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Dispersive optical bistability: Stability of the steady states

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We study the semiclassical mean-field theory of single-mode dispersive optical bistability in the limit of a fully developed hysteresis cycle (i.e., in the limit of large values of the bistability parameter C) and when all decay rates have comparable magnitude. We locate (i) on the upper branch instabilities leading to time periodic regimes, (ii) on the lower branch instabilities leading to a switching before the limit point is reached. None of these instabilities are observed in the case of purely absorptive optical bistability.

I. INTRODUCTION

Since the first observation of optical bistability (OB) in 1976 (Ref. 1) a large number of investigations have appeared in the literature.[2] They were motivated either by the potential applications of OB to optical signal processing[3] or by the large variety of new states emerging from instabilities which can be predicted and are found experimentally. This paper deals with the stability properties of steady states in single-mode OB. We consider a homogeneously broadened nonlinear passive medium modeled by two-level atom. We retain nonlinear absorptive and dispersive processes. In the case of purely absorptive OB three main limits have been investigated analytically:

(i) The good-cavity limit[4] in which all atomic variables are adiabatically eliminated and the system is governed by a differential equation for the electric field.

(ii) The bad-cavity limit[5] in which the field and the atomic polarization are adiabatically eliminated and the system is governed by a differential equation for the atomic population.

(iii) The intermediate case[6] where all decay rates are of the same order of magnitude but the bistability parameter is very large.

The emphasis on absorptive OB was justified because it provided a first approach to a set of equations which cannot be analyzed in full generality. In the single-mode limit of absorptive OB, however, linear stability analyses did not show instabilities of the steady states. Hence the focus was mainly on temporal evolution properties. By contrast, a wealth of instabilities have been found in dispersive OB. In the single-mode mean-field limit, instabilities to small perturbations of the steady states were found only on the high transmission branch.[7] For the lower transmission branch the only reported instability was anomalous switching[8,9] which requires large perturbations of the steady state.

The purpose of this paper is to extend to the dispersive case our previous analysis of absorptive OB.[6] To this end we shall consider the asymptotic limit of large values of the bistability parameter C, corresponding to a fully developed hysteresis. Our analysis of the dispersive OB equations in this limit is motivated by the numerical results which have been obtained for very large values of C.[7] However, in contrast to these studies, we consider arbitrary $O(1)$ quantities for the cavity and atomic relaxation constants as well as the cavity and atomic detunings.

In Sec. II, we present the mean-field equations for OB in a ring cavity. Sections III and IV analyze the cases of absorptive and dispersive OB, respectively. Section V summarizes the main results.

II. FORMULATION

An ensemble of homogeneously broadened two-level atoms in a ring cavity is excited by a monochromatic external field of amplitude y. In the single-mode case, the mean-field model for dispersive OB is obtained from the Maxwell-Bloch equations and describes the evolution of the complex cavity field x, the atomic polarization p, and the population difference d. They satisfy the following system of ordinary differential equations:[7]

$$\begin{aligned} x_t &= y-(1+i\theta)x-2Cp \;, \\ p_t &= d_\perp[-(1+i\Delta)p+xd] \;, \\ d_t &= d_{\parallel}[1-d-\tfrac{1}{2}(p^*x+px^*)] \;, \end{aligned} \tag{2.1}$$

where $f_t \equiv df/dt$ and t is a dimensionless time scaled by the cavity relaxation constant κ. $d_\perp \equiv \gamma_\perp/\kappa$ and $d_{\parallel} \equiv \gamma_{\parallel}/\kappa$ are defined as the ratios of the transversal and the longitudinal atomic decay rates to the cavity relaxation constant, respectively. Δ and θ are the atomic and cavity mistunings, respectively.

The steady state-solutions of (2.1) are

$$p=\frac{(1-i\Delta)x}{1+\Delta^2+|x|^2}, \quad d=\frac{1+\Delta^2}{1+\Delta^2+|x|^2} \tag{2.2}$$

and

$$y^2 = |x|^2 \left[\left[1 + \frac{2C}{1+\Delta^2+|x|^2} \right]^2 + \left[\theta - \frac{2C\Delta}{1+\Delta^2+|x|^2} \right]^2 \right] . \quad (2.3)$$

From (2.3), we can analyze the condition for multiple steady states.[10] As $C \to \infty$, the lower and upper transmission branches and their limit points are characterized by the following orders of magnitude:

Lower transmission branch

$$x = O(1),\ p = O(1),\ d = O(1) \text{ and } y = O(C) . \quad (2.4)$$

Upper transmission branch

$$x = O(C^{1/2}),\ p = O(C^{-1/2}) ,$$
$$d = O(C^{-1}),\ y = O(C^{1/2}) . \quad (2.5)$$

The purpose of our asymptotic analysis is to describe the time-dependent response of the optical system when it is initially near either the lower branch or the upper branch. Thus, we shall take into account the scales (2.4) or (2.5) and propose a multitime perturbation analysis of Eqs. (2.1). Specifically, we seek a solution of (2.1) satisfying either (2.4) or (2.5) which is a function of two independent time-variables T and t. T is the fast time defined by

$$T \equiv C^{1/2} \int_0^t \omega(s) ds , \quad (2.6)$$

where $\omega = \omega(t)$ must be determined by the perturbation procedure. The $C^{1/2}$ scale for T is suggested by our previous study of the absorptive case ($\theta = \Delta = 0$) (Ref. 6) and was justified by the results of the linear stability analysis of the steady states. In order to contrast the differences between the absorptive and dispersive cases, we first summarize the results obtained for absorptive OB and then analyze the properties of the dispersive system.

III. ABSORPTIVE OB

In this section, we analyze Eqs. (2.1) with $\theta = \Delta = 0$. If at $t = 0$, the optically bistable system is near the lower or upper branch of the steady states, it is possible to determine the approximate solution of the time-dependent problem. We summarize the principal results.

A. The lower branch

Motivated by (2.4), we first assume

$$y = C[Y + O(C^{-1})] \quad (3.1)$$

and seek a solution of (2.1) by expanding x, p, and d in power series of $C^{-1/2}$. Using (2.6), the perturbation analysis leads to the following expressions for x, p, and d:

$$x = (\alpha e^{iT} + \beta e^{-iT}) + \frac{d_\perp Y}{\omega^2} + O(C^{-1/2}) , \quad (3.2)$$

$$p = \frac{Y}{2} - C^{-1/2}\frac{\omega}{2}(i\alpha e^{iT} - i\beta e^{-iT}) + O(C^{-1}) , \quad (3.3)$$

$$d = d_0 + C^{-1/2}\left[\frac{d_\parallel Y}{4\omega}[i(\alpha+\beta^*)e^{iT} - i(\alpha^*+\beta)e^{-iT}] + d_1 \right] + O(C^{-1}) , \quad (3.4)$$

where the time-dependent frequency ω of the oscillations is given by

$$\omega = (2d_\perp d_0)^{1/2} . \quad (3.5)$$

The real functions $d_0(t)$, $d_1(t)$ and the complex functions $\alpha(t)$, $\beta(t)$ satisfy the following equations obtained from the solvability conditions

$$d_{0t} = d_\parallel \left[-d_0 + 1 - \frac{Y^2}{4d_0} \right] , \quad (3.6)$$

$$d_{1t} = d_\parallel \left[-1 + \frac{Y^2}{4d_0^2} \right] d_1 , \quad (3.7)$$

$$\alpha_t = -\alpha \left[\frac{1+d_\perp}{2} + \frac{d_{0t}}{4d_0} + \frac{d_\parallel Y^2}{16 d_0^2} \right] - \frac{d_\parallel Y^2}{16 d_0^2}\beta^* + \frac{i d_\perp}{\omega}\alpha d_1 , \quad (3.8)$$

$$\beta_t = -\beta \left[\frac{1+d_\perp}{2} + \frac{d_{0t}}{4d_0} + \frac{d_\parallel Y^2}{16 d_0^2} \right] - \frac{d_\parallel Y^2}{16 d_0^2}\alpha^* - \frac{i d_\perp}{\omega}\beta d_1 . \quad (3.9)$$

Equation (3.6) admits two steady states given by

$$d_0 = d_\pm = \tfrac{1}{2} \pm \tfrac{1}{2}(1+Y^2)^{1/2} (Y \le 1) , \quad (3.10)$$

where $d_+ (d_-)$ corresponds to the lower (upper) transmission branch of x. If $Y < 1$ and $d_0(0) > d_-$, Eq. (3.6) describes the approach to the stable steady state $d_0 = d_+$. Then as $t \to \infty$, we note from (3.7) that $d_1 \to 0$ and a study of (3.8) and (3.9) indicates that $|\alpha| \to 0$, $|\beta| \to 0$. Therefore, the solution described by (3.2)—(3.5) represents rapid oscillations about a stable steady state which slowly decays on the time scale t. On the other hand, if $Y > 1$ or if $Y < 1$ but $d_0(0) < d_-$, the amplitude of the solution increases as $t \to \infty$ and the solution (3.2)—(3.5) describes the first stage of the jump to the upper transmission branch. Assuming $\beta = \alpha^*$ (i.e., real x and p), we find the results described by (3.19)—(3.22) in Ref. 6.

B. The upper branch

Motivated by (2.5), we now assume that

$$y = C^{1/2}[Y + O(C^{-1})] \quad (3.11)$$

and seek a solution of (2.1) by expanding $X = C^{-1/2}x$, $P = C^{1/2}p$, and $D = Cd$ in power series of $C^{-1/2}$. Using (2.6), we obtain

$$x = C^{1/2}[X_0 + O(C^{-1/2})] , \quad (3.12)$$

$$p = C^{-1/2}[P_0 + O(C^{-1/2})] , \quad (3.13)$$

$$d = C^{-1}\{(\alpha e^{iT} + \text{c.c.}) + \frac{d_{\parallel}}{\omega^2}[d_\perp + \tfrac{1}{2}(P_{0t}X_0^* + P_{0t}^*X_0)] + O(C^{-1/2})\}\ , \tag{3.14}$$

where c.c. denotes complex conjugate and the frequency ω is now given by

$$\omega = (d_{\parallel} d_\perp X_0 X_0^*)^{1/2}\ . \tag{3.15}$$

The complex functions $P_0(t)$, $X_0(t) = re^{i\varphi}$ and $\rho(t) = |\alpha(t)|$ satisfy the following conditions:

$$P_0 = \tfrac{1}{2}(Y - X_0 - X_{0t})\ , \tag{3.16}$$

$$r_t = -\frac{2}{r} - r + Y\cos\varphi\ , \tag{3.17}$$

$$r^2\varphi_{tt} + \varphi_t[r^2(1+d_\perp) + (r^2)_t] + d_\perp Yr\sin\varphi = 0\ , \tag{3.18}$$

$$\rho_t = -\rho\left\{\frac{d_{\parallel}+d_\perp}{2} + \frac{1}{X_0X_0^*} + \frac{d_{\parallel}d_\perp}{4\omega} \times \left[\left[\frac{X_0}{\omega}\right]_t X_0^* + \left[\frac{X_0^*}{\omega}\right]_t X_0\right]\right\}\ . \tag{3.19}$$

From (3.17)—(3.19), we find that the steady-state solutions are given by

$$r = r_\pm = \frac{Y}{2} \pm \tfrac{1}{2}(Y^2 - 8)^{1/2},\quad \varphi = \rho = 0,\quad (Y > \sqrt{8})\ . \tag{3.20}$$

Then an analysis of the time-dependent equations shows that if $r(0) > r_-$ and $Y > \sqrt{8}$, $r \to r_+$, $\rho \to 0$, and $\varphi \to 0$ as $t \to \infty$. Otherwise [i.e., if $r(0) < r_-$ or $Y < \sqrt{8}$], the system jumps to the lower transmission state.

In contrast with the analysis of the following sections, we emphasize that in this absorptive case the only instabilities of the steady states are the two limit points.

IV. DISPERSIVE OB

We now consider the case $\Delta \neq 0$ and $\theta \neq 0$. We again assume that the optically bistable system is either near the lower or the upper branch of the steady states and analyze its time-dependent behavior by a multitime perturbation analysis. We first examine the lower branch.

A. The lower branch

Assuming (3.1), we seek a solution of (2.1) by expanding x, p, and d in power series of $C^{-1/2}$. After introducing these expansions into (2.1) and using (2.6), we find that p and d are still given by (3.3) and (3.4) while x slightly differs from (3.2) by a term involving Δ

$$x = (\alpha e^{iT} + \beta e^{-iT}) + \frac{(1+i\Delta)}{\omega^2} d_\perp Y + O(C^{-1/2})\ . \tag{4.1}$$

The expressions (4.1), (3.3), and (3.4) for x, p, and d depend on the real functions ω, d_0, d_1 which are defined by (3.5), (3.6), and (3.7), respectively. The major difference between the absorptive and dispersive problems appears in the equations for α and β. They are now given by

$$\alpha_t = \alpha(P + iQ) + \beta^*(R + iS) + \frac{id_\perp}{\omega}\alpha d_1\ , \tag{4.2}$$

$$\beta_t = (P + iQ) + \alpha^*(R + iS) - \frac{id_\perp}{\omega}\beta d_1\ , \tag{4.3}$$

where

$$P = -\left[\frac{(1+d_\perp)}{2} + \frac{d_{\parallel}Y^2}{16d_0^2} + \frac{d_{0t}}{4d_0}\right]\ , \tag{4.4}$$

$$Q = -\left[\frac{(\theta + d_\perp\Delta)}{2} + \frac{d_{\parallel}Y^2\Delta}{16d_0^2}\right]\ , \tag{4.5}$$

$$R = -\frac{d_{\parallel}Y^2}{16d_0^2},\quad S = -\frac{d_{\parallel}Y^2\Delta}{16d_0^2}\ . \tag{4.6}$$

If $\theta = \Delta = 0$, Eqs. (4.2) and (4.3) reduce to (3.8) and (3.9) and we know from the discussion in Sec. III that the steady state $\alpha = \beta = d_1 = 0$ is always stable. However, a different conclusion is possible if $\Delta \neq 0$, $\theta \neq 0$. We first observe from (3.7) that $d_1 \to 0$ as $t \to \infty$ if $d_0 \to d_+$ where d_+ is defined by (3.10). Thus, if d_0 approaches the stable steady state $d_0 = d_+$, $d_1 = 0$ and consequently Eqs. (4.2) and (4.3) reduce to two linear equations in α and β:

$$\alpha_t = \alpha(P + iQ) + \beta^*(R + iS)\ , \tag{4.7}$$

$$\beta_t = \beta(P + iQ) + \alpha^*(R + iS)\ , \tag{4.8}$$

where P, Q, R, and S are given by

$$P = -\left[\frac{(1+d_\perp)}{2} + \frac{d_{\parallel}Y^2}{16d_+^2}\right],\quad Q = -\left[\frac{(\theta + d_\perp\Delta)}{2} + \frac{d_{\parallel}Y^2\Delta}{16d_+^2}\right]\ , \tag{4.9}$$

$$R = -\frac{d_{\parallel}Y^2}{16d_+^2},\quad S = -\frac{d_{\parallel}Y^2\Delta}{16d_+^2}\ . \tag{4.10}$$

After rewriting (4.7), (4.8) while $\alpha = ae^{i\varphi}$ and $\beta = be^{i\mu}$, we observe that since $P < 0$ and if $a(0) \neq b(0)$, $|a^2(t) - b^2(t)| \to 0$ as $t \to \infty$. Thus, to determine the stability of the zero solution, it is sufficient to examine the case $a = b$. With $a = b$, (4.7) and (4.8) become

$$a_t = a(P + R\cos\Psi + S\sin\Psi)\ , \tag{4.11}$$

$$a\Psi_t = 2a(Q + S\cos\Psi - R\sin\Psi)\ , \tag{4.12}$$

where

$$\Psi \equiv \varphi + \mu\ . \tag{4.13}$$

Defining U and Ψ_0 by $U\cos\Psi_0 = S$ and $U\sin\Psi_0 = R$, these equations can be rewritten as

$$a_t = a(P + U\sin\xi)\ , \tag{4.14}$$

$$a\xi_t = 2a(Q + U\cos\xi)\ , \tag{4.15}$$

where

$$\xi \equiv \Psi + \Psi_0 \text{ and } U \equiv (R^2 + S^2)^{1/2}\ . \tag{4.16}$$

Assuming $a \neq 0$, the solution of Eq. (4.15) is either a steady-state or a time-dependent function of t depending on whether the condition

$$|Q/U| \le 1 \tag{4.17}$$

is satisfied or not. We thus consider two cases. If $|Q/U|>1$, we find from (4.14) that the amplitude a is given by

$$\begin{aligned} a(t) &= a(0)e^{Pt}\{Q+U\cos[\xi(t)]\}^{-1/2} \\ &= a(0)e^{Pt}(\xi_t/2)^{-1/2}, \end{aligned} \tag{4.18}$$

where $\xi(t)$ satisfies Eq. (4.15). Since $P<0$ and ξ_t is a time-periodic bounded function of t, $|a| \to 0$ exponentially as $t \to \infty$. If (4.17) is satisfied, Eq. (4.15) admits a stable steady state $\xi=\xi_s$ defined by

$$\xi=\xi_s \equiv \arccos\left[-\frac{Q}{U}\right], \ \sin(\xi_s)>0 . \tag{4.19}$$

Then from Eq. (4.14), we find that $|a| \to 0$ as $t \to \infty$ provided that

$$P+U\sin(\xi_s)<0 . \tag{4.20}$$

Using (4.19) and since $P<0$, (4.20) requires that

$$P+(U^2-Q^2)^{1/2}<0 \text{ or } P^2>U^2-Q^2 . \tag{4.21}$$

Using the definitions of P, U, and Q given by (4.9), (4.10), and (4.16), respectively, Eq. (4.21) becomes

$$\frac{Y^2 d_{\parallel}}{4d_+^2}[1+d_\perp(1+\Delta^2)+\theta\Delta] > -(d_\perp+1)^2-(\theta+\Delta d_\perp)^2 . \tag{4.22}$$

This condition is always verified when $\theta\Delta>0$. If θ and Δ have opposite signs, however, the inequality may be violated. This will be the case when

$$\begin{aligned} &\theta\Delta<0 , \\ &|\theta| > \frac{1+d_\perp(1+\Delta^2)}{|\Delta|} , \\ &Y^2 > Y_c^2 \equiv \frac{4\gamma d_{\parallel}}{(\gamma+d_{\parallel})^2} , \end{aligned} \tag{4.23}$$

where

$$\gamma = \frac{(1+d_\perp)^2+(\theta+\Delta d_\perp)^2}{-\Delta\theta-1-d_\perp(1+\Delta^2)} > 0$$

so that $Y_c^2<1$. Provided that (4.17) is satisfied, the conditions (4.23) define a domain in parameter space where the lower branch becomes unstable before the limit point. The nature of this bifurcation is not determined by the Eqs. (4.7) and (4.8) since they are linear. As an illustration of this instability we have solved numerically Eqs. (2.1) with $C=200$, $\theta=-4$, $\Delta=3$, $d_{\parallel}=1.3$, and $d_\perp=0.1$. These parameters were chosen in order to ensure that Y_c^2 is $O(1)$; in this case $Y_c^2=0.9954$. The limit point has coordinates $(y_M, |x_M|)=(213.46, 3.39)$. The initial conditions were taken as $x(t=0)=|x_s|$, $p(t=0)=|p_s|$, $d(t=0)=d_s$, where $|x_s|$, $|p_s|$, and d_s are the steady-state solutions. Figure 1 shows the time evolution of the intensity for $y=211.07$ corresponding to $|x_s|=2.89$. It displays an approach to a stable steady state on the lower branch via damped oscillations. Figure 2 shows the time evolution of the intensity for $y=211.18$ corresponding to $|x_s|=2.90$. In this case we observe a transition to the upper branch and the short time behavior ($t \le 10$) is characterized by amplified oscillations. The same type of instability occurs for all values of y larger than 211.18 up to the limit point.

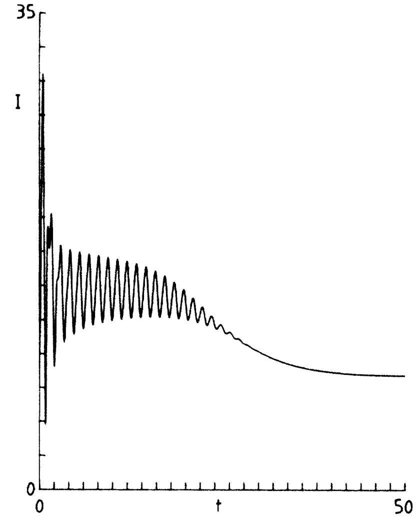

FIG. 1. Time evolution of the intensity for $y=211.07$, $C=200$, $\Delta=3$, $\theta=-4$, $d_\perp=0.1$, and $d_{\parallel}=1.3$. The final steady state is on the low transmission branch.

The instability described here is very different from the so-called anomalous switching found by Hopf *et al.*[8] and discussed analytically by Lugiato *et al.*[9] These authors have analyzed the stability of the lower branch when y is suddenly changed by a large amount, typically starting at

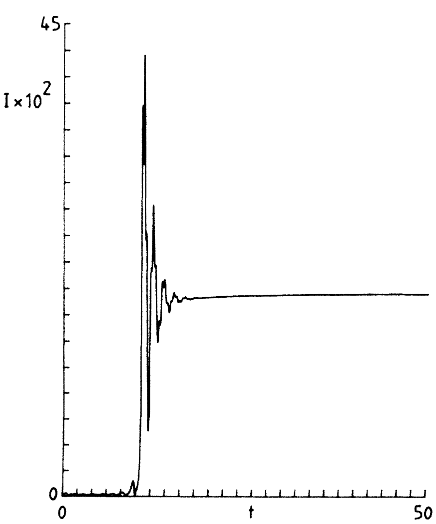

FIG. 2. Time evolution of the intensity for $y=211.18$; all other parameters as in Fig. 1. The final steady state is on the high transmission branch.

$y \simeq 0$ and ending in the bistable domain. On the contrary, the instability defined by (4.22) is of a more classical type corresponding to an instability to infinitesimal perturbations.

B. The upper branch

Assuming (3.11), we seek a solution of (2.1) by expanding $X = C^{-1/2}x$, $P = C^{1/2}p$, and $D = Cd$ in power series of $C^{-1/2}$. Using (2.6), we find that (3.12) and (3.13) remain unchanged while (3.14) involves an additional term proportional to Δ

$$d = C^{-1}\left[(\alpha e^{iT} + \text{c.c.}) + \frac{d_{\|}}{\omega^2}\left[d_\perp + \tfrac{1}{2}(P_{0t}X_0^* + P_{0t}X_0^*) + \frac{i\Delta}{2} d_\perp (P_0 X_0^* - P_0^* X_0)\right]\right] + O(C^{-1/2}) , \qquad (4.24)$$

where ω is given by (3.15). The complex functions $P_0(t)$, $X_0(t) = re^{i\varphi}$, and $\rho(t) = |\alpha(t)|$ which appear in the expressions of x, p, and d are obtained from the solvability conditions. Equations (3.17) and (3.19) remain unchanged but Eqs. (3.16) and (3.18) for P_0 and φ contain additional terms

$$P_0 = \tfrac{1}{2}[Y - (1+i\theta)X_0 - X_{0t}] , \qquad (4.25)$$

$$r^2\varphi_{tt} + \varphi_t[r^2(1+d_\perp) + (r^2)_t] + d_\perp rY\sin\varphi + d_\perp\theta r^2 - 2d_\perp\Delta + \frac{\theta}{2}(r^2)_t = 0 . \qquad (4.26)$$

Equations (3.17) and (4.26) describe the evolution of the amplitude r and the phase of the complex output field x. Since p is related to x by (4.25) and $\rho = |\alpha| \to 0$ as $t \to \infty$, the stability of the steady-state solutions will be determined from (3.17) and (4.26). The steady-state solution (r_s, φ_s) are

$$Y\cos\varphi_s = \left[\frac{2}{r_s} + r_s\right], \quad Y\sin\varphi_s = \left[\frac{2\Delta}{r_s} - \theta r_s\right] . \qquad (4.27)$$

Eliminating φ_s, we obtain

$$Y^2 = r_s^2\left[\left[\frac{2}{r_s^2} + 1\right]^2 + \left[\theta - 2\frac{\Delta}{r_s^2}\right]^2\right] \qquad (4.28)$$

which corresponds to (2.3) with $y = CY$ and $|x| = C^{1/2}r_s$. Equation (4.28) describes two branches of steady states which are connected by a limit point (r_c, Y_c)

$$r_c^2 = 2\left[\frac{1+\Delta^2}{1+\theta^2}\right]^{1/2} , \qquad (4.29a)$$

$$Y_c^2 = 4\{[(1+\Delta^2)(1+\theta^2)]^{1/2} + 1 - \theta\Delta\} , \qquad (4.29b)$$

and the steady states only exist if

$$Y \geq Y_c . \qquad (4.30)$$

The stability of (r_s, φ_s) is determined from the linearized theory. We obtain the following characteristic equation:

$$\omega^3 - T_1\omega^2 + T_2\omega - T_3 = 0 , \qquad (4.31)$$

where

$$T_1 = \frac{1}{r_s^2}[2 - r_s^2(1+d_\perp)] , \qquad (4.32)$$

$$T_2 = \frac{1}{r_s^2}[r_s^2(1+\theta^2+2d_\perp) - 2(1+\theta\Delta)] , \qquad (4.33)$$

$$T_3 = d_\perp\left[4\left[\frac{1+\Delta^2}{r_s^4}\right] - (1+\theta^2)\right] , \qquad (4.34)$$

and r_s is obtained from (4.28). The steady state is stable if $\text{Re}(\omega) < 0$ or if

$$T_3 < 0,\ T_1 < 0 \text{ and } T_1T_2 - T_3 < 0 . \qquad (4.35)$$

Since $T_3 > 0$ when $r_s < r_c$, we conclude that the second branch of steady states with $r_s < r_c$ is always unstable. When $r_s > r_c$, $T_3 < 0$ and the second or third condition must be violated for (r_s, φ_s) to be unstable. These two conditions require that

$$r_s^2 > 2/(2+d_\perp) \qquad (4.36)$$

and

$$-r_s^4[(1+d_\perp)^2 + \theta^2] + r_s^2[3(1+d_\perp) + \theta^2 + \theta\Delta(2+d_\perp)] - 2[1 + \theta\Delta + d_\perp(1+\Delta^2)] < 0 . \qquad (4.37)$$

We illustrate our stability results by considering the special case

$$\theta = \Delta . \qquad (4.38)$$

Then the steady states are stable if

$$r_s^2 > r_c^2 = 2 \qquad (4.39)$$

and

$$-r_s^4[(1+d_\perp)^2 + \Delta^2] + r_s^2[3(1+d_\perp) + \Delta^2(3+d_\perp)] - 2(1+d_\perp)(1+\Delta^2) < 0 . \qquad (4.40)$$

Note that condition (4.36) is always satisfied. In Fig. 3,

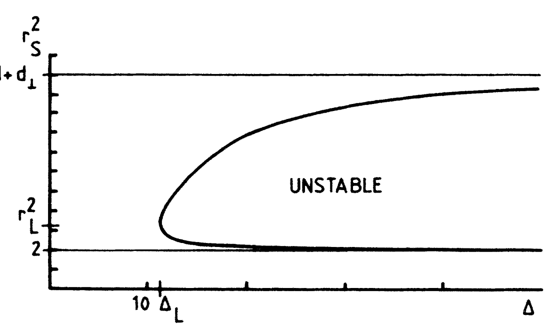

FIG. 3. Stability diagram r_s^2 vs Δ. The stability diagram is obtained by solving Eq. (4.40) with $\theta = \Delta$ and $d_\perp = 10$. r_s corresponds to the steady-state amplitude and is related to Y by (4.28). $r_s^2 = r_c^2 = 2$ represents the limit point of the upper branch $(r_s \geq r_c)$. We observe zero, one or two Hopf bifurcations points if $\Delta < \Delta_L$, $\Delta = \Delta_L$ or $\Delta > \Delta_L$, respectively. As $\Delta \to \infty$, the two Hopf bifurcation points approach the asymptotic limits $r_s^2 = 2$ and $1 + d_\perp$.

we represent the regions of stability and instability of the steady states when $d_\perp > 1$. The parabolic neutral stability curve is given by the roots of Eq. (4.40). It can be shown that they correspond to Hopf bifurcation points to time-periodic solutions. This suggests the existence of stable time-periodic solutions when the steady states are unstable.

V. SUMMARY AND DISCUSSION

We have analyzed the stability of the steady transmission states in the limit $C \to \infty$.

In the case of the lower branch of steady states, we have shown that the steady states are stable if conditions (4.17) and (4.22) are satisfied. Otherwise an instability leading to diverging oscillations appears. This suggests the existence of a Hopf bifurcation. However, since the amplitude equations are linear [Eqs. (4.7) and (4.8) for α and β], it is unclear if a bifurcation to stable time-periodic regimes is possible.

The case of the upper branch of steady states is more completely analyzed. We have shown that the steady states may become unstable and a Hopf bifurcation to time-periodic solutions is possible. This bifurcation can be studied from the nonlinear amplitude equations (3.17) and (4.26) for r and φ, respectively.

Another asymptotic approach of the OB equation has been proposed by Lugiato *et al.*[7] They first consider the limit $d_\perp \to \infty$ (or $d_\perp \gg C$), then they analyze numerically the simplified evolution equations when $C \to \infty$. However, the limit $d_\perp \to \infty$ of our results is singular [both frequencies (3.5) and (3.15) are proportional to $d_\perp$] which suggest that the simultaneous limit $d_\perp \to \infty$ and $C \to \infty$ may lead to different approximations depending on their relative orders of magnitude.

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