sigma_T)^{-1/2}\exp\{-[q_c-(v_0/\nu)(1-e^{-\nu t})]^2/\sigma_T\} \tag{17a}$$

where

$$\sigma_T(t)=2k_BT(2\nu t-3+4e^{-\nu t}-e^{-2\nu t})/m\nu^2 \tag{17b}$$

$$\simeq(4k_BT/m\nu)t \quad \text{(for large times)} \ . \tag{17c}$$

Hence

$$\bar\rho(x,t)=\int_{-\infty}^{+\infty}\rho(x-q_c)\,W(q_c,t)\,dq_c$$
$$=(\pi\Sigma)^{-1/2}\exp\{-[x-(v_0/\nu)(1-e^{-\nu t})]^2/\Sigma\} \ , \tag{18a}$$

where

$$\Sigma(t)=(\hbar/m\lambda)+(2k_BT/m\nu^2)(2\nu t-3+4e^{-\nu t}-e^{-2\nu t}) \ , \tag{18b}$$

and

$$v=\dot q_c(t)=e^{-\nu t}\left[v_0+\int_0^t e^{\nu\tau}[A(\tau)/m]\,d\tau\right] \tag{19}$$

are the probability density and velocity of the quantum Brownian fluid particle. Thus, when a stochastic external force $[A(t)]$ is present, the center of gravity of the fluid-particle wave packet travels along the respective Brownian classical path in a viscous medium, with its breadth spreading out as the time goes on (the motion undergoes friction with dispersion). Therefore, no solitonlike solution exists in this case.

Case 2. For $F(t)=g$ (a constant), we call the process $q(t)$ [as viewed through (16)] a nonstochastic external field (zero-temperature) Nelson-Langevin process.

Hence

$$\rho(x,t)=(\pi\sigma)^{-1/2}\exp\left[-\left\{x-\frac{1}{\nu}\{gt+[v_0-(g/\nu)](1-e^{-\nu t})\}\right\}^2\Big/\sigma\right] , \tag{20a}$$

where $\sigma=\hbar/m\lambda$, and

$$v=\dot{q}_c(t)=e^{-\nu t}[v_0+(g/\nu)(e^{\nu t}-1)] . \tag{21}$$

Thus, in this case, the Gaussian wave packet has a shape independent of time, and its centroid follows the motion which a damped classical particle in constant field would. In other words, this is a damped-solitonlike solution of Gaussian shape (the motion undergoes friction without dispersion).[30]

This result answers in a very simple way the question of whether friction ought or ought not to operate on the *(zero-temperature)* lower-bound energy-state set by the "$(\hbar\lambda/2)$ nonlinearity": Frictional forces do not apply to this "zero-point" energy.[31] In fact, this question has been addressed and answered previously by Griffin and Kan,[12] whom we refer to for a rigorous and complete analysis of this point. Although their entire conclusion relates to the case of the damped harmonic-oscillator ground-state problem, it still holds true here, inasmuch as we have shown that the "$(\hbar\lambda/2)$ nonlinear potential" has striking similarities to the harmonic potential as far as setting a lower-bound energy state [see Eqs. (16), and compare with those analogous in the work of Ruggiero and Zannetti[26]].

In conclusion, we believe that our GNLSLE may be used as a clue to a deeper understanding of important nuclear physical processes where friction (nuclear viscosity) plays a fundamental role,[12,32,33] e.g., in scattering of heavy ions on each other with distances of smallest approach that are comparable to their spatial extensions, that is, for overlapping systems, such as nucleons bound together within a nucleus.[9] Thus, further study in this direction is of immediate interest.

I am very grateful to Professor E. Nelson, Professor J. C. Zambrini, Professor K. Yasue, and Professor J. Messer for correspondence. I am indebted to Professor S. Putterman for encouragement and advice. I also wish to thank Professor F. Guerra, Professor C. P. Enz, and Dr. E. F. Hefter for sharing many of their unpublished results. I would like to express special thanks to R. Berg and F. L. A. Machado for their critical reading of the manuscript.

[1] I. Bialynicki-Birula and J. Mycielski, Ann. Phys. (N.Y.) **100**, 62 (1976); Phys. Scr. **20**, 539 (1979).
[2] G. C. Marques and I. Ventura, Phys. Rev. D **14**, 1056 (1976); J. Math. Phys. **19**, 838 (1978).
[3] I. L. Bogolyubskii, Zh. Eksp. Teor. Fiz. **76**, 422 (1979) [Sov. Phys. JETP **49**, 213 (1979)]; see also V. G. Makhankov, I. L. Bogolyubskii, G. Kumer, and B. Schvachka, Phys. Scr. **23**, 767 (1981).
[4] T. A. Minelli and A. Pascolini, Lett. Nuovo Cimento **27**, 413 (1980).
[5] G. Rosen, Phys. Rev. **183**, 1186 (1969).
[6] A. Shimony, Phys. Rev. A **20**, 394 (1979).
[7] C. G. Shull, D. K. Atwood, J. Arthur, and M. A. Horne, Phys. Rev. Lett. **44**, 765 (1980).
[8] R. Gähler, A. G. Klein, and A. Zeilinger, Phys. Rev. A **23**, 1611 (1981).
[9] E. F. Hefter, Phys. Rev. A **32**, 1201 (1985).
[10] M. D. Kostin, J. Chem. Phys. **57**, 3589 (1972).
[11] L. Brüll and H. Lange, J. Math. Phys. **25**, 786 (1984).
[12] J. J. Griffin and K. K. Kan, Rev. Mod. Phys. **48**, 467 (1976).
[13] B. K. Skagerstam, Phys. Lett. **58B**, 21 (1975).
[14] K. Yasue, Phys. Lett. **64B**, 239 (1976).
[15] J. H. Weiner and R. E. Forman, Phys. Rev. B **10**, 314 (1974); **10**, 325 (1974).
[16] K. Yasue, Ann. Phys. (N.Y.) **114**, 479 (1978).
[17] A. O. Caldeira and A. J. Leggett, Phys. Rev. A **31**, 1059 (1985).
[18] A. B. Nassar, Phys. Lett. **109A**, 1 (1985); J. Phys. A **18**, L509 (1985).
[19] E. Nelson, Phys. Rev. **150**, 1079 (1966); *Quantum Fluctuations* (Princeton Univ. Press, Princeton, 1985).
[20] L. de la Peña and A. M. Cetto, Found. Phys. **5**, 355 (1975); Phys. Rev. D **3**, 795 (1971).
[21] F. Guerra, Phys. Rep. **77**, 263 (1981).
[22] J. P. Vigier, Lett. Nuovo Cimento **24**, 265 (1979).
[23] A. B. Nassar, Phys. Rev. A **32**, 1862 (1985).
[24] R. W. Hasse, Phys. Rev. A **25**, 583 (1982); Z. Phys. B **37**, 83 (1980).
[25] S. M. Roy and V. Singh, J. Phys. A **14**, 2927 (1981).
[26] P. Ruggiero and M. Zannetti, Phys. Rev. Lett. **48**, 963 (1982).
[27] J. Messer, Acta Phys. Austriaca **50**, 75 (1979).
[28] A. B. Nassar, Phys. Lett. **106A**, 43 (1984).
[29] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[30] It is very interesting to contrast our GNLSLE solution with, e.g., that of the NLSE with damping describing high-frequency surface acoustic solitons (phonons) in a semiconductor (the dissipation corresponds to the attenuation rate of the surface phonon). In this model the damped-soliton velocity is constant, while its amplitude decreases exponentially in time. See T. Sakuma and Y. Kawanami, Phys. Rev. B **29**, 880 (1984).
[31] At absolute zero the system causing the noise must be in its ground state, and no energy can be extracted from it; see also H. Metiu and G. Schön, Phys. Rev. Lett. **53**, 13 (1984), for an alternative discussion of this point.
[32] J. D. Immele, K. K. Kan, and J. J. Griffin, Nucl. Phys. **A241**, 47 (1975).
[33] R. W. Hasse, Rep. Prog. Phys. **41**, 1027 (1978).