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Hamiltonian systems with three degrees of freedom, singular-point analysis, and chaotic behavior

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The singular-point analysis for two Hamiltonian systems with three degrees of freedom is performed. The connection with their chaotic behavior is discussed. To characterize the chaotic behavior we have calculated the one-dimensional Lyapunov exponent.

In the present Brief Report we perform a singular-point analysis (Painlevé test) for Hamiltonians with three degrees of freedom. We compare our results with numerical studies, i.e., we calculate the maximal one-dimensional Lyapunov exponents which serve to characterize chaotic behavior. The models under investigation are the Hamiltonian

$$H(x,p)=\sum_{i=1}^{3}\tfrac{1}{2}p_i^2+\tfrac{1}{2}(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2)\,, \qquad (1)$$

which arises in connection with Yang-Mills equations[1] and the Contopoulos Hamiltonian[2,3]

$$H(x,p)=\sum_{i=1}^{3}(\omega_i/2)(p_i^2+x_i^2)+x_1^2x_2+x_1^2x_3 \qquad (2)$$

($\omega_1=1$, $\omega_2=2^{1/2}$, $\omega_3=3^{1/2}$), which arises in an astronomical context. The equations of motion for the Hamiltonian (1) are

$$\ddot{x}_1=-x_1(x_2^2+x_3^2)\,, \qquad (3a)$$

$$\ddot{x}_2=-x_2(x_1^2+x_3^2)\,, \qquad (3b)$$

$$\ddot{x}_3=-x_3(x_1^2+x_2^2)\,, \qquad (3c)$$

and for the Hamiltonian (2) we find

$$\ddot{x}_1=\omega_1(-\omega_1x_1-2x_1x_2-2x_1x_3)\,, \qquad (4a)$$

$$\ddot{x}_2=\omega_2(-\omega_2x_2-x_1^2)\,, \qquad (4b)$$

$$\ddot{x}_3=\omega_3(-\omega_3x_3-x_1^2)\,. \qquad (4c)$$

To see what new aspects enter when Hamiltonians with three degrees of freedom are studied let us discuss the case with two degrees of freedom. The behavior of nonlinear Hamiltonian systems with two degrees of freedom of the form $H(x,p)=\frac{1}{2}(p_1^2+p_2^2)+U(x)$ has been widely discussed in literature (compare Ref. 4, and references therein). In the following we assume that U is a polynomial with rational coefficients and the associated equations of motion are similarity invariant.[5,6] When we perform a singular-point analysis[5-11] (the equations are considered in the complex domain) and determine the resonances (Kowalevski's exponents) we find that $r=-1$ is always a resonance and represents the arbitrariness of t_1. Another resonance is related to the Hamiltonian (which is a first integral).[5,6] Since we have assumed that U is a polynomial and the system is similarity invariant, one of the resonances must be a rational number. For the remaining two resonances we have to solve a quadratic equation with rational coefficients. The following cases can arise: (i) The resonances become complex (complex conjugate pair). (ii) The resonances are irrational numbers. (iii) The resonances are rational numbers. Yoshida[5,6] showed that the appearance of at least one irrational or complex resonance means the nonexistence of another algebraic (and analytic) first integral independent of the Hamiltonian (theorem of Yoshida). Adler and van Moerbeke[7] proved, assuming certain technical conditions, that the Painlevé property (a necessary condition that an ordinary differential equation has the Painlevé property is that it passes the Painlevé test) is a necessary condition for algebraically complete integrability in terms of Abelian functions. On the other hand, it is not necessary for a Hamiltonian system to pass the Painlevé test in order to be completely integrable. For example, let $U(x)=x_1^5/5+2x_1^3x_2^2+x_1x_2^4$. Besides the Hamiltonian we have the first integral $h(x,p)=p_1p_2+x_1^4x_2+2x_1^2x_2^3+x_2^5/5$. The Painlevé test leads to the dominant behavior $x_1(t)\propto(t-t_1)^{-2/3}$ and $x_2(t)\propto(t-t_1)^{-2/3}$. The resonances are -1, $\frac{2}{3}$, $\frac{5}{3}$, and $\frac{10}{3}$. The essential point is that we find a psi-series[12] without logarithmic terms. This is called the "weak Painlevé property."[10]

For a Hamiltonian with three degrees of freedom the situation becomes more complicated because the number of the resonances is six. As described above $r=-1$ is related to the arbitrariness of t_1 and another resonance must be a rational number (associated with the Hamiltonian being a first integral). We continue to assume that the potential U is a polynomial. Thus we must discuss the remaining four resonances. Among other possibilities we can find that (i) all remaining resonances are complex (complex conjugate pairs), (ii) there is a pair of conjugate complex resonances and the remaining two are rational numbers, and (iii) all remaining resonances are rational numbers.

This motivates us to perform a singular-point analysis for the two Hamiltonians given above, since Hamiltonian (1) leads to case (iii) (at the resonances we must introduce logarithmic terms for the present case) and Hamiltonian (2) leads to case (ii). Then we compare the results with numerical studies.

First let us perform the singular-point analysis for system

(3) (compare also Ref. 11). Putting $\dot{x}_k = p_k$ we find that the resulting Hamiltonian equations of motion are similarity invariant. Determining the dominant behavior we find the branch $x_k(t) \propto a_{k0}(t-t_1)^{-1}$ $(k=1,2,3)$, where $a_{10}^2 + a_{20}^2 = -2$, $a_{10}^2 + a_{30}^2 = -2$, and $a_{20}^2 + a_{30}^2 = -2$. Since the Hamiltonian equations of motion are scale invariant under $t \to \alpha^{-1} t$, $x_i \to \alpha x_i$, $p_i \to \alpha^2 p_i$ we obtain $H(\alpha x, \alpha^2 p) = \alpha^4 H(x,p)$. Consequently, $r=4$ is a resonance.[5,6] For the main branch[11] the resonances are given by -1, 1, 1, 2, 2, 4. At the resonance $r=2$ (twofold) we have to introduce logarithmic terms. This means the general solution of system (3) (considered in the complex domain) is expressed as logarithmic psi-series.[12] On the other hand, if we put $x_3(t)=0$ in system (3) and study the remaining system we find complex resonances,[13] namely, $\frac{3}{2} \pm i7^{1/2}/2$ (besides -1 and 4). Due to the theorem of Yoshida[5,6] Eq. (3) is not algebraically integrable.

The singular-point analysis can give only a decision that there is no further algebraic first integral besides H. However, the system can admit transcendental first integrals. With the help of the Lie theory of extended vector fields we have proved that besides H there is no further first integral. The approach has been described by Leach[14] for n degrees of freedom and has been applied to the Henon-Heiles system. In our case only the symmetry generator $S = \partial/\partial t$ arises which is associated with the conservation of energy. The constants of motion are obtained from the symmetry generator and the Cartan form $\alpha = \sum_i p_i dx_i + H(x,p)dt$.[15]

Let us now perform our numerical studies. We solve the Hamiltonian equations of motion numerically. The integration has been performed with the help of Lie series.[16] The maximal one-dimensional Lyapunov exponent λ_{max} is calculated using an approach described by Contopoulos, Galgani, and Giorgilli.[2] This means we integrate directly the equations of motion and the variational system. For the initial conditions we put $x_i(0)=0$ and $p_i(0)=(2E_i)^{1/2}$ for given values E_1, E_2, and E_3. Consequently, $E = E_1 + E_2 + E_3$. In Table I we give the maximal one-dimensional Lyapunov exponents for $E=0.15$. Due to the initial conditions the orbit 1 is periodic. Consequently, $\lambda_{max}=0$. This coincides with our numerical results. Orbits 2–9 are chaotic, since the maximal one-dimensional Lyapunov exponent is positive. We have also calculated the autocorrelation functions. We find for the orbits 2–9 that the autocorrelation functions decay.

TABLE I. Maximal one-dimensional Lyapunov exponent for system (4). The initial conditions are $q_i(t=0)=0$ and $p_i(t=0)=(2E_i)^{1/2}$ $(i=1,2,3)$ with $E=E_1+E_2+E_3=0.15$.

Orbit	E_1	E_2	E_3	λ_{max}
1	0.05	0.05	0.05	0
2	0.01	0.13	0.01	0.19
3	0.02	0.11	0.02	0.19
4	0.02	0.10	0.03	0.23
5	0.03	0.08	0.04	0.29
6	0.04	0.06	0.05	0.25
7	0.01	0.09	0.05	0.28
8	0.02	0.08	0.05	0.25
9	0.03	0.07	0.05	0.22

Now the Hamiltonian equations of motion and the associated variational system are invariant under

$$t \to \alpha^{-1} t, \quad x_i \to \alpha x_i, \quad p_i \to \alpha^2 p_i \ , \tag{5a}$$

and

$$y_i \to \alpha y_i \ (i=1,2,3), \quad y_i \to \alpha^2 y_i \ (i=4,5,6) \tag{5b}$$

(scale invariance). For the Hamiltonian (1) we have $H(\alpha x, \alpha^2 p) = \alpha^4 H(x,p)$. The one-dimensional Lyapunov exponent is not scale invariant but $\lambda \to \alpha\lambda$. This means the following: Given two sets of initial values $(x_{10}, x_{20}, x_{30}, p_{10}, p_{20}, p_{30}, y_{10}, y_{20}, \ldots, y_{60})$ and $(\bar{x}_{10}, \bar{x}_{20}, \bar{x}_{30}, \bar{p}_{10}, \bar{p}_{20}, \bar{p}_{30}, \bar{y}_{10}, \bar{y}_{20}, \ldots, \bar{y}_{60})$, where $\bar{x}_{i0} = \alpha x_{i0}$, $\bar{p}_{i0} = \alpha^2 p_{i0}$, $\bar{y}_{i0} = \alpha y_{i0}$ $(i=1,2,3)$, $\bar{y}_{i0} = \alpha^2 y_{i0}$ $(i=4,5,6)$. Then it follows that $\bar{E} = \alpha^4 E$ and $\bar{\lambda} = \alpha\lambda$, where $E = H(x_{i0}, p_{i0})$. Due to the scale invariance we have to do our calculation only for one energy shell.

Let us now discuss our results for the system (3). Let $x_3(t)=0$. In this case the numerical investigations[17] strongly suggest that there is no regular region and that the motion is always irregular except for special orbits, like the orbits that form a set of measure zero. This coincides with the result from the Toda Brumer criterion.[18] From our numerical investigations we conjecture that the same holds for system (3). This means the "motion of a particle" in the potential $U(x) = \frac{1}{2}(x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2)$ is irregular except for a set of measure zero. Notice, however, the region which can contain regular motion can be very small.

To answer this conjecture it would be helpful to study the stability of periodic solutions. A periodic solution of system (3) can be found by setting $x_1(t) = x_2(t) = x_3(t) = x(t)$. Then we have the equation $\ddot{x} = -2x^3$. The solution to this equation is given by $x(t) = 2^{-1/2} A\,\mathrm{cn}(A(t-t_0), 2^{-1/2})$, where A and t_0 are the constants of integration and $\mathrm{cn}(\cdot)$ is the elliptic function. To study the stability we have to solve the variational equations $(i=1,2,3)$

$$\dot{y}_i = y_{i+3} \ , \tag{6a}$$

$$\dot{y}_{i+3} = -2x^2(y_1 + y_2 + y_3) \ , \tag{6b}$$

where $x(t)$ is the solution given above. It is known that a solution to system (6) is of the form $y_i(t) = \sum_{k=1} \times \exp(\alpha_k t) S_{ik}(t)$, where the quantities S_{ik} denote periodic functions of t with the period $2K$. The three quantities α_k are constants, which are called the characteristic exponents of the periodic solution. A necessary condition for stability of the periodic orbit is that all the characteristic exponents must be purely imaginary.[18] Since system (3) does not depend explicitly on t and, moreover, H is a first integral; two of the characteristic exponents are equal to zero. Let $y = y_1 + y_2 + y_3$. Then we find $\ddot{y} + 6x^2(t) y = 0$, where $\ddot{x} + 2x^3 = 0$. A solution is given by $y(t) = \dot{x}(t)$ and another independent solution is obtained by quadrature, both of which show no exponential instability [$x(t)$ and $\dot{x}(t)$ are periodic functions]. To prove the instability we put $x_3(t)=0$. Then we perform the canonical transformation

$$x_1 + ix_2 = \exp(-i\pi/4)(X_1 + iX_2) \ , \tag{7a}$$

and

$$p_1 + ip_2 = \exp(-i\pi/4)(P_1 + iP_2) \ , \tag{7b}$$

and find $H(P,X) = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{8}(X_1^4 + X_2^4 - 2X_1^2 X_2^2)$. The periodic solution $x_1(t) = x_2(t) = x(t)$ with $\ddot{x} = -x^3$ is related to $X_1=0$. Then $\ddot{X}_2=-X_2^3/2$ and the variational equations are $\ddot{y}_1=[X_2^2(t)/2]y_1$ and $\ddot{y}_2=[-3X_2^2(t)/2]y_2$. By a simple change of time scale $t=2^{1/2}\bar{t}$ we obtain $y_1''=X_2^2(t)y_1$ and $y_2''=-3X_2^2(t)y_2$. Calculating the index of stability[19,20] $\mathrm{Tr}M(T)$ we find $|\mathrm{Tr}M(T)|>2$ for $y_1''=X_2^2(t)y_1$. Now the solution is stable if $|\mathrm{Tr}M(T)|<2$ and exponentially unstable if $|\mathrm{Tr}M(T)|>2$. Consequently, X_2 is unstable.

Consider now the Contopoulos Hamiltonian (2). The Hamiltonian equations of motion are not similarity invariant. Performing a singular-point analysis we find the branch $x_k(t)\propto a_{k0}(t-t_1)^{-2}$, where

$$a_{10}^2\omega_1(\omega_2+\omega_3)=18 \ , \tag{8a}$$

$$a_{20}\omega_1(\omega_2+\omega_3)=-3\omega_2 \ , \tag{8b}$$

$$a_{30}\omega_1(\omega_2+\omega_3)=-3\omega_3 \ . \tag{8c}$$

The system with the dominant terms is

$$\ddot{x}_1=-2\omega_1(x_1x_2+x_1x_3) \ , \tag{9a}$$

$$\ddot{x}_2=-\omega_2x_1^2 \ , \tag{9b}$$

$$\ddot{x}_3=-\omega_3x_1^2 \ . \tag{9c}$$

Putting $\dot{x}_i=\omega_ip_i$ we find that the Hamiltonian equations of motion are similarity invariant ($t\rightarrow\alpha^{-1}t$, $x_i\rightarrow\alpha^2x_i$, $p_i\rightarrow\alpha^3p_i$). The Hamiltonian is given by

$$H(x,p)=\sum_{i=1}^{3}(\omega_i/2)p_i^2+x_1^2x_2+x_1^2x_3 \ . \tag{10}$$

Then we find $H(\alpha^2x,\alpha^3p)=\alpha^6H(x,p)$. Hence, $r=6$ is a resonance. For the resonances we find -1, 2, 3, 6, $\frac{5}{2}\pm(i/2)23^{1/2}$. Due to the theorem of Yoshida[5,6] we conclude that the system cannot be algebraically integrable. To find out the behavior at the resonance $r=2$ we insert the Laurent expansion

$$x_k(t)=(t-t_1)^{-2}\sum_{j=0}^{\infty}a_{kj}(t-t_1)^j \tag{11}$$

($k=1,2,3$) into Eqs. (4). We find that $a_{11}=a_{21}=a_{31}=0$ and

$$\begin{pmatrix} 6 & -2\omega_1a_{10} & -2\omega_1a_{10} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}=\begin{pmatrix} \omega_1a_{10} \\ \frac{1}{12}\omega_2^2a_{10} \\ \frac{1}{12}\omega_3^2a_{10} \end{pmatrix} \ . \tag{12}$$

Since $\omega_2^2\neq\omega_3^2$ it follows that the ansatz (11) does not work and we find a logarithmic psi-series.

The search for symmetry generators with the help of the theory of extended vector fields has no success. Only the symmetry generator $S=\partial/\partial t$ arises which is associated with the conservation of energy.

A detailed numerical analysis for the system (4) has been performed by Contopoulos *et al.*[2] and Pettini and Vulpiani.[3] At least three disjoint regions with a different "degree of stochasticity" have been observed. The escape energy for system (4) is $E_{\mathrm{esc}}=0.097$. Thus the numerical calculations are performed for $E=0.090$. The (at least) three disjoint and invariant regions for system (4) are (i) an ordered region with $\lambda_{\max}=0$, (ii) a large stochastic region with $\lambda_{\max}=0.03$, and (iii) a stochastic region with $\lambda_{\max}=0.002$ and 0.005, respectively. In our singular-point analysis for system (4) we find complex resonances and at the integer resonance $r=2$ we have to introduce logarithmic terms. Both the logarithmic terms and the complex resonances can indicate chaotic behavior. System (4) is not scale invariant. Thus different energy shells can show different behavior.

For system (3) we have only to study one energy shell. For the special case $x_3(t)=0$ the distribution of the singularities in the complex t plane has been calculated.[13] For the nonlinear oscillator $\ddot{x}+x^3=0$ the periodic solution is described by Jacobi elliptic functions. Its singularities (in the complex plane) are characterized as simple poles of order one and are distributed doubly periodically in the whole complex t plane. Such a regular distribution of the singularities reflects faithfully on periodicity of the solution. When a system shows chaotic behavior the singularities are distributed at random.

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