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Exact quantum-electrodynamics results for scattering, emission, and absorption from a Rydberg atom in a cavity with arbitrary Q

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Effect of cavity leakage on the emission and absorption of radiation from an atom in a cavity, with the cavity mode in vacuum, is studied exactly by solving density-matrix equations and by evaluating the relevant two-time quantum-mechanical correlation functions. Analytic expressions for the widths and position of various spectral lines are given. Widths are found to depend on detuning. Manifestation of the vacuum-field Rabi oscillations on different types of spectra are discussed. Vacuum-field Rabi oscillations are also apparent in the coherent scattering of external radiation from the cavity.

I. INTRODUCTION

Recently it has become possible to experimentally study[1–3] the novel features of the behavior of atoms contained in a cavity.[4–10] The dynamical behavior is determined by the relative magnitudes of the time scales κ^{-1} and g^{-1} associated, respectively, with the cavity field decay and with the vacuum-field Rabi oscillations. Most theoretical predictions deal with idealized situations. For example, the Jaynes-Cummings model is exactly soluble.[11] However, in the presence of cavity relaxation no exact solutions for various spectra appear to be known, although cavity relaxation is very important since studies on fluorescence, etc., can be carried out only if photons leak out. Thus it is desirable to have solutions for the Jaynes-Cummings model in the presence of cavity relaxation, so that the effect of the cavity Q on vacuum-field Rabi oscillations can be studied. The purpose of this paper is to present exact quantum-electrodynamic results for the behavior of an atom contained in a cavity with finite Q. Many manifestations of the vacuum-field coupling in a cavity will be presented.

The organization of this paper is as follows. In Sec. II we discuss the basic dynamical equations for the model. We show how closed sets of equations can be obtained for a certain set of density-matrix elements assuming that initially the cavity with finite Q is in vacuum state. The influence of Q on the vacuum-field Rabi oscillations is studied. Vacuum-field Rabi oscillations can be monitored using cw methods by considering the response of the atom contained in the cavity to an external field.[7] In Sec. III we study the absorption spectra, which in general have a doublet structure; the resolution of the doublet depends on the magnitude of the detuning and cavity Q in relation to the field-atom coupling g. In Sec. IV we examine the effect of cavity relaxation on the transient spontaneous spectra. Section V is devoted to the coherent scattering from the atom in the cavity. The intensity of the coherent scattering can be used to study the characteristic features of the vacuum-field Rabi oscillations. Since our analysis is valid for arbitrary Q it is possible to study the dynamics of the atom in the cavity in various regimes.

II. MODEL AND SOLUTION FOR DENSITY-MATRIX ELEMENTS

We consider the solution for the Jaynes-Cummings model in the presence of cavity-relaxation effects. This model consists of the interaction of a single two-level system with frequency ω_0 with a single mode of the radiation field of frequency ω in a cavity. In addition, we assume that the field can decay at the rate κ. The dynamical equation for the density matrix of the combined system of field and atom is

$$\frac{\partial\rho}{\partial t}=-i[H,\rho]-\kappa(a^{\dagger}a\rho-2a\rho a^{\dagger}+\rho a^{\dagger}a)\equiv L\rho\ , \quad (2.1)$$

where, in terms of the spin-$\frac{1}{2}$ operators, H is

$$H=\hbar\omega_0 S^z+\hbar\omega a^{\dagger}a+\hbar g(a^{\dagger}S^{-}+\text{H.c.})\ . \quad (2.2)$$

The relaxation part of Eq. (2.1) (κ-dependent terms) can be derived by considering the interaction of the cavity mode with a heat bath consisting of infinity of modes into which the field leaks out. This interaction representing the leakage gives the cavity mode a finite width κ. Exact eigenstates and eigenvalues of H are well known,[12]

$$H\,|\,0,g\rangle=\hbar\left[\frac{-\omega_0}{2}\right]|\,0,g\rangle\;,$$

$$H\,|\,\psi_n\pm\rangle=\hbar\omega_n^{\pm}\,|\,\psi_n^{\pm}\rangle,\quad \omega_n^{\pm}=\omega(n+\tfrac{1}{2})\pm\frac{\Omega_{n\Delta}}{2}$$

$$\Omega_{n\Delta}^2=4g^2(n+1)+\Delta^2,\quad \Delta=\omega_0-\omega \tag{2.3}$$

$$|\,\psi_n^{\pm}\rangle=\begin{bmatrix}\cos\theta_n\\-\sin\theta_n\end{bmatrix}|\,n+1,g\rangle+\begin{bmatrix}\sin\theta_n\\ \cos\theta_n\end{bmatrix}|\,n,e\rangle\;,\qquad n=0,1,2,\ldots,\infty$$

$$\tan\theta_n=2g\sqrt{n+1}/(\Omega_{n\Delta}-\Delta)\;.$$

Here $|\,g\rangle, |\,e\rangle$ are the ground and excited states of the atom and $|\,n\rangle$ the Fock state of the field. The Hamiltonian H causes transitions between the states $|\,n,e\rangle$ and $|\,n+1,g\rangle$. Field and atom occupation numbers change at the same time. The relaxation in the cavity changes only the photon number. For example, if the initial state of the system is $|\,n,g\rangle$, then the system can be found in any of the states

$$|\,m,g\rangle, m=0,1,\ldots,n,\quad |\,p,e\rangle, p=0,1,\ldots,n-1\;.$$

The total number of states to be dealt with is $(2n+1)$ and thus one needs to work with $(2n+1)^2\times(2n+1)^2$ matrices. Thus with increasing n values one has to work with matrices of higher and higher dimensions. Simpler analytical solutions are possible for the initial state $|\,0,e\rangle$, i.e., for the empty-cavity case and when the atom is in the excited state. In this case the dressed states corresponding to $\psi_n^{\pm}, n\neq 0$ do not enter the dynamical equations.

For the initial state $|\,0,e\rangle$, the states to be considered are $|\,1,g\rangle, |\,0,g\rangle$. The density-matrix elements now satisfy

$$\langle 0,g\,|\dot\rho\,|\,0,e\rangle=i(\omega+\Delta)\langle 0,g\,|\rho\,|\,0,e\rangle+ig\langle 0,g\,|\rho\,|\,1,g\rangle\;, \tag{2.4}$$

$$\langle 0,g\,|\dot\rho\,|\,1,g\rangle=(i\omega-\kappa)\langle 0,g\,|\rho\,|\,1,g\rangle+ig\langle 0,g\,|\rho\,|\,0,e\rangle\;. \tag{2.5}$$

The induced dipole moment will be given by

$$\mathbf{P}=\mathrm{Tr}(\rho\mathbf{d})=\mathbf{d}_{eg}\sum_n\langle n,g\,|\rho\,|\,n,e\rangle+\text{c.c.}$$
$$=\mathbf{d}_{eg}\langle 0,g\,|\rho(t)\,|\,0,e\rangle+\text{c.c.}\;, \tag{2.6}$$

since the states $|\,n,e\rangle, n>0$ do not participate in transitions. The equations relevant for calculating population distributions are found to be

$$\left[\frac{d}{dt}+\begin{bmatrix}0 & +ig & -ig & 0\\ +ig & \kappa-i\Delta & 0 & -ig\\ -ig & 0 & +\kappa+i\Delta & +ig\\ 0 & -ig & +ig & +2\kappa\end{bmatrix}\right]\begin{bmatrix}\langle 0,e\,|\rho\,|\,0,e\rangle\\ \langle 1,g\,|\rho\,|\,0,e\rangle\\ \langle 0,e\,|\rho\,|\,1,g\rangle\\ \langle 1,g\,|\rho\,|\,1,g\rangle\end{bmatrix}=0\;. \tag{2.7}$$

These equations are solved by Laplace transforms (denoted by carets). The following results are obtained from (2.4) and (2.5):

$$\langle 0,g\,|\hat\rho\,|\,0,e\rangle=\frac{1}{(z_1-z_2)}\{[(z_1-i\omega+\kappa)\langle 0,g\,|\rho(0)\,|\,0,e\rangle+ig\langle 0,g\,|\rho(0)\,|\,1,g\rangle]e^{z_1t}$$
$$-[(z_2-i\omega+\kappa)\langle 0,g\,|\rho(0)\,|\,0,e\rangle+ig\langle 0,g\,|\rho(0)\,|\,1,g\rangle]e^{z_2t}\}\;, \tag{2.8}$$

$$\langle 0,g\,|\hat\rho\,|\,1,g\rangle=\frac{1}{(z_1-z_2)}\{[(z_1-i\omega-i\Delta)\langle 0,g\,|\rho(0)\,|\,1,g\rangle+ig\langle 0,g\,|\rho(0)\,|\,0,e\rangle]e^{z_1t}$$
$$-[(z_2-i\omega-i\Delta)\langle 0,g\,|\rho(0)\,|\,1,g\rangle+ig\langle 0,g\,|\rho(0)\,|\,0,e\rangle]e^{z_2t}\}\;, \tag{2.9}$$

$$z_{1,2}=i\left[\omega+\frac{\Delta}{2}\right]-\frac{\kappa}{2}\pm\tfrac{1}{2}(\kappa^2-\Delta^2-4g^2+2i\Delta\kappa)^{1/2}\;. \tag{2.10}$$

The Laplace transforms of the results following from (2.7) are given by the matrix relation

$$\hat\psi(z)=P^{-1}(z)\begin{bmatrix}z_0(z_0^2-\Delta^2+4g^2) & i\Delta(z_0^2-\Delta^2) & 2\Delta g\kappa & 2\Delta g z_0\\ i\Delta(z_0^2-\kappa^2) & z_0(z_0^2-\kappa^2) & -2ig\kappa z_0 & -2igz_0^2\\ 2g\Delta\kappa & -2ig\kappa z_0 & z_0(4g^2+\Delta^2+z_0^2) & \kappa[\Delta^2+z_0^2]\\ 2g\Delta z_0 & -2igz_0^2 & (\Delta^2+z_0^2)\kappa & z_0[\Delta^2+z_0^2]\end{bmatrix}\psi(0)\;,$$

$$z_0=z+\kappa,\quad \hat\psi_{\substack{1\\2}}=\langle 1,g\,|\hat\rho\,|\,0,e\rangle\pm\langle 0,e\,|\hat\rho\,|\,1,g\rangle\;, \tag{2.11}$$

$$\hat\psi_{\substack{3\\4}}=\langle 0,e\,|\hat\rho\,|\,0,e\rangle\pm\langle 1,g\,|\hat\rho\,|\,1,g\rangle\;,$$

where the polynomial $P(z)$ is

$$P(z)=(z+\kappa)^4+(z+\kappa)^2(\Delta^2+4g^2-\kappa^2)-\Delta^2\kappa^2 . \tag{2.12}$$

Above results (2.8)–(2.12) will determine the dynamical aspects of the radiation matter interaction in a cavity with finite Q. Equations (2.8)–(2.10) will be useful in our considerations of the absorption and emission spectra, whereas (2.11) will yield transition probabilities. The roots of (2.12) govern the time dependence of the transition probabilities, whereas (2.10) will yield the positions and widths of various lines in the spectra. From (2.10), the following limiting cases are obtained.

(1) Exact resonance $\Delta=\omega_0-\omega=0$,

$$z_{1,2}=i\omega-\frac{\kappa}{2}\pm\tfrac{1}{2}(\kappa^2-4g^2)^{1/2} , \tag{2.13}$$

which for the bad-cavity case $\kappa^2 \gg 4g^2$ becomes approximately equal to

$$i\omega-\kappa+O(g^2/\kappa^2),\;\; i\omega-O(g^2/\kappa^2) , \tag{2.14}$$

and which for the good cavity case $\kappa^2 \ll 4g^2$ yields

$$z_{1,2}\approx i\omega-\frac{\kappa}{2}\pm ig . \tag{2.15}$$

These oscillation frequencies correspond to the transitions from the dressed states $\psi_n^{\pm}$, $n=0$ to $|0,g\rangle$, i.e., to $\langle 0,g\,|\rho(t)|\,\psi_0^{\pm}\rangle\neq 0$.

(2) $\Delta \gg g,\kappa$. In this limit the two roots are approximately $i\omega_0, i\omega-\kappa$. The root at the field frequency has the width of the cavity decay. The root at the atomic frequency has no width. This is as expected since no other source of the atomic decay has been included in the model.

It should be noted that our results have been obtained for arbitrary values of κ and thus one can study in detail how the physics of the radiation matter interaction changes with change in the relaxation time of the cavity. In particular, we can study the effect of the leakage of photons on vacuum-field Rabi oscillations. Figures 1 and 2 show these oscillations in the transition probability p for

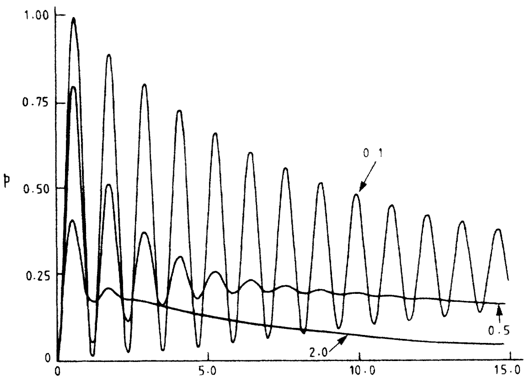

FIG. 2. Same as in Fig. 1 but now cavity is detuned $\Delta=5g$.

the transition $|0,e\rangle \rightarrow |1,g\rangle$. In Fig. 1, the distance between two zeros is roughly π. Note that in the absence of cavity relaxation and for $\Delta=0$, the transition probability has the form $\sin^2 gt$. Figure 1 shows that with an increase in κ, the oscillations quickly damp out, and for $\kappa\sim 2g$ one has already reached, in a sense, the bad-cavity limit. Note also that in the limit $t\rightarrow\infty$, $\langle 1,g\,|\rho|(t)|\,1,g\rangle\rightarrow 0$ as the system has to end up in the steady state $\rho(\infty)=|0,g\rangle\langle 0,g\,|$. A similar result has been known from Ref. 9. For $\Delta\neq 0$ the scale of oscillations changes as shown in Fig. 2. This is in accordance with the formula

$$\frac{4g^2}{\Delta^2+4g^2}\sin^2[(g^2+\Delta^2/4)^{1/2}t]$$

for the transition probability for $\kappa=0$. A recent experiment[2] looks at the average of the occupation probability over several Rabi cycles. Hence in Fig. 3, we have plotted the time average

$$T^{-1}\int_0^T\langle 0,e\,|\rho(\tau)|\,0,e\rangle d\tau\equiv q$$

as a function of the detuning parameter for several values of κ. The results are sensitive to the value of T used in averaging. Note that for $\kappa=0$,

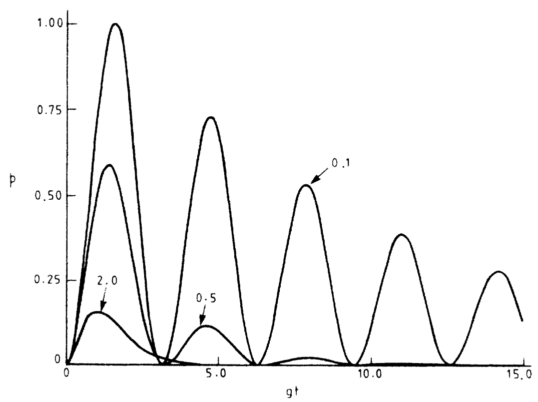

FIG. 1. The probability p of finding the atom in the ground state and the field with one photon as a function of time for $\omega=\omega_0$ and for different values of κ. All frequencies are in units of g. Curves are marked by κ values.

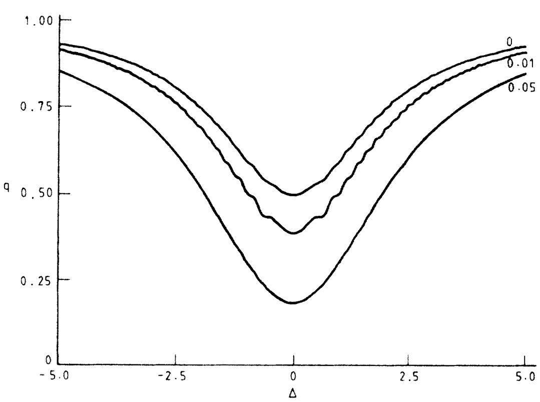

FIG. 3. Time average of the probability $q=T^{-1}\int_0^T d\tau\langle 0,e\,|\rho(\tau)|\,0,e\rangle$ of finding the atom in the excited state as a function of detuning.

$$q=1-\frac{2g^2}{4g^2+\Delta^2}, \quad gT \gg 1 . \tag{2.16}$$

Figure 3 clearly shows how the resonance dip in the average excitation probability is affected by the cavity damping.

III. EFFECT OF CAVITY RELAXATION ON ABSORPTION SPECTRUM

The dynamics of the atom in a cavity can be studied in several ways. One way to study this interaction will be through absorption experiments[7] where one monitors the rate of absorption of energy from a weak-probe field. For Rydberg atoms the probe field will be a microwave field of frequency ν. The density matrix Eq. (2.1) will be modified to

$$\frac{\partial \rho}{\partial t}=L\rho-i[(GS^+e^{-i\nu t}+\text{H.c.}),\rho] , \tag{3.1}$$

where the coupling constant G with the microwave field $\boldsymbol{\epsilon}$ is

$$G=-\mathbf{d}\cdot\boldsymbol{\epsilon}/\hbar . \tag{3.2}$$

The rate of absorption of energy can be calculated in the usual manner by evaluating the induced dipole moment to first order in G. It can be shown from (3.1) that

$$\rho=\rho^{(0)}+\rho^{(i)}+\cdots, \quad L\rho^0=0 ,$$
$$\rho^{(1)}(t)=-i\int_{t_0}^{t} e^{L(t-\tau)}[(GS^+e^{-i\nu\tau}+\text{H.c.}),\rho^{(0)}(\tau)]d\tau . \tag{3.3}$$

The expectation value of the dynamical variable S^+ in the long-time limit is

$$\langle S^+(t)\rangle=\lim_{t\to\infty}\text{Tr}[\rho^{(1)}(t)S^+]$$
$$=-iG^*e^{i\nu t}\text{Tr}\left[S^+\int_0^\infty d\tau e^{L\tau}[e^{-i\nu\tau}S^-,\rho^{(0)}]\right]-iGe^{-i\nu t}\text{Tr}\left[S^+\int_0^\infty d\tau e^{L\tau}[e^{i\nu\tau}S^+,\rho^{(0)}]\right] . \tag{3.4}$$

The time-average rate of absorption W is

$$W=\frac{d}{dt}\langle p\rangle\cdot\boldsymbol{\epsilon}=i\nu(\mathbf{d}\cdot\boldsymbol{\epsilon})\langle S^+\rangle e^{-i\nu t}+\text{c.c.} , \tag{3.5}$$

which on using (3.4) can be reduced to

$$W=-2\nu\left|\frac{\mathbf{d}\cdot\boldsymbol{\epsilon}}{\hbar}\right|^2\text{Re}\int_0^\infty d\tau e^{-i\nu\tau}\text{Tr}(S^+e^{L\tau}[S^-,\rho^{(0)}]) . \tag{3.6}$$

As our cavity is at zero temperature, the initial density matrix $\rho^{(0)}$ is

$$\rho^{(0)}=|0,g\rangle\langle 0,g| , \tag{3.7}$$

and hence

$$W=2\nu\left|\frac{\mathbf{d}\cdot\boldsymbol{\epsilon}}{\hbar}\right|^2\text{Re}\int_0^\infty d\tau e^{-i\nu\tau}\text{Tr}[S^+e^{L\tau}(|0,g\rangle\langle 0,e|)] . \tag{3.8}$$

The operators $e^{L\tau}|0,g\rangle\langle 0,e|$, $e^{L\tau}|0,g\rangle\langle 1,g|$ satisfy the same equation as (2.4) and (2.5) and hence

$$e^{L\tau}|0,g\rangle\langle 0,e|=\alpha(\tau)|0,g\rangle\langle 0,e|+\beta(\tau)|0,g\rangle\langle 1,g| , \tag{3.9}$$

where $\alpha(\tau)$ and $\beta(\tau)$ can be read from (2.8) and (2.9). On substituting (3.9) in (3.8) and on simplification we get

$$W=2\nu\left|\frac{\mathbf{d}\cdot\boldsymbol{\epsilon}}{\hbar}\right|^2\text{Re}\hat{\alpha}(i\nu) , \tag{3.10}$$

where $\hat{\alpha}(z)$ is the Laplace transform of $\alpha(\tau)$. In terms of the roots z_1,z_2 (3.10) can be written as

$$W=2\nu\left|\frac{\mathbf{d}\cdot\boldsymbol{\epsilon}}{\hbar}\right|^2\text{Re}\left[\frac{1}{(z_1-z_2)}\left[\frac{(z_1-i\omega+\kappa)}{(i\nu-z_1)}-\frac{(z_2-i\omega+\kappa)}{(i\nu-z_2)}\right]\right] . \tag{3.11}$$

Thus the absorption spectrum will exhibit resonances at the complex frequencies $\nu=-iz_1,-iz_2$. For the case of a good cavity on resonance, Eq. (2.15) shows that the spectrum is a doublet $\nu=\omega\pm g$, the width of each doublet being $\kappa/2$. These doublets can be associated with the transitions from the state $|0,g\rangle$ to the two dressed states $|\psi_0^\pm\rangle$. Vacuum-field Rabi oscillations lead to a splitting of the absorption spectra. For the bad-cavity case, the spectra only have a single peak. It is remarkable to note that the spectra in this case also consist of a line at $\nu=\omega$,

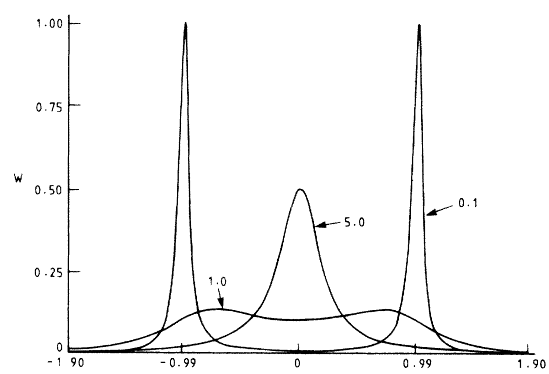

FIG. 4. Absorption spectrum [Eq. (3.11)] as a function of $\omega-\nu$ for $\omega=\omega_0$ and for different values of κ. The spectrum goes from a doublet to a singlet as the cavity κ is changed.

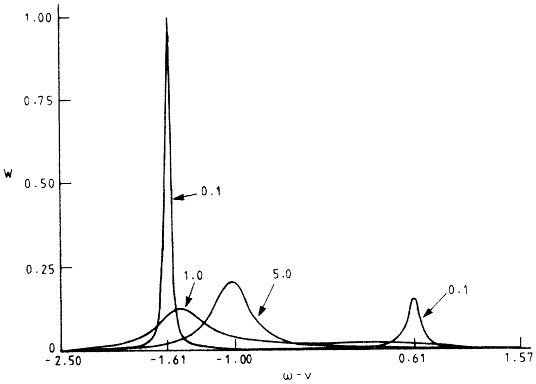

FIG. 5. Same as in Fig. 4 but with finite detuning $\Delta=1.0g$.

which is much narrower than the cavity-relaxation rate. That is, it has a width $[\kappa/2-\frac{1}{2}(\kappa^2-4g^2)^{1/2}]$. Thus for a bad cavity the absorption measurements will result in the linewidth $\sim(g^2/\kappa)$. Such a linewidth has been seen in the experiments of Goy *et al.* These general features are confirmed by the numerical solutions of (3.11) which are shown in Figs. 4–6. In Fig. 4, the atom and the cavity are at exact resonance. For small $\kappa(\ll g)$ the vacuum-field Rabi splitting is prominent. For large κ, one observes a single-peak absorption spectrum with a half-width $\sim[\kappa/2-\frac{1}{2}(\kappa^2-4g^2)^{1/2}]$ which is equal to $\frac{1}{2}$ (life of the atom in the cavity)$^{-1}$. Figures 5 and 6 are, respectively, for positive and negative values of the detuning $\omega_0-\omega$. For small κ, the vacuum-field Rabi splitting is seen. Note also that the width of the line depends on the detuning. For large κ, the peak at the atomic frequency, i.e., at $\nu=\omega_0$ is seen. The symmetry property of the absorption spectra with respect to the sign of Δ is apparent from Figs. 5 and 6.

A quantum treatment of the external field will yield the same result as long as single-photon absorption from the external field is considered. A problem in the observation of the absorption spectra of this section can arise from the saturation of the atom by the external microwave field and this can mask the effects of the vacuum-field Rabi splittings. However, in principle it is possible to imagine

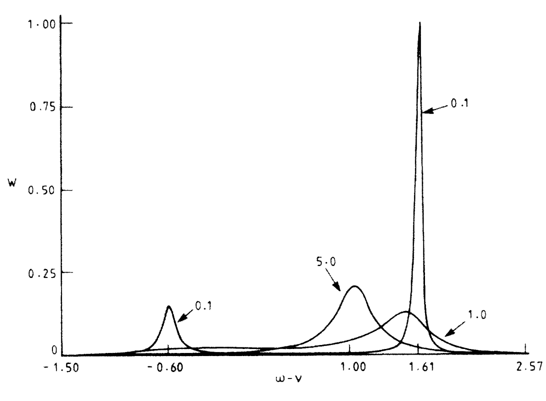

FIG. 6. Same as in Fig. 5 but $\Delta=-1.0g$.

a purely quantum absorption problem involving the energy transfer from an excited atom to an unexcited atom; which may or may not be identical to the excited atom.

Since the roots $-iz_1$ and $-iz_2$ determine the structure of the absorption and emission spectra, we have carried out a detailed study of these roots for various values of κ and Δ. Figure 7 shows how the roots $(-iz_i=x_i+iy_i)$ change as κ is changed. Observe that for $\Delta=0$, $x_1^2+y_1^2=g^2$ if $\kappa<2$, whereas for $\kappa>2$, $x_1=0$, $y_1\to 0$ as $k\to\infty$. On the other hand, for $\kappa\geq 2$, $\Delta=0$, $x_2=0$, $y_2\to\infty$ as $\kappa\to\infty$. Note that $x_i(-\Delta)=-x_i(\Delta)$, $y_i(-\Delta)=y_i(\Delta)$. An interesting feature of Fig. 7(a) is that the effect of cavity damping can be partially offset by cavity detuning, i.e., for a given value of κ the width of one of the peaks will be smaller if the detuning is finite. This effect is similar to inhibited spontaneous emission where large offset reduces the free decay of the atom. The crossing of the curves suggests that a range of (Δ,κ) values can lead to the same width as one of the spectral features.

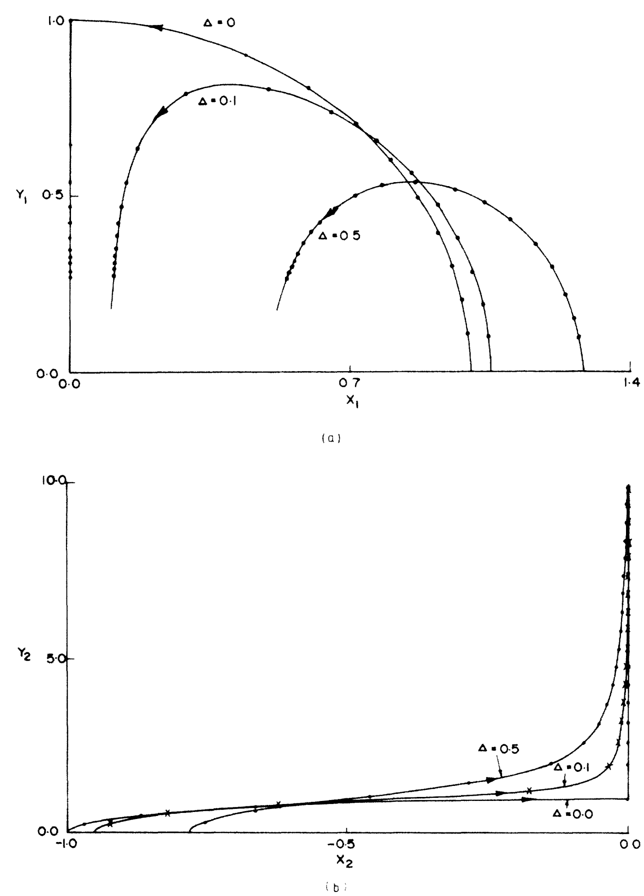

FIG. 7. Complex roots $x_i+iy_i=\Delta/2+i\kappa/2\mp\frac{1}{2}i(\kappa^2-\Delta^2-4g^2+2i\Delta\kappa)^{1/2}$ that determine spectral features of absorption and emission for different values of Δ and κ. The arrow indicates the direction of increasing κ. In (a) [(b)] each dot [cross] represents an increment of κ equal to 0.1 [0.5].

IV. EMISSION SPECTRUM FOR ARBITRARY Q VALUES

We have already seen how the coupling of the radiation field in vacuum state with the atom can affect in an important manner the absorption spectra. Such a coupling can also change drastically the spontaneous-emission spectra. Kleppner showed, for example, that spontaneous emission can be inhibited by choosing the cavity dimensions such that part of the contribution to the density of states gets switched off. Thus the spontaneous line can be narrowed. Eberly and co-workers[6] examined the spontaneous radiation from an idealized cavity assuming no source of relaxation so that in their model the density of states was a delta function. They[6] showed that when the cavity is on resonance with the atom, then the spontaneous-emission spectra have doublet structure. The only source of line width in their model was the width of the detector.

In the present section we account for an important source of relaxation, i.e., leakage of photons from the cavity or the finite Q of the cavity. This is very important because realistic cavities have finite Q and in addition to the measuring process will require that photons should leak out. Following Ref. 13 we define the transient spectrum of the radiation that leaks out as

$$S(\nu,T)=2\Gamma\beta\int_0^T dt_1\int_0^T dt_2 e^{-(\Gamma-i\nu)(T-t_1)-(\Gamma+i\nu)(T-t_2)} \times\langle a^\dagger(t_1)a(t_2)\rangle\ , \tag{4.1}$$

where Γ is the bandwidth of the detector, T is the time at which the spectrum is evaluated, and β is a measure of the leakage of the field energy. Expression (4.1) can be transformed so that we have to only evaluate the time-ordered correlation function $\langle a^\dagger(t+\tau)a(t)\rangle$, $\tau>0$:

$$S(\nu,T)=2\Gamma\beta\,\mathrm{Re}\int_0^T d\tau\, e^{(\Gamma-i\nu)\tau}\int_0^{T-\tau}dt'\, e^{-2\Gamma(T-t')}\langle a^\dagger(t'+\tau)a(t')\rangle\ . \tag{4.2}$$

If the correlation function has the structure

$$\langle a^\dagger(t+\tau)a(t)\rangle=\sum_{i,j}A_{ij}e^{\lambda_i\tau+\eta_j t}\ , \tag{4.3}$$

then the transient spectrum will be

$$S(\nu,T)=2\Gamma\beta\,\mathrm{Re}\sum_{i,j}A_{ij}(2\Gamma+\eta_j)^{-1}[(\Gamma+\eta_j+i\nu-\lambda_i)^{-1}(e^{\eta_j T}-e^{-T(\Gamma-\lambda_i+i\nu)})-(\Gamma+\lambda_i-i\nu)^{-1}(e^{-(\Gamma+i\nu-\lambda_i)T}-e^{-2\Gamma T})]\ . \tag{4.4}$$

We will now evaluate the field correlation function using the exact solutions from Sec. II. Using regression theorem, one can show that

$$\langle a^\dagger(t+\tau)a(t)\rangle=\mathrm{Tr}[a^\dagger e^{L\tau}ae^{Lt}\rho(0)]\ , \tag{4.5}$$

where the initial density matrix $\rho(0)$ is $|0,e\rangle\langle 0,e|$. This initial state is chosen keeping in view the problem of pure spontaneous emission. The operator $e^{Lt}|0,e\rangle\langle 0,e|\equiv(|0,e\rangle\langle 0,e|)_t$ can be evaluated by examining equations of motion

$$\frac{d}{dt}e^{Lt}|0,e\rangle\langle 0,e|=e^{Lt}L\,|0,e\rangle\langle 0,e|\ , \tag{4.6}$$

which on using L from (2.1) reduces to

$$\frac{d}{dt}(|0,e\rangle\langle 0,e|)_t=-ig(|1,g\rangle\langle 0,e|)_t-\kappa(|0,e\rangle\langle 1,g|)_t\ . \tag{4.7}$$

Resulting is a closed set of equations

$$\left[\frac{d}{dt}+\begin{bmatrix}0 & ig & -ig & 0 & 0\\ ig & -i(\omega_0-\omega)+\kappa & 0 & -ig & 0\\ -ig & 0 & i(\omega_0-\omega)+\kappa & ig & 0\\ 0 & -ig & ig & 2\kappa & -2\kappa\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\right]\begin{bmatrix}|0,e\rangle\langle 0,e|\\ |1,g\rangle\langle 0,e|\\ |0,e\rangle\langle 1,g|\\ |1,g\rangle\langle 1,g|\\ |0,g\rangle\langle 0,g|\end{bmatrix}=0\ . \tag{4.8}$$

If we denote by M the 4×4 square matrix in (4.8) formed out of the first four rows and four columns, then it can be shown that

$$e^{L\tau}ae^{Lt}\rho(0)=(e^{-Mt})_{12}e^{L\tau}|0,g\rangle\langle 0,e| +(e^{-Mt})_{14}e^{L\tau}|0,g\rangle\langle 1,g|\ . \tag{4.9}$$

One can further show that

$$\left[\frac{d}{dt}+\begin{bmatrix}-i\omega_0 & -ig\\ -ig & -i\omega+\kappa\end{bmatrix}\right]\begin{bmatrix}(|0,g\rangle\langle 0,e|)_\tau\\ (|0,g\rangle\langle 1,g|)_\tau\end{bmatrix}=0\ . \tag{4.10}$$

Let us denote the 2×2 square matrix in (4.10) by N. Then, using the solution of (4.10) in (4.9), one can show that

$$\langle a^{\dagger}(t+\tau)a(t)\rangle = (e^{-Mt})_{12}(e^{-N\tau})_{12} + (e^{-Mt})_{14}(e^{-N\tau})_{22} \,. \tag{4.11}$$

The matrices M and N are the same as those appearing in Eqs. (2.7), (2.4), and (2.5). The Laplace transform of e^{-Mt} can be read from (2.11). The relevant elements of $e^{-N\tau}$ are given by (2.8) and (2.9):

$$(e^{-N\tau})_{12} = \frac{ig}{(z_1-z_2)}(e^{z_1\tau}-e^{z_2\tau}) \,,$$

$$(e^{-N\tau})_{22} = \frac{1}{(z_1-z_2)}[(z_1-i\omega-i\Delta)e^{z_1\tau} - (z_2-i\omega-i\Delta)e^{z_2\tau}] \,. \tag{4.12}$$

Complete spectrum of spontaneous emission can now be obtained using (4.11) and (4.3) in (4.4). If we consider the long-time limit of (4.4), $\Gamma T \gg 1$, then

$$S(\nu,T) \rightarrow 2\Gamma\beta \,\mathrm{Re} \sum_{i,j} A_{ij}(2\Gamma+\eta_j)^{-1} \times e^{\eta_j T}(\Gamma+\eta_j+i\nu-\lambda_i)^{-1} \,. \tag{4.13}$$

From (2.12) we see that if $\kappa \rightarrow 0$, then the roots of (2.12) are simply $z=0, \pm i(\Delta^2+4g^2)^{1/2}$. Thus η_j can be zero or $\pm i(\Delta^2+4g^2)^{1/2}$. If $\eta_j T$ is large, then the terms corresponding to $\eta_j = \pm i(\Delta^2+4g^2)^{1/2}$ can be ignored and thus in (4.13) we can retain only those terms such that $\eta_j \approx 0$,

$$S(\omega,T) \rightarrow 2\Gamma\beta \,\mathrm{Re} \sum_{\eta_j \sim 0} A_{ij}(2\Gamma+\eta_j)^{-1} \times e^{\eta_j T}(\Gamma+\eta_j+i\nu-\lambda_i)^{-1} \,. \tag{4.14}$$

In this limit the spontaneous-emission spectra consists of several lines whose positions and widths are determined by $\mathrm{Im}(\lambda_i-\eta_j)$, $\Gamma+\mathrm{Re}(\eta_j-\lambda_i)$. In the special case of exact resonance $\Delta=0$, such positions and widths can be obtained from (2.15)

$$\nu = \omega \pm g + i\left[\Gamma+\frac{\kappa}{2}\right] \,. \tag{4.15}$$

On the other hand, for large Δ, spontaneous-emission lines occur at

$$\nu = \omega_0 + i\Gamma, \quad \nu = \omega + i(\kappa+\Gamma) \,. \tag{4.16}$$

The fluorescence line acquires only the width of the detector, the Rayleigh peak has width due to both the detector and the cavity leakage. Figures 8 and 9 give the spectrum of the radiation that leaks out for the resonant as well as the offresonant case. For $\omega_0=\omega$, the spectrum is a doublet which is a consequence of vacuum-field Rabi oscillations. For nonzero Δ, and $\kappa \rightarrow 0$, the spectrum is a symmetric function of $\nu-\omega$. The cavity damping very quickly reduces the overall magnitude of the spectral peaks.

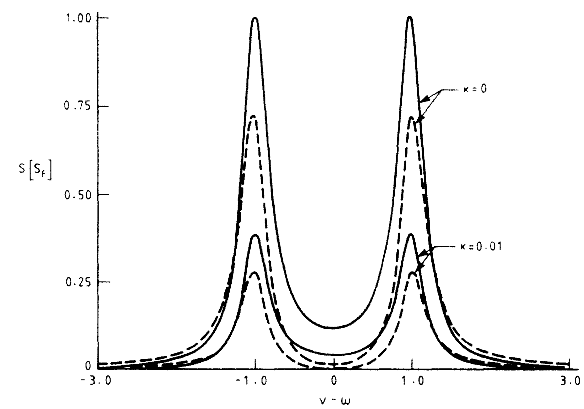

FIG. 8. Emission spectra for leaked-out photons (solid curves) and fluorescence photons (dashed curves) as a function of $(\nu-\omega)$. Different curves are labeled by κ values. Other parameters are detector width $\Gamma=0.2g$ and the counting time $T=100g^{-1}$.

For moderate values of κ or for $\kappa \gg g$ (bad cavity), the transients die out very quickly and the spectrum takes the shape shown in Fig. 10. The curve corresponding to $\kappa=g$ shows considerable transit-time broadening.

Finally note that the atom in the cavity will also emit by spontaneous emission into other modes although this channel will be very weak. Thus, weak fluorescence in a direction perpendicular to the cavity axis can also be monitored. The spectrum of such a fluorescence will be

$$S_F(\nu,T) = 2\Gamma\beta' \int_0^T dt_1 \int_0^T dt_2\, e^{-(\Gamma-i\nu)(T-t_1)-(\Gamma+i\nu)(T-t_2)} \times \langle S^+(t_1)S^-(t_2)\rangle \,. \tag{4.17}$$

Here β' will be related to the measure of spontaneous emission into other modes. The atomic correlation function $\langle S^+(t_1)S^-(t_2)\rangle$ can be shown to have the form

$$\langle S^+(t+\tau)S^-(t)\rangle = (e^{-Mt})_{11}(e^{-N\tau})_{11} + (e^{-Mt})_{13}(e^{-N\tau})_{21} \,, \tag{4.18}$$

and thus S_F can also be computed. We exhibit S_F in Figs. 8 and 9 by dashed lines. S_F shares many of the

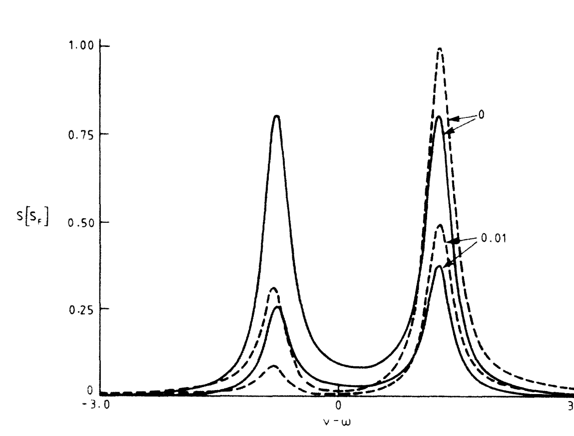

FIG. 9. Same as in Fig. 8 but now detuning is finite $\Delta=0.5g$.

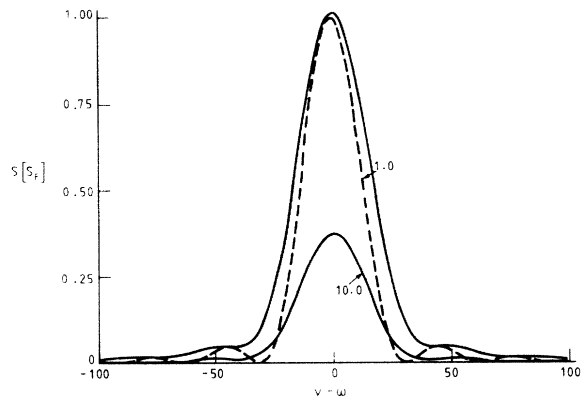

FIG. 10. Emission spectra for the bad-cavity case $\Delta=0$ $\Gamma=0.2g$, $T=0.2g^{-1}$.

features of S though for $\kappa=0,\Delta\neq 0$, S_F is not a symmetric function of $\nu-\omega$. Thus for the detuned case, S_F has dominant asymmetries in contrast to S. It may be added that the work of Sanchez-Mondragon *et al.*[6] focuses on the spectrum S_F in an ideal cavity. The bad-cavity result for S_F is also shown in Fig. 10. Here S_F is narrower than the corresponding S.

V. COHERENT SCATTERING OF RADIATION

For an atom in free space, it is well known[14] that the spectrum of resonance fluorescence from a system driven by a weak monochromatic field consists of a line at the applied frequency with zero width. The atom thus scatters coherently. In this section we show how coherent scattering of the external radiation at the frequency ν from the atom in the cavity can be used to monitor the vacuum-field Rabi splittings.

The mathematical development is somewhat similar to that used in Sec. III. We have to evaluate the mean value of the field amplitude associated with the cavity mode $\langle a^\dagger\rangle$. Thus, in place of (3.4) we will get

$$\langle a^\dagger\rangle=-iG^*e^{i\nu t}\mathrm{Tr}\left[a^\dagger\int_0^\infty d\tau\, e^{L\tau}[e^{-i\nu\tau}S^-,\rho^{(0)}]\right] -iGe^{-i\nu t}\mathrm{Tr}\left[a^\dagger\int_0^\infty d\tau\, e^{L\tau}[e^{i\nu\tau}S^+,\rho^{(0)}]\right], \quad (5.1)$$

which on simplification [cf. Eq. (3.8)] leads to

$$\langle a^\dagger\rangle=iG^*e^{i\nu t}\mathrm{Tr}\left[a^\dagger\int_0^\infty d\tau\, e^{-i\nu\tau}e^{L\tau}(\,|0,g\rangle\langle 0,e\,|\,)\right]. \quad (5.2)$$

On using the result (3.9), (4.2) simplifies to

$$\langle a^\dagger\rangle=iG^*e^{i\nu t}\tilde{\beta}(i\nu)\,, \quad (5.3)$$

which on using (2.8) leads to

$$\langle a^\dagger\rangle=iG^*e^{i\nu t}\frac{ig}{(z_1-z_2)}\left[\frac{1}{(i\nu-z_1)}-\frac{1}{(i\nu-z_2)}\right]. \quad (5.4)$$

The intensity I of the coherent scattering will be[15]

$$I(\nu)=|\langle a^\dagger\rangle|^2=|G|^2g^2/|(i\nu-z_1)(i\nu-z_2)|^2\,. \quad (5.5)$$

Thus a scan of the coherent scattering as a function of applied frequency will have spectral features determined by the roots z_1,z_2 [Eq. (2.10)]. Hence another cw method of studying the vacuum-field Rabi oscillations is provided by the coherent scattering from the atom in the cavity. One can also show,[16] using second-order perturbation on (3.1), that

$$\lim_{t\to\infty}\langle a^\dagger(t+\tau)a(t)\rangle=e^{i\nu\tau}I(\nu)\,. \quad (5.6)$$

It turns out that the coherent scattering $I(\nu)$ as proportional to the absorption spectrum of Sec. III and hence various characteristics of the vacuum-field Rabi oscillations will also be seen in the coherent scattering of radiation.

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[16] These calculations are rather long and that is why they are not presented here.