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# Adiabatic switching and long-time corrections to the exponential decay law

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Standard theory predicts that the familiar exponential decay law must be modified at long times, a result that was recently challenged by Mittleman and Tip, who claimed, in the context of a model problem, that the modification was strongly attenuated if the artificial assumption of an interaction that is suddenly switched on is replaced by a gradual onset. We have studied the same question, and find that while this reduction is present, it is of a different form than that predicted by Mittleman and Tip, and could be negligible in certain cases. We suggest that their model might actually be a promising candidate problem for an experimental investigation of departures from exponential decay.

It is well known that if an interaction, suddenly switched on and constant thereafter, has the property of coupling an embedded bound state to a continuum, the survival probability of the initial bound state evolves, to a very good approximation, according to the exponential decay law $P_s = \exp(-\gamma t)$, where $\gamma$ is given by Fermi's golden rule. Theory further predicts departures from the dominant exponential behavior at very long and very short times, and the questions whether these deviations are real and, if so, what their experimental implications are and what their precise functional forms are have received renewed attention in recent years.[1-5] The present paper examines the decay law for times $t \gg 1/\gamma$, where the standard theoretical result[2,6] gives a survival probability that behaves like an inverse power law, $t^{-n}$, where the precise value of $n$ is specific to each particular problem.

Ordinarily, the changeover from exponential to power law is not predicted to occur until the signal has decreased to a value so minute that it is completely unobservable.[2] An exception might occur if the decay were to take place at a nominal energy just above the threshold for the continuum, and these processes might be the best candidates for an experimental search for long-time breakdown of the exponential decay law. Recently, however, Mittleman and Tip[3] have presented a calculation in the context of a model photoionization problem which indicates that the size of the correction to the exponential that is predicted by the usual theory may even be too large. Their conclusion is that if the artificial choice of a sudden turn-on for the interaction is replaced by the assumption that the onset occurs gradually, the correction term becomes a function of the smoothness of the switching, and is strongly attenuated in general. If their ideas are correct, any residual chance of seeing the long-time, nonexponential decay would be virtually eliminated. This work was undertaken to check their results in a general way, i.e., without recourse to any specific model. Our conclusion proves only to partially agree with Mittleman and Tip.[3] We find that while a nonzero rise time *can* reduce the power-law term, it need not. This disagreement occurs as a consequence of differing results for the function that attenuates the correction to the exponential decay law.

We begin our analysis by rederiving the law of decay in a form that is independent of the details of the turn-on. Assume a nondegenerate continuum-embedded bound state $|0\rangle$ to be coupled to that continuum $|k(\omega_k)\rangle$ by an interaction whose matrix elements are given by $H'_{0k} = \mu_k f(t)$. The preparation function $f(t) \to 0$ as $t \to -\infty$, and approaches unity as $t \to +\infty$, and is characterized by a rise time $T$, assumed to be $\ll 1/\gamma$. Using a system of units where $\hbar = 1$, the equations of motion for the state amplitudes $a_0, a_k$ are

$$i\dot{a}_k = \mu_k f(t) \exp(i\omega_k t) \exp(\epsilon t) a_0 , \tag{1a}$$

$$i\dot{a}_0 = \int_{\Omega_{\rm th}}^{\infty} f(t) \mu_k^* \rho(\omega_k) \exp(-i\omega_k t + \epsilon t) d\omega_k a_k , \tag{1b}$$

where $\rho(\omega_k)$ is the continuum state density, where the zero of energy is that of the initial bound state, and the energy threshold $\Omega_{\rm th} < 0$. The convergence factors of $\exp(\epsilon t)$ are inserted to facilitate integration by parts, and do not affect the results in the implied limit $\epsilon \to 0$. We may formally integrate Eq. (1a), and substitute in Eq. (1b) to obtain an expression for $a_0$ alone, namely,

$$\dot{a}_0 = -\int_{\Omega_{\rm th}}^{\infty} \phi(\omega_k) \exp(-i\omega_k t + \epsilon t) f(t) d\omega_k \int_{-\infty}^{t} f(t') a_0(t') \exp(i\omega_k t' + \epsilon t') dt' ,$$

where $\phi(\omega_k)=|\mu_k|^2\rho(\omega_k)$. Integrating by parts with respect to $t'$, we obtain

$$\dot{a}_0=i\left[\int_{\Omega_{\rm th}}^{\infty}\{\phi(\omega_k)\exp(2\epsilon t)[f(t)]^2a_0(t)/(\omega_k-i\epsilon)\}d\omega_k\right.$$
$$\left.-\int_{\Omega_{\rm th}}^{\infty}\phi(\omega_k)f(t)\exp(-i\omega_k t+\epsilon t)d\omega_k/(\omega_k-i\epsilon)\int_{-\infty}^{t}\exp(i\omega_k t'+\epsilon t')(d/dt')[f(t')a_0(t')]dt'\right] . \quad (2)$$

The second term on the right-hand side of Eq. (2) vanishes if one evaluates it in the Weisskopf-Wigner (WW) approximation, i.e., extends the lower limit of the $\omega_k$ integration to $-\infty$ and treats $\phi(\omega_k)$ as a constant. The justification for this ansatz is obvious—the integrand is large only near $\omega_k=0$, so that we expect our result to be insensitive to its form in the wings. Our task later will be to calculate the leading nonvanishing contribution to this second integral.

To see that the WW approximation causes this integral to go to 0, perform the $\omega_k$ integration in the complex plane, using a contour that extends along the real axis between $\pm V$, with $V\to\infty$, closed by a semicircular arc in the lower half-plane. The contribution from the arc vanishes, since $t>t'$, while the contour has no singularities except for a simple pole in the *upper* half-plane. Thus, within the WW approximation, we immediately have

$$\dot{a}_0=-\left[\pi\phi(0)-i\mathrm{P}\int_{\Omega_{\rm th}}^{\infty}\phi(\omega_k)d\omega_k/\omega_k\right][f(t)]^2a_0(t)$$

or

$$a_0=\exp\left[-(\gamma/2+iS)\int_{-\infty}^{t}[f(t')]^2dt'\right] , \quad (3)$$

which reduces to the standard $a_0=\exp[(-\gamma/2+iS)t]$ for the case where $f$ is a unit step function. The parameter $S$ is the shift induced by the interaction.

We note in passing that the analysis leading to Eq. (3) is also valid if $f(t)\to 0$ as $t\to+\infty$, i.e., if the interaction is a pulse. Thus, Eq. (3) may be interpreted as a generalized WW approximation for time-dependent couplings.

We shall now calculate the dominant contribution to the second integral in Eq. (2) for the case of finite $\Omega_{\rm th}$ and variable $\phi$. It will be sufficient to evaluate the integral

$$I=-i\int_{\Omega_{\rm th}}^{\infty}[\phi(\omega_k)f(t)\exp(-i\omega_k t+\epsilon t)/(\omega_k-i\epsilon)]d\omega_k\int_{-\infty}^{t}\exp(i\omega_k t'+\epsilon t')(d/dt')[f(t')a_0(t')]dt'$$

to lowest order in $\phi$. If, in addition, we assume that the absolute values of $\gamma$ and $S$ are sufficiently small, we may simply take $a_0=1$ for the time interval where $\dot{f}$ is nonzero, while the contribution arising from $\dot{a}_0$ is of higher order in $\phi$ and will be neglected. To the indicated level of approximation,

$$I=-i\int_{\Omega_{\rm th}}^{\infty}[\phi(\omega_k)\exp(-i\omega_k t+\epsilon t)d\omega_k/(\omega_k-i\epsilon)]\int_{-\infty}^{t}\exp(i\omega_k t'+\epsilon t')\dot{f}(t')dt'$$
$$=-i\int_{\Omega_{\rm th}}^{\infty}(2\pi)^{1/2}\phi(\omega_k)\exp(-i\omega_k t+\epsilon t)g_i(\omega_k,t)d\omega_k/(\omega_k-i\epsilon) ,$$

where the function $g_i(\omega_k,t)$ will be designated the "incomplete" Fourier transform of $\dot{f}(t')$. Considered as a function of $\omega_k$, the product $P(\omega_k)=\exp(-i\omega_k t+\epsilon t)g_i$ is analytic in the half-plane $\mathrm{Im}\,\omega_k<0$, provided that the condition $t\geq t'$ obtains, as it does here. It is convenient to make the change of variable $\nu=\omega_k-\Omega_{\rm th}$, yielding

$$I=-i(2\pi)^{1/2}\int_0^{\infty}d\nu\,P(\nu+\Omega)\phi(\nu+\Omega)/(\nu+\Omega-i\epsilon) ,$$

where we have dropped the subscript "th."

We shall formally integrate the expression for $I$ in the complex plane, using the same contour as Robiscoe; namely, the real axis from the origin to $R$, a circular arc clockwise to $(0,-iR)$, with return to the origin along the imaginary axis, in the limit $R\to\infty$. The contribution from the arc vanishes, while the evaluation of the contour integral requires a knowledge of the singularity structure of $\phi$. We shall make the *assumption* that either $\phi$ is analytic in this quadrant or the singularities are of such character and location that their effect is negligible. This ansatz applies to the problems considered by Robiscoe and by Mittleman and Tip,[2,3] but would not obtain if there were a resonance near $\nu=-\Omega$. Thus, the dominant contribution in this approximation arises from the integral along the negative imaginary axis. Since $t$ is very large, the integrand contains a factor that is exponentially small except for a small region near the origin. Accordingly, it is correct to approximate the factor $g_i(\nu+\Omega,t)/(\nu+\Omega-i\epsilon)$ by $g_i(\Omega,t)/(\Omega-i\epsilon)$, and move this product outside the integral sign. Furthermore, since $\dot{f}\to 0$ for $t\gg T$, we may replace $g_i(\Omega,t)$ by $g(\Omega)=\lim_{t\to\infty}g_i(\Omega,t)$, the normal or "complete" Fourier transform. In addition, if $|\Omega T|\gg 1$, it would be appropriate to represent $g$ by its asymptotic form. Thus, the entire difference between corrections that apply when the preparation function is varied is contained in the factor $g(\Omega)$, the Fourier transform of $\dot{f}$. The asymptotic form of $g$ will be a power law in $(\Omega T)^{-1}$ unless all derivatives of $f$ exist. In the latter case, $g$ will be a function of $\Omega T$ that vanishes more rapidly than any power law as $|\Omega T|\to\infty$. These conclusions are similar to those presented in Ref. 3 in so far as the functional form of the attenuation factor is concerned, but differ in that this factor is a function of $T$, not $t$. Only if there is no distinction made between the rise time $T$ of the coupling and the observation time $t$ will we get the same results as Mittleman and Tip.[3] Furthermore, if $\Omega T\ll 1$, it is obvious that the expression for a step-function switch is recovered. Thus, if an experimentalist wished to look for long-time corrections to the exponential decay law, he might, with a suitable choice of problem, succeed in avoiding a case where adiabatic switching is predicted to attenuate the signal. This would, of course, still leave the very formidable difficulty of surmounting the small size of the correction in the presence of sudden onset.

It is relevant to inquire whether, in fact, there are cases where $|\Omega T| \ll 1$. An example immediately presents itself—the photoionization problem discussed by Mittleman and Tip.[3] If the exciting laser is of nominal frequency $\leq 1$ cm$^{-1}$ above the ionization threshold, and is switched on in a time $\leq 10^{-14}$ sec, a choice which seems compatible with the state of the art, $|\Omega T|$ would be $<0.1$, which should be satisfactorily sudden. Moreover, the photoionization cross section is finite at threshold (in contradistinction to that for the photodetachment of negative ions, which vanishes at least as fast as $\Omega^{3/2}$). This means that, since the integral $I$ is proportional to $\gamma/\Omega$, and $\gamma$ does not vanish more rapidly than $\Omega$ as $\Omega$ goes to zero, the correction term has a relatively large intrinsic amplitude. That is, a decay which involves photoionization near threshold appears to be a promising candidate for an experimental investigation of long-time deviations from the exponential law.

To summarize, we have found, in agreement with Mittleman and Tip,[3] that adiabatic switching can attenuate the correction term to the exponential decay law at long times. However, we differ from those authors in that the reduction depends on the rise time of the coupling potential, instead of the observation time, a result that suggests the existence of problems where the attenuation is absent or not important. Photoionization, the problem analyzed in Ref. 3, may be a promising candidate for a search for this effect.

Finally, we note that our treatment does not address the question of what effect couplings other than those between the one bound state and the single continuum have on the correction term, or on the main exponential law itself. This matter should be studied before any experimental investigation is undertaken, should anyone be bold enough to consider doing so.

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