Physical mechanisms in the plasma wake-field accelerator

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The acceleration of trapped electrons in the relativistic plasma waves (wake fields) produced by specially shaped charged-particle beams is described by physical, analytic, and two-dimensional simulation models. The effects of competing instabilities, imperfect bunch shapes, transverse dynamics, and dephasing of trapped particles are considered.

I. INTRODUCTION

The plasma wake-field accelerator (PWA) has been proposed[1] as a means of coupling relativistic electron-beam energy into high phase velocity plasma waves. A trailing bunch of fewer electrons can then ride the wave electric field and accelerate to high energy. Just as for other plasma schemes,[2] such as the laser-driven beat-wave accelerator (BWA),[3] the acceleration gradient in the plasma waves can be very high [$E \le (n_0)^{1/2}$ V/cm, n_0 is plasma density in cm^{-3}]. The difference is that here the plasma wave is driven not by the pondermotive force of two lasers, but by the space-charge force of a charged particle beam. The space charge of the beam perturbs the plasma electrons and leaves behind it a wake of plasma oscillations at the plasma frequency ω_p. The phase velocity of these waves is exactly the velocity of the driving beam (even though their group velocity is nearly zero), just as the wake of a boat follows at the velocity of the boat.

Chen *et al.*[1] first studied the plasma wake-field accelerator scheme with a model consisting of a series of short driving bunches separated by an integer number of plasma wavelengths. Such short bunches are subject to the fundamental wake-field theorem[4,5] which limits the energy gain of trailing electrons to $2\gamma_b mc^2$, where $\gamma_b mc^2$ is the energy of the driving electrons. Bane *et al.*[6] have shown that this limit can be overcome by employing properly shaped driving bunch densities of finite longitudinal extent. Chen *et al.*[7] consider the plasma response to linearly rising bunch densities (of length $N\lambda_p$, $\lambda_p = 2\pi c/\omega_p$), which are sharply cut off at the tail of the bunch. Using a 1D model they find that the maximum energy gained by trailing electrons can be R times the driving beam energy, where the transformer ratio R is between $2\pi N$ and πN, depending on whether or not a precursor space-charge kick is added to the head of the bunch as in Fig. 1.

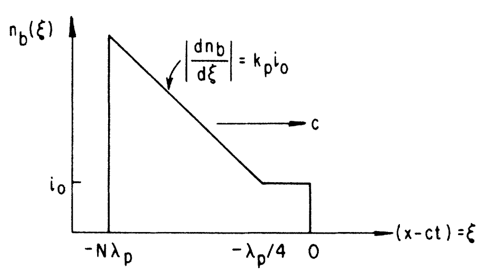

FIG. 1. The ideal "doorstep" bunch shape considered by Chen *et al.* (Ref. 7).

In this paper, we examine physical mechanisms involved in the PWA with shaped driving bunches. We will include several effects not considered in the idealized 1D model of Chen *et al.*,[7] which will be important to any experimental realization of the PWA. Instabilities of the driving beam, nonideal bunch shapes, transverse plasma dynamics, and dephasing of accelerated particles will be treated.

In Sec. II, we give a physical interpretation of the transformer ratio R. In Sec. III, we examine the plasma response to smooth (Gaussian) rather than piecewise linear-bunch shapes. We also consider the effect of a nonzero cutoff length of the driving bunch. In Sec. IV we examine the competing two-stream instability of the driving bunch and derive the limit to the transformer ratio imposed by this instability. In Sec. V, the effects of transverse plasma dynamics are demonstrated through 2D particle-in-cell simulations. In Sec. VI, a revised transformer ratio is derived which includes the effect of the slowing of the driving beam on the dephasing of accelerated particles in the wake. A new mechanism is proposed in order to avoid dephasing. Finally, we summarize the design equations for the PWA and present some examples in Sec. VII.

II. PHYSICAL DERIVATION OF THE PLASMA WAKE-FIELD TRANSFORMER RATIO

The transformer ratio R of wake-field accelerator is defined by the ratio of the maximum energy gain ($\Delta\gamma$) of accelerated particles to the initial energy (γ_b) of driving particles ($R \equiv \Delta\gamma/\gamma_b$). This limit arises because the driving beam is decelerated by an electric field E_- from its initial energy to zero in some length $\Delta x \approx \gamma_b mc^2/eE_-$. During that same distance, the driven electrons can only gain energy $eE_+ \Delta x = (E_+/E_-)\gamma_b mc^2 \equiv R\gamma_b mc^2$, where E_+ is the peak accelerating field of the wake behind the driving beam.

Chen *et al.*[7] have shown that the transformer ratio of the PWA driven by shaped driving bunches that are slowly ramped in density and sharply cut off can be as high as

$2\pi N$ where N is the length of the bunch in plasma wavelengths ($2\pi c/\omega_p$). This result was obtained by evaluating a convolution integral over the wake-field functions of single particles.

Here we present a physical derivation of this transformer ratio which sheds some light on the wake-field generation mechanism. The basic physical assumptions are as follows: As the driving electron bunch enters a given region, the plasma there sees an excess of negative charge. Since the charge builds up slowly (as long as $n_b/Nn_0 \ll 1$, where n_b is the peak density at the tail of the bunch and n_0 is the background plasma density), the plasma moves in order to shield or neutralize the bunch field. This adiabatic shielding of the bunch field reduces the retarding field E_- and ultimately allows for high transformer ratios. The shielding continues until the tail of the bunch exits the region. Then suddenly the plasma, which was nearly neutral, is left with a non-neutral space charge of amplitude n_1 equal to the charge density at the tail of the driver (n_b). Each plasma particle then acts like a spring pulled out to its maximum amplitude and released, setting up an oscillation at frequency ω_p, amplitude n_b, and phase velocity tied (like the wake of a boat) to the driving-bunch velocity.

Unlike the beat-wave accelerator, which resonantly drives up the plasma wave over many cycles, the wake-field accelerator simply displaces and releases the background plasma once. Thus, an advantage of the PWA is that it is not necessary to fine tune the plasma density to satisfy a resonance condition as in the BWA.

Once the amplitude n_1 of the plasma oscillation (and hence E_+) is known ($n_1 = n_b$), the retarding field E_- on the driving bunch and therefore the transformer ratio can be computed from energy conservation. The energy density in the wake is $E_+^2/8\pi$ and is left by the drivers in a volume ct times the area (nearly all this energy is left behind because the group velocity of the wake is order $V_{th}^2/c \approx 0$). Thus, energy balance requires

$$\frac{d}{dt}\left[\frac{E_+^2}{8\pi}\right]ctA = -\frac{d}{dt}N_b\gamma_b mc^2 = N_b e\langle E_-\rangle ,$$

where A is area, N_b and γ_b are the number and Lorentz factor of the driving electrons, and the average retarding field on the driving bunch $\langle E_-\rangle$ we take to be some fraction α of its peak value E_-. Differentiating on the left and rearranging yields

$$R \equiv \frac{E_+}{E_-} = \frac{8\pi N_b e\alpha}{E_+ cA} .$$

Now, the peak density of the driving bunch can be expressed as $n_b = 2N_b/Al$ where l is the effective length of the bunch.

From Poisson's equation, the plasma waves have electric-field amplitude $eE_+/m\omega_p c = n_1/n_0$ where n_1 is the amplitude of the density oscillation ($\omega_p^2 = 4\pi n_0 e^2/m$). Substituting these into the above expression, we find

$$R = 2\pi N\frac{n_b}{n_1}\alpha ,$$

where

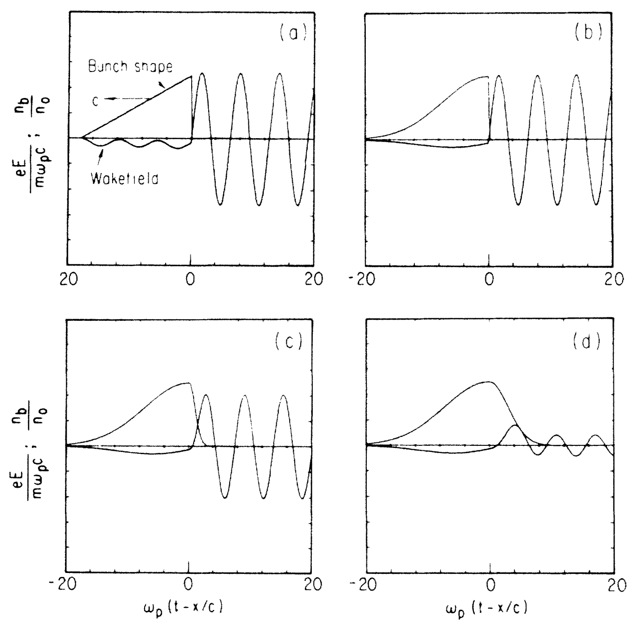

FIG. 2. Numerical solutions of 1D wake fields produced by various bunch shapes: (a) Triangular bunch; (b) Gaussian rise, $\sigma_r = 7.2c/\omega_p$. Gaussian fall, $\sigma_f = 0.1c/\omega_p$; (c) $\sigma_r = 7.2c/\omega_p$, $\sigma_f = 1c/\omega_p$; (d) $\sigma_f = 3c/\omega_p$.

$$N \equiv l/\lambda_p = l\omega_p/2\pi c .$$

Invoking the assumption made previously that n_1 of the plasma wave equals n_b gives

$$R \approx 2\pi N\alpha . \tag{1}$$

The ideal bunch shape is one for which E is uniform within the bunch, so that $E_- = \langle E_-\rangle$ and $\alpha = 1$. In this case, we obtain the optimal transformer ratio $R = 2\pi N$. Such an ideal situation can only be created with a delta function rise at the head of the driving bunch. For a triangular bunch, $\langle E_-\rangle = \frac{1}{2}E_-$ [see Fig. 2(a)], and we recover the result $R = N\pi$ found by Chen *et al.*[7] by the conventional wake-field analysis.

In both the 1D and 2D simulations of Sec. V, we have seen n_1 to be slightly higher than n_b (by about 20%). The transformer ratios are correspondingly lower by about this factor. The reason n_1 is larger than n_b may be that the shielding of the bunch charge by the background plasma is not quasistatic as in our previous stretched and released spring argument. Instead, the spring is moving when released and slightly overshoots the amplitude at which it is released.

III. WAKE FIELDS OF NONIDEAL BUNCHES

The linearly ramped and sharply cut off bunches considered by Chen *et al.*[7] (see Fig. 1), can only be approximated experimentally. In this section, we examine the effect of more realistic (Gaussian) rise and fall of the driving-bunch density.

The response of a 1D cold plasma to a relativistic charge bunch of arbitrary density shape can be found from the Green's function response to a single charge. Neglecting the change in velocity of the driving particles, the space-time dependence of both beam and plasma quantities can be expressed as functions of the single variable $\xi \equiv x - V_b t$, where V_b is the beam velocity ($\approx c$). Then Poisson's equation, the equation of motion and the continuity equation become

$$\partial_\xi E_1 = -4\pi e n_1 + 4\pi \rho_b(\xi) , \tag{2}$$

$$-V_b \partial_\xi v_1 = -eE_1/m , \tag{3}$$

$$-V_b \partial_\xi n_1 + n_0 \partial_\xi v_1 = 0 , \tag{4}$$

where $\rho_b(\xi)$ is the charge density of the driving bunch. Taking ξ derivatives of (3) and (4) and combining with (2) gives

$$\partial_\xi^2 n_1 + k_p^2 n_1 = k_p^2 \rho_b(\xi)/e , \tag{5}$$

where $k_p \equiv \omega_p / V_b$.

For $\rho_b(\xi) = \delta(\xi)$, the solution to (5) is simply the Green's function of a simple harmonic oscillator [$n_1 = (k_p/e)\sin k_p\xi$ for $\xi < 0$ and $n_1 = 0$ for $\xi > 0$]. The response must be zero ahead of the bunch ($\xi > 0$) by causality. The electric field behind the bunch ($\xi < 0$) is simply [from (3) and (4)] $E_1 \equiv G(\xi) = 4\pi e \partial_\xi n_1 / k_p^2 = 4\pi \cos k_p \xi$.

From the above Green's function, the response of the plasma to arbitrary bunch density is simply

$$E_1(\xi) = \int_\infty^\xi \rho_b(\xi') G(\xi - \xi') d\xi' .$$

Consider a bunch density consisting of a Gaussian rise of width σ_r and a Gaussian fall of width σ_f:

$$\rho_b(\xi) = \begin{cases} -en_b e^{-\xi^2/2\sigma_r^2}, & \xi > 0 \\ -en_b e^{-\xi^2/2\sigma_f^2}, & \xi < 0 . \end{cases}$$

Then the wake-field response of the plasma is

$$E_1(\xi) = \begin{cases} 4\pi e n_b \int_\infty^\xi e^{-(\xi')^2/2\sigma_r^2} \cos[k_p(\xi - \xi')] d\xi', & \xi > 0 \\ 4\pi e n_b \left[\int_\infty^0 e^{-(\xi)'^2/2\sigma_r^2} \cos[k_p(\xi-\xi')] d\xi' + \int_0^\xi e^{-(\xi')^2/2\sigma_f^2} \cos[k_p(\xi - \xi')] d\xi' \right], & \xi < 0. \end{cases} \tag{6}$$

Analytic expressions for the above integrals are not particularly revealing, but numerical solutions for various values of σ_r and σ_f are shown in Fig. 2. Also shown for comparison is the solution for the linearly ramped bunch density. From the figures we are able to draw three conclusions about the Gaussian bunches.

(1) The wake *behind* a sharply cut off Gaussian is nearly identical to that of a triangle bunch of the same number of particles [(length of triangle) $=(2\pi\sigma_r)^{1/2}$, height of triangle = height of Gaussian].

(2) The wake field (E_-) inside the Gaussian rise is smoother than in the triangular bunch. This is preferred since it will lead to less distortion of the bunch shape, a higher transformer ratio, and ultimately higher coupling efficiency of beam energy to plasma waves. From the analysis of Sec. II and the uniformity of E_- in the numerical solutions, one could expect the transformer ratio of the Gaussian bunch to be between πN and $2\pi N$ where $N = \sigma_r \omega_p / c\sqrt{2\pi}$.

(3) The sharp cutoff of the driving bunch is not too critical as long as it is shorter than c/ω_p. In Fig. 2(c), $\sigma_f = c/\omega_p$ leads to only a 10% reduction in the wake amplitude compared to the case of $\sigma_f = 0.1c/\omega_p$. Furthermore, the smeared out cutoff will be "self-sharpening" since it is apparent from Fig. 2(c) that electrons in the cutoff region feel an accelerating field which will help them to catch up to the bulk of the driving bunch. The cutoff requirement may prove to be one of the largest technological barriers to realization of a plasma wake-field accelerator. For example, in a plasma of density 10^{16} cm^{-3}, the cutoff length must be of order 0.1 mm (only 0.3 ps).

IV. TWO-STREAM INSTABILITY OF THE WEDGE-SHAPED BUNCH

The distance which the driving bunch can travel through the plasma may be limited by the plasma two-stream instability. The instability feeds energy from the driving bunch to a plasma wave which can modulate and degrade the driving bunch. Fortunately, the instability can be suppressed if the density gradient of the driving beam is high enough. In this section, we calculate the gradient threshold for two-stream instability in an inhomogeneous plasma and apply the result to the ramped driving beam. Since a high gradient implies a short ramp-up length (low N), the gradient threshold corresponds to an upper limit of the transformer ratio ($2\pi N$) which we will determine.

The two-stream instability of a spatially uniform beam-plasma system has the following well-known frequency, growth rate, wave number, and group velocity:[8]

$$\begin{aligned} &\omega \simeq \omega_p(1-\delta) , \\ &\nu \simeq \sqrt{3}\delta\omega_p , \\ &k \simeq \omega_p / V_b , \\ &V_g \simeq d\omega/dk = 2V_b/3 , \end{aligned} \tag{7}$$

where V_b is the driving beam speed, $\delta = (n_b/16\gamma_b^3 n_0)^{1/3}$, n_b/n_0 is the ratio of beam to background density, and $\gamma_b = (1 - V_b^2/c^2)^{-1/2}$.

Consider a system in which either the beam density, background density, or beam energy is varying with an inverse spatial scale length

$$\kappa = \frac{1}{n_b}\frac{\partial n_b}{\partial x}, \quad \frac{1}{n_0}\frac{\partial n_0}{\partial_x},$$

or

$$\frac{3}{\gamma_b}\frac{\partial \gamma_b}{\partial x}.$$

This spatial dependence causes a wave which convects from point $x=0$ to $x=x$ to be out of phase with a wave growing locally at point x by an amount

$$\Delta\phi = \int_0^x [k(x)-k(0)]dx \approx \int_0^x k'(0)x\,dx = \tfrac{1}{2}k'(0)x^2, \tag{8}$$

where we have Taylor expanded

$$k(x) \cong k(0) + xk'(0).$$

Now

$$k' = (dk/d\omega)(d\omega/dx) = \omega'/V_g.$$

From Eq. (7) and the definition of κ, $\omega \approx \omega_p[1-\delta(0)(1+\frac{1}{3}\kappa x)]$ so that $\omega' = \frac{1}{3}\omega_p\kappa\delta$. Substituting into Eq. (8) and defining $x=L$ when $\Delta\phi=\pi$ gives

$$L = (2\pi/k')^{1/2} = (4\pi V_b/\omega_p\kappa\delta)^{1/2}.$$

Since the wave cannot grow if the distance L is less than an e-folding length of the homogeneous instability (Ref. 9) $L_e = V_g/\nu$, the plasma will be stable to the two-stream instability if $L < 2V_b/3\sqrt{3}\omega_p\delta$ or if

$$\kappa^{-1} < \frac{2^{4/3}}{27\pi}\frac{V_b}{\omega_p}\frac{\gamma_b}{(n_b/n_0)^{1/3}}. \tag{9}$$

We apply this gradient threshold to the wedge-shaped driving bunches. These are assumed to be ramped linearly over N plasma wavelengths, so

$$\kappa = (1/n_0)(\partial n_b/\partial x) = \omega_p/2\pi N V_b$$

at the peak of the bunch density. Substituting for κ in Eq. (9) and recalling that the maximum transformer ratio R is approximately $2\pi N$, we find that

$$R \approx 2\pi N < \frac{2^{4/3}}{27\pi}\frac{\gamma_b}{(n_b/n_0)^{1/3}}. \tag{10}$$

This is the peak transformer ratio we can obtain while still remaining stable to the two-stream instability. For typical parameters, this will be slightly less than the maximum transformer ratio from particle dephasing considerations discussed in Sec. VI ($R_d \lesssim 2\epsilon\gamma_b$, $\epsilon = eE/m\omega_p c$). Although the limit (10) can be very small for modest driving beams, for high-energy driving beams R is still quite large. For example, if $\gamma_b = 10^4$ and $n_b/n_0 = 0.1$, $R \approx 600$; while for $\gamma_b \sim 10$ and $n_b/n_0 = 0.01$, R is less than 1.

The actual transformer ratio obtained in modest energy simulations is higher than (10) since wake-field excitation can take place even in a regime where the two-stream instability begins to grow. However, for a high-energy accelerator where the beam must travel over long distances, the limitation imposed by (10) (or a few times this value if a few e foldings can be tolerated) will probably apply.

We have only considered waves parallel to $\mathbf{V}_b$. We note that for the two-stream instability at angles nearly perpendicular to the beam direction ($\hat{\mathbf{x}}$), the growth rate may be as much as $\gamma_b^{2/3}$ higher than that for the 1D instability (see the discussion towards the end of Sec. V), while $\mathbf{V}_g \cdot \hat{x}$ remains $2V_b/3$. The peak value of R corresponding to gradient stabilization of the oblique two-stream instability is then smaller than Eq. (10) by the factor $\gamma_b^{2/3}$. Thus, the longitudinal beam gradient is not strongly stabilizing for the oblique two-steam instability. On the other hand, the growth of nearly perpendicular waves may be limited by the finite radial extent of the driving beam.

V. TRANSVERSE DYNAMICS AND 2D SIMULATIONS

The one-dimensional wake-field analysis has been verified recently with 1D computer simulations.[7] For beams of finite radius one might expect that the 1D mechanism would be greatly altered by transverse plasma dynamics, particularly since the electric-field lines from a very relativistic charge in vacuum are primarily transverse.[10,11]

In order to model the PWA including transverse dynamics, a 2D electromagnetic particle-in-cell simulation code (WAVE)[12] was used. Typically, 10^5 particles were followed on a cartesian simulation grid $50c/\omega_p$ long by $10c/\omega_p$ wide (250×25 grids). The transverse driving beam profile was trapezoidal (flat for $2c/\omega_p$ in the center, falling linearly to zero in $1c/\omega_p$, on each side, occupying $4c/\omega_p$ in total). Both metallic and periodic transverse boundary conditions were used with little difference observed. The longitudinal driving beam profile was chosen to make comparisons to the 1D simulations of Chen *et al.*[7] Namely, $\rho_b(\xi) = 0.01 n_0$ for the first $\lambda_p/4$ and $\rho_b(\xi)$ rose linearly to $0.2n_0$ over the next $3\lambda_p$ [see Fig. 3(c)]; $\gamma_b = 7$.

Figure 3(a) shows a contour plot of the plasma potential indicating plane waves behind the driving bunch. The electric field down the axis is shown in Fig. 3(b) and the corresponding electric field from 1D simulation is shown in Fig. 3(c). Comparison of Figs. 3(b) and 3(c) reveals that the 2D wake field behind the driving bunch is nearly identical to that of the 1D case. (The decaying fields near the left and right edges are due to the simulation boundaries and should be ignored.) The transverse-beam fields did not significantly alter the wake-field generation mechanism which still arises primarily from the *longitudinal* displacement of plasma electrons.

Subsequent acceleration of particles injected behind the wake is depicted in the phase space plots of Fig. 4. A low-density beam of particles of energy $\gamma_b = 7$ was distributed uniformly in x and y behind the driving beam. Depending on their phase relative to the wake field, the trailing particles were either decelerated or accelerated. The maximum energies at time $90\omega_p^{-1}$ are near $\gamma \approx 30$ (consistent with $\Delta\gamma mc^2 = eE_+\Delta x$, $eE_+ \approx 0.26 mc\omega_p$, $\Delta x \approx 90c/\omega_p$).

Besides verifying the 1D analysis, the 2D simulations provide insight into important transverse effects both in and behind the driving beam. First, we discuss the focus-

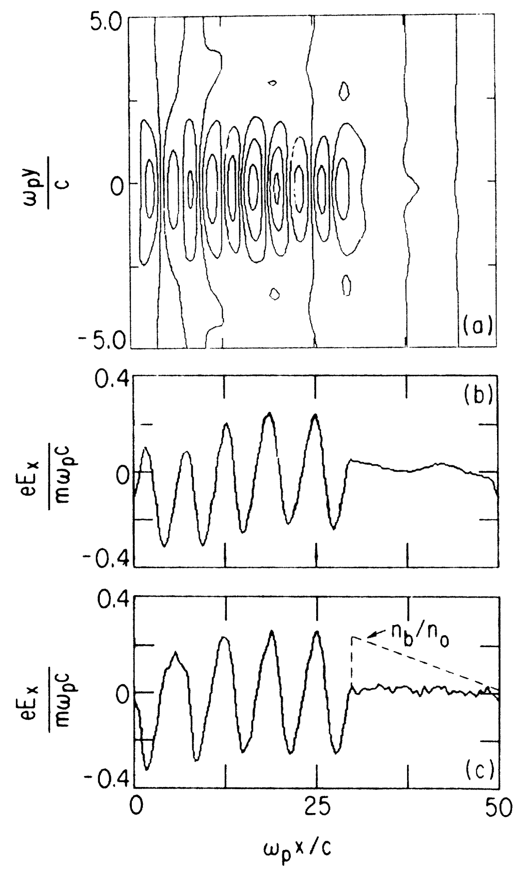

FIG. 3. Plasma wakes in simulations. (a) Contours of equipotential in 2D simulation; (b) electric-field slice down the $y=0$ axis of (a); (c) electric field from a 1D simulation corresponding to (b). For all cases $t=30\omega_p^{-1}$, bunch density was $\rho_b(\xi)=0.01n_0$ for the first $\lambda_p/4$ and rose to $0.2n_0$ over the next $3\lambda_p$ [as indicated in Fig. 3(a)]. For the 2D cases, the bunch width was trapezoidal over $y=-2c/\omega_p$ to $+2c/\omega_p$.

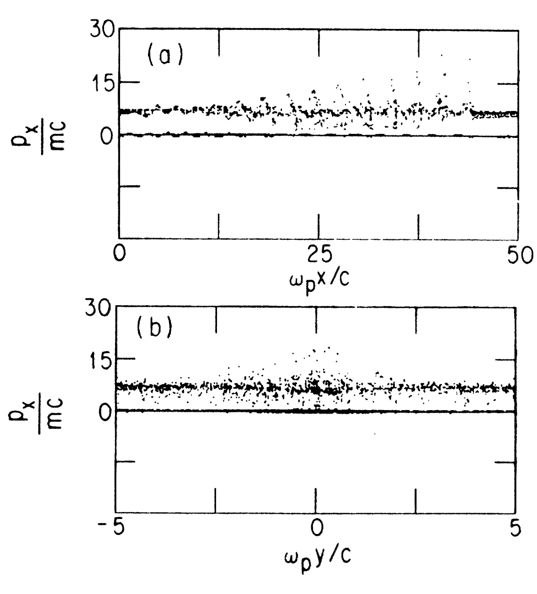

FIG. 4. Phase space of accelerated particles in 2D simulations of Fig. 3 at $t=90\omega_p^{-1}$. (a) p_x vs x, (b) p_y vs y. Particles were initially uniformly distributed in space behind the driving bunch with energy $\gamma=7=\gamma_b$ and density $0.005n_b$.

ing fields associated with the finite-width plasma wave behind the beam. Then, we address the evolution of the transverse driving beam envelope.

The focusing/defocusing fields associated with longitudinal plasma waves of finite width arise simply because electric fields point toward the density compression of the plasma waves, as depicted in Fig. 5. Focusing fields produced by driving beams of parabolic transverse profile have been discussed by Ruth *et al.*[5] Gaussian driving profiles have been considered by Fedele *et al.*[13] for the beat-wave accelerator.

Consider a plasma-density wave of finite cross section similar to that depicted in Fig. 5. In particular, take the transverse profile to be parabolic out to radius a, i.e.,

$$n_1(x,y,t,)\propto(1-y^2/a^2)\sin[k_p(x-ct)] \quad (2D) .$$

If a is large compared to $1/k_p(c/\omega_p)$, then the potential ϕ will be of the same functional dependence as n_1 [since $\nabla^2\phi=(\partial_x^2+\partial_y^2)\phi=-4\pi e n_1(x,y,t)$ and $\partial_y^2\sim O(1/a^2) \ll \partial_x^2 \sim O(k_p^2)$].

The longitudinal and transverse plasma-wave fields are simply given by $\mathbf{E}=-\nabla\phi$, so

$$E_x=E_{x0}(1-y^2/a^2)\cos(k_p\xi) , \quad (11a)$$

$$E_y=-2E_{x0}(y/k_p a^2)\sin(k_p\xi) , \quad (11b)$$

where E_{x0} is a constant representing the longitudinal field amplitude on axis.

From Eqs. (11) and Fig. 5, it is apparent that half of a plasma wave's accelerating phase is focusing, while the other half is defocusing to trapped particles. For an eventual accelerator, it will probably be necessary to avoid the regions of defocusing. Means for accomplishing this are discussed in Sec. VI.

The focusing/defocusing fields predicted by Eq. (11) can be quite large for wave widths comparable to the plasma wavelength and realistic plasma-wave parameters. For example, a 10% density wave in a 10^{18} cm^{-3} density plasma produces a longitudinal field E_{x0} of order 10^8 V/cm at a wavelength of 30 μ. If the wave is one plasma

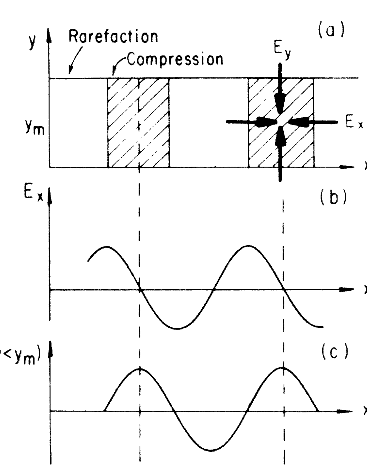

FIG. 5. In finite-width plasma waves, electric fields point toward the density compressions as shown in (a) giving rise to the longitudinal and radial fields shown in (b) and (c), respectively.

wavelength wide, Eq. (11) predicts a focusing field equivalent to a quadrupole magnetic field of 10 MG/cm.

Although our beam profile in the simulations was trapezoidal rather than parabolic, we can make an approximate comparison between the prediction of Eqs. (11) and the simulation results. In Fig. 6, we have plotted the linear prediction of E_y versus y from Eq. (11) on the simulation results. In determining the slope, we have used $k_p a=2$, $E_{x0}=0.26 m\omega_p c/e$ [from Fig. 7(a)], and a value of x for the slice plot such that $\sin k_p(x-ct)\approx 1$ in Eq. (11).

We turn now from the plasma waves to the driving beam, and its transverse profile evolution. By partially neutralizing the space charge of the beam, the background plasma enables the beam to be self-trapped by its own azimuthal magnetic field. Without some self-trapping, the space-charge repulsion and nonzero emittance of real driving beams would cause them to diverge in a distance far too short to make an effective accelerator. The self-focusing of a particle beam by this mechanism is also of interest for conventional accelerators. Higher luminosity might be obtained merely by designing the final stage of a collider to pass through a *passive* plasma; no plasma waves or applied currents would be necessary.

The fields acting on the driving bunch (a slice at $x=42c/\omega_p$, $t=45\omega_p^{-1}$) of the 2D simulation are shown in Fig. 8. Figure 8(a) is the retarding field E_- slowing the driver beam; comparison to Fig. 3(b) suggests a transformer ration E_+/E_- of about 10 ($\approx\pi N$). Figures 8(b) and 8(c) show the space charge field E_y and the "azimuthal" (in 2D Cartesian geometry) magnetic field B_z due to the bunch current. Note that E_y is nearly an order of magnitude smaller than B_z, substantiating the argument that the plasma shields the beam's space-charge repulsion.

In Fig. 9, we see that the profile of the driving beam at times 0, $45\omega_p^{-1}$, and $90\omega_p^{-1}$ shows a definite narrowing trend. Self-focusing can also be inferred from a slight increase (a few percent from $t=30\omega_p^{-1}$ to $45\omega_p^{-1}$) of the wake-field amplitude behind the bunch. This is probably due to an increase in beam density on axis caused by the beam self-focusing. The modulation of axial beam density leads to a slight oscillation of the decelerating field E_- inside the beam which does not appear in 1D [see Figs. 3(b) and 3(c)].

The beam self-focusing is often associated with the Weibel or filamentation instability.[14] This instability

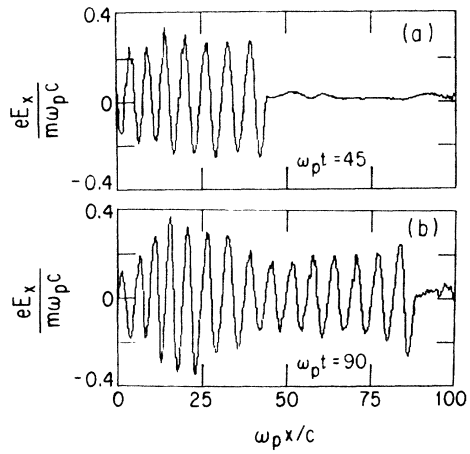

FIG. 7. Slice plots of 2D wake fields (E_x vs x at $y=0$) at (a) $t=45\omega_p^{-1}$ and (b) $t=90\omega_p^{-1}$ showing changes in the wake-field amplitude.

arises physically because two parallel currents attract. The growth rate of the Weibel instability is[15]

$$\nu/\omega_p=(n_b/\gamma_b n_0)^{1/2} kc(k^2c^2+\omega_p^2)^{-1/2}$$

which is roughly 0.1 for our simulation parameters. However, the presence of the true Weibel instability would be indicated by a growth of B_z perturbations, and these were not evident in the simulations [see Fig. 8(c)]. Instead, modulations are evident in the E_y versus y plot of Fig. 8(b). These may be indicative of the electrostatic two-stream instability propagating obliquely to the beam

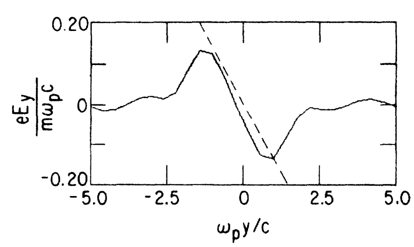

FIG. 6. Focusing field (E_y vs y) from the 2D computer simulations ($t=45\omega_p^{-1}, x=41.7c/\omega_p$) and the corresponding linear prediction from Eq. (11).

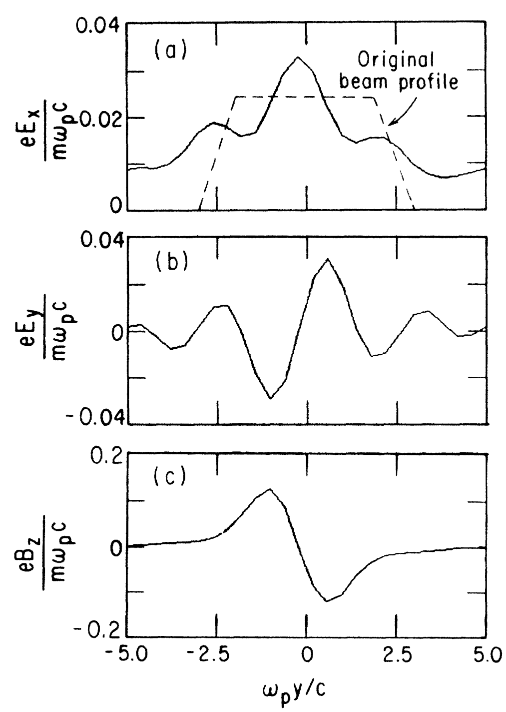

FIG. 8. Fields in the driving bunch ($t=45\omega_p^{-1}$, $x=41.7c/\omega_p$). (a) E_x vs y, (b) E_y vs y, (c) B_z vs y.

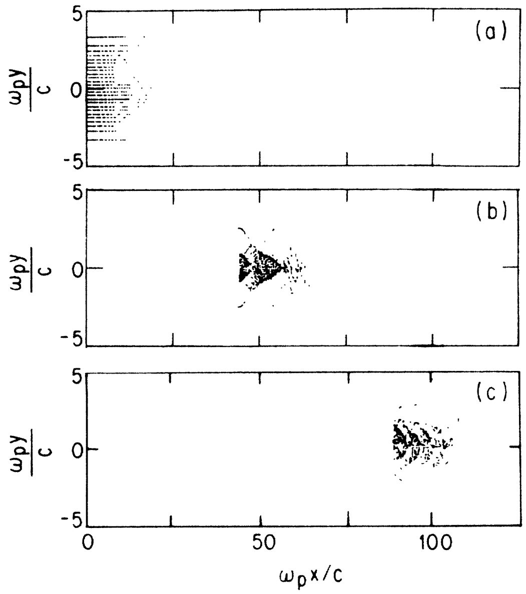

FIG. 9. Real space (y vs x) of driving beam particles at (a) $t=0$, (b) $t=45\omega_p^{-1}$, (c) $t=90\omega_p^{-1}$ in the 2D simulations.

direction. This instability has a growth rate at an angle θ to the beam of[15]

$$\nu/\omega_p = (\sqrt{3}/2)^{4/3}(n_b/2\gamma_b n_0)^{1/3}(\sin^2\theta + \cos^2\theta/\gamma_b^2)^{1/3} .$$

From Fig. 8(b), $k_y \approx 2.2\omega_p/c$, suggesting that $\tan\theta = k_y/k_x \approx 2.2$ and $\nu/\omega_p \approx 0.2$. The time of Fig. 8(b) ($50\omega_p^{-1}$) corresponds to about 10 e foldings of the instability.

As the beam self-focuses, its perpendicular temperature rises. To what extent the self-focusing can offset the natural divergence of the beams and what will be the final beam profile are important questions that require further investigation.

VI. DEPHASING OF TRAPPED PARTICLES

Since accelerated particles move at nearly c and the accelerating waves move at $V_b < c$, the particles outrun the waves. As discussed in Sec. V, only one-fourth of a plasma wave is both focusing and accelerating, so the effective accelerator length is determined by the distance it takes the particles to phase slip by $\lambda_p/4$. In the following, we derive a revised transformer ratio which takes into account both dephasing and the considerations of Sec. II. We also present a means for increasing the effective dephasing length by tailoring the plasma density.

A particle moving at nearly c will slip past a wave moving at $V_{ph} = V_b(x)$ by an incremental amount ds in a time dt or a distance $dx \approx c\,dt$ given by

$$dx = \frac{c\,ds}{c - V_b(x)} \approx 2\gamma_b^2(x)ds . \qquad (12)$$

Now γ_b is decreasing according to $\gamma_b(x) = \gamma_b(1 - x/L_R)$, where L_R is determined by the decelerating field (E_-) acting on the bunch ($eE_- L_R = \gamma_b mc^2$). Dividing both sides of Eq. (12) by $\gamma_b^2(x)$ and integrating determines the effective acceleration length L corresponding to phase slippage by $\lambda p/4$:

$$\int_0^{\lambda_p/4} ds = \int_0^L \frac{dx}{2\gamma_b^2(x)} = \frac{1}{2\gamma_b^2}\int_0^L \frac{dx}{(1-x/L_R)^2} .$$

Evaluating the integral, we obtain for L,

$$L = \frac{\pi\gamma_b^2 c/\omega_p}{1+\epsilon\pi\gamma_b/R} ,$$

where we have substituted $L_R = \gamma_b mc^2/eE_-$, $\lambda_p = 2\pi c/\omega_p$, $R = E_+/E_-$, and $\epsilon = eE_+/m\omega_p c$ is the normalized plasma wave amplitude ($\epsilon \approx n_b/n_0 < 1$).

The maximum energy that an accelerated particle can gain is $\Delta\gamma mc^2 \approx e\langle E_+\rangle L$ (where the brackets denote the average accelerating field on the particle) or

$$\Delta\gamma mc^2 \approx \frac{\pi\langle\epsilon\rangle\gamma_b^2 mc^2}{1+\pi\epsilon\gamma_b/R} = R\gamma_b mc^2 \frac{\pi\langle\epsilon\rangle\gamma_b}{R+\pi\epsilon\gamma_b} . \qquad (13)$$

Alternatively, we can express (13) in terms of a revised transformer ratio R_d which includes dephasing,

$$R_d \equiv \frac{\Delta\gamma}{\gamma_b} = R\frac{\pi\langle\epsilon\rangle\gamma_b}{R+\pi\epsilon\gamma_b} . \qquad (14)$$

For high-γ driving beams, such that $\pi\epsilon\gamma_b \gg R$, the phase slippage is small until the beam is nearly depleted. In this case, the particle can stay at the peak accelerating field for most of the accelerator (so $\langle\epsilon\rangle \approx \epsilon$) and R_d approaches the value R ($\approx 2\pi N\alpha$) given in Sec. II. For example, if $\gamma_b \approx 10^4$ and ϵ is of order 0.1, then dephasing is negligible for R less than 1000 or energy gains less than 5 TeV.

Although driving electron bunches of very high energy avoid the dephasing problem, lower energy driving electrons may be desirable because they can be produced more efficiently. Alternatively, one could even use proton beams. In these cases, the dephasing can dominate ($\pi\epsilon\gamma_b \ll R$) and R_d approaches the value $\pi\langle\epsilon\rangle\gamma_b$, where $\langle\epsilon\rangle \approx \epsilon/\sqrt{2}$. The corresponding limit on energy gain is roughly $R_d\gamma_b mc^2 \approx 2\epsilon\gamma_b^2 mc^2$, which is completely analogous to the BWA limit given by Tajima and Dawson[3] (with replacement of γ_b by $\omega^{\text{laser}}/\omega_p \approx \gamma_{ph}$).

Particles in the beat-wave accelerator can be phase locked in regions which are both accelerating and focusing by imposing a dc magnetic field (the surfatron scheme).[16] Since the magnetic field would influence the driving beam, as well as the trapped particles, the surfatron does not appear to be applicable to wake-field accelerators. However, there is another way to maintain PWA particles' phase in the waves which is not possible with the resonant beat-wave scheme.

Consider the wake fields generated by a bunch moving up a plasma density gradient. If we assume $R \gg \pi\epsilon\gamma_b$ so that the beam speed can be considered constant, then the wake phase velocity is fixed at $V_b \approx c$. Its wavelength will decrease according to

$$\lambda_p \approx 2\pi c/\omega_p(x) \propto n_0^{-1/2}(x)$$

where

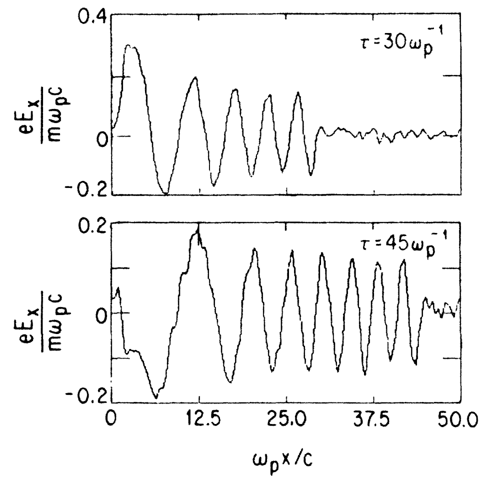

FIG. 10. One-dimensional simulations of wake fields produced in a linear plasma-density gradient at (a) $t=30\omega_p^{-1}$ and (b) $t=45\omega_p^{-1}$. The plasma density at the right boundary is four times that at the left boundary (n_0); the driving bunch was a doorstep rising to $0.2n_0$ in $1\lambda_p$.

$$\omega_p^2(x)=4\pi n_0(x)e^2/m \ .$$

As the driving beam progresses up the density gradient, the plasma wakes will appear to catch up to the tail of the beam (see Fig. 10). Highly relativistic particles ($V \approx c$) that are trapped "w" wavelengths behind the bunch will stay in the same phase of their trapping potential if the rate of advance of the particles toward the driving beam ($c-V_b$) matches the rate of advance of the wake

$$\frac{w\Delta\lambda}{\Delta t}=wc\frac{\Delta\lambda}{\Delta x}=wc\frac{\partial\lambda}{\partial n_0}\frac{\Delta n_0}{\Delta x}$$

or if the density gradient satisfies

$$\left[\frac{dn}{n(dx)}\right]^{-1}=w\lambda_p\gamma_b^2 \ . \tag{15}$$

In Fig. 10, we show the wakefield produced in a 1D simulation with a density gradient increasing to the right $[n_0(x=50)=4n_0(x=0)]$. At time $45\omega_p^{-1}$, the distance from the wth peak of the wake field to the tail of the driving beam is less than its corresponding distance at $t=30\omega_p^{-1}$. The wake appears to be catching up to the beam (although the phase velocity at fixed x is still V_b). A decrease in the accelerating wake field at higher plasma densities follows from Poisson's Eq. (2):

$$E=(m\omega_p c/e)(n_1/n_0)\approx(m\omega_p c/e)(n_b/n_0)$$
$$\approx(n_b/n_0)(n_0)^{1/2}\ \text{V/cm} \ , \tag{16}$$

which is proportional to $n_0^{-1/2}$ for fixed n_b ($<n_0$).

VII. SUMMARY

Slowly ramped and sharply cutoff driving beams can produce high-phase-velocity high-gradient-plasma waves which can trap and accelerate charged particles. The acceleration gradient and maximum energy gain of this plasma wake-field accelerator can be summarized by the following two equations:

$$E\approx(n_b/n_0)(n_0)^{1/2}\ \text{V/cm} \quad (n_b<n_0) \ , \tag{16'}$$

$$\Delta\gamma \lesssim R_d\gamma_b=R\gamma_b\frac{\pi\epsilon\gamma_b}{R+\pi\epsilon\gamma_b} \ , \tag{13'}$$

where $R\leq 2\pi N=\omega_p l_b/c$, l_b is the driving bunch length, n_b is the peak density, γ_b is its energy, and $\epsilon\approx n_b/n_0$.

For example, these equations predict that a ramped driving bunch consisting of 5×10^{10} 50-GeV electrons 3 mm long and 20 μm in radius sent into a 10^{18} cm^{-3} plasma could accelerate a properly phased trailing bunch to 1 TeV in 20 m.

Implicit in these expressions are a number of assumptions or requirements considered in this paper. We have shown that for physically realizable bunch shapes, a transformer ratio between πN and $2\pi N$ can be expected if the bunch is cutoff in a distance shorter than c/ω_p. The above equations neglect competing instabilities such as the plasma two-stream instability. We found the parallel two-stream instability to be gradient stabilized for transformer ratios smaller than that given by Eq. (10). For the 1 TeV example, the transformation ratio is 20, well below the maximum of 3700 given by Eq. (10). Thus, one would not expect this instability to occur in such a high-γ case.

The design equations were derived from a 1D model which from our 2D simulations appears to be justified for beam widths greater than c/ω_p. Our 2D simulations substantiate the 1D models at early times. At later times, the simulations indicate that transverse motion can distort the driving beam and modify the wake fields, both in and behind the bunch. To what extent this would occur in the 1 TeV accelerator example is not clear because limitations on computing time preclude simulations of such length (order 10^6 c/ω_p). We note that both γ_b and the acceleration length scale up by a factor of 10^4 in comparison to the simulations we have performed so that bulk focusing of the stiffer high-γ beam may be qualitatively similar to the simulations. Recently, Su at UCLA has developed a 1D simulation code which follows the driving beam and enables modeling of more realistic parameters. Energy gains from 70 MeV to 1 GeV have been simulated and are in agreement with the analytic model.[17]

We have proposed a plasma density gradient (15) as a means of avoiding limitations imposed by the dephasing of accelerated particles (13). This scheme may enable the use of proton beams or moderate energy electron beams as the driving source.

Accelerator issues such as luminosity and emittance are the subjects of continuing investigation for all plasma-wave schemes[18] and have not been addressed in this paper. Among plasma schemes, the wake-field accelerator is particularly attractive because of its comparatively efficient free-energy source and because it does not require the plasma density to be fine tuned to a resonant frequency. Further investigation of transverse driving beam stability and particle loading appears key to realizing the promise of a useful high-gradient high-energy plasma wake-field accelerator.

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