Theory of the extended-ratio method and its application to lattice models

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The theory of the extended-ratio method (ERM) and its extensive applications to various lattice models are presented here in detail. We find the following interesting features in contrast with the traditional ratio method (TRM): the results of the ERM very often (except for a very special example encountered among the studied lattice models) appear to be smoother and better or at least comparable to those of TRM; the odd-even oscillations appearing in the loose-packed lattices are reduced greatly; the Neville tables often provide sequences with a faster convergence rate and appear steadier and more reliable; while the computational efforts needed for the ERM are the same as those needed for the TRM.

I. INTRODUCTION

The series-expansion method has played an important role historically in the development of the study of critical phenomena. Like the Monte Carlo and some renormalization methods, it is still very important in the theoretical investigations of critical phenomena nowadays. The reason is very obvious: only very few problems can be solved exactly, while the series expansion itself often provides a reliable but rather simple and straightforward method. It is well known that it is very time-consuming to get any valuable results from the exact enumeration, which is the starting point of the series-expansion method, of configurations for a given lattice model. The rate of increase of computer time used for every additional step (or site or bond) of the enumeration, generally speaking, is as rapid as (or even faster than) that of a geometric series. Thus, in the interest of reducing the time consumption in the exact enumeration process, how to improve the convergence of the series-expansion method is a very significant subject.

The ratio method has succeeded remarkably in determining the location and nature of the singularity which lies on the real axis and is dominant. For example, the value of the exponent γ ($=1.75$) characterizing the high-temperature susceptibility of the two-dimensional Ising model close to the Curie temperature was determined on the basis of up to nine terms of the exact-series-expansion method, some time before its exact theoretical justification was known. Because of its simplicity and efficiency, one often prefers to use the ratio method first rather than any other.

An extended-ratio method (ERM) has been proposed by these authors[1,2] and successfully applied to self-avoiding-path walks (SAPW's) and the spiral-bond–animal model very recently.

In this paper, the mathematical basis for the ERM, using two mathematical theorems (Stoltz's theorem[3] and Darboux's first theorem[4]), is given. The results for the application of ERM to various lattice models are presented in detail. Our results show that in almost every case studied, the results from the ERM are better than or at least comparable to those from the traditional ratio method, while the computer time consumed is almost the same for both.

II. THEORETICAL FRAMEWORK

We assume the following conventional asymptotic relations for large N for some lattice model:

$$C_N \equiv \sum_{\{\omega_N\}} 1 \approx A\mu^N N^{\gamma-1} \tag{1}$$

and

$$\rho_N \equiv \sum_{\{\omega_N\}} \rho_{\omega_N}/C_N \approx BN^{2\nu} , \tag{2}$$

where $\{\omega_N\}$ and ρ_{ω_N} are, respectively, the distinguishable N-step (or site or bond) configurations and square end-to-end distances obtained by the exact enumeration in the given lattice model. In Appendix B, we show generally the validity of the following three equations (we have proved some of these only for the SAPW case specifically[1]) provided the Eqs. (1) and (2) are fulfilled:

$$\mu_N^S \equiv \sum_{m=0}^{N} C_m \Big/ \sum_{m=0}^{N-1} C_m = \mu\left[1+\frac{\gamma-1}{N}+O\left[\frac{1}{N^2}\right]\right] , \tag{3}$$

$$\nu_N^{S1} \equiv \frac{N}{2}\left[\frac{\sum^{N+1} C_m\rho_m}{\sum^{N+1} C_m} \Big/ \frac{\sum^{N} C_m\rho_m}{\sum^{N} C_m} - 1\right] = \nu\left[1+O\left[\frac{1}{N}\right]\right] , \tag{4}$$

$$\nu_N^{S2} \equiv \frac{N}{2}\left[\sum_{m=0}^{N+1}\rho_m \Big/ \sum_{m=0}^{N}\rho_m - 1\right] - 0.5 = \nu\left[1+O\left[\frac{1}{N}\right]\right] . \tag{5}$$

The method in which the ratios (3)–(5) are used, instead of those used in the tradiational ratio method (TRM), is called the extended-ratio method in this paper, and a quantity with a superscript S represents a result of ERM, using Stoltz's theorem.

From Eqs. (3)–(5), one finds that these equations give the same asymptotic behaviors as those of the TRM (but give no information about the rate of convergence, especially when N is not large). Thus all the steps for the further numerical analysis in the TRM can be adapted to obtain the critical quantities. This is just the content of the ERM. At first sight, the surprising simplicity of the ERM seems to be trivial, while it does in practice often improve the convergence rate.

Defining the generating function of C_N as usual,

$$f(z)=\sum_{N=0}^{\infty} C_N z^N \approx A(1-\mu z)^{-\gamma} , \tag{6}$$

one defines $C'_N \equiv \sum_{m=0}^{N} C_m$; it is obvious that

$$f(z)(1-z)^{-1}=\sum_{N=0}^{\infty} C'_N z^N . \tag{7}$$

By use of Darboux's first theorem,[4] one can show that these two sequences of C_N and C'_N have the same asymptotic behaviors. That is, the ERM and TRM certainly will lead to the same critical quantities eventually when $N\rightarrow\infty$. It is very interesting to note that the convergence rate of the original sequence can be improved by such a simple procedure: multiplying the generating function by a proper function [here the $(1-z)^{-1}$ is used in the ERM]. One can consider the rate of convergence as a functional of the multiplied function with some constraint conditions so that it should not change the location and nature of the interesting singularity of the generating function. The problem of determining which function to give the highest rate of convergence for the new sequence is a significant subject.

We can prove that the odd-even oscillation, a trouble encountered in the loose-packed lattices, is reduced in the ERM. According to Sykes *et al.*,[5] the O_N's in the loose-packed lattice can be written as

$$C_N \approx A a_N + A'(-1)^N b_N , \tag{8}$$

where

$$a_N=\mu^N N^{\gamma-1} \tag{9}$$

and

$$b_N=\mu^N N^{\gamma'-1} . \tag{10}$$

We can prove (see Appendixes B and C)

$$\sum_{m=0}^{N} a_m \approx \frac{\mu}{\mu-1}\mu^N N^{\gamma-1}\left[1+O\left[\frac{1}{N}\right]\right] \tag{11}$$

and

$$\sum_{m=0}^{N} (-1)^m b_m \approx \frac{\mu}{\mu+1}\mu^N N^{\gamma'-1}\left[1+O\left[\frac{1}{N}\right]\right] . \tag{12}$$

Notice that $\mu>1$; thus taking the sum will strengthen the weight of the smooth terms and depress the oscillating ones. The analogous conclusion also holds for $\sum^N C_m \rho_m$; since if we have

$$\rho_N \approx BN^{2\nu}+B'(-1)^N N^{2\nu'} \tag{13}$$

then we can prove the next line in the same way (see Appendix C),

$$\sum_{m=0}^{N} \rho_m = \frac{B}{2\nu+1}N^{2\nu+1}\left[1+O\left[\frac{1}{N}\right]\right] + \frac{B'}{2}(-1)^N N^{2\nu'}\left[1+O\left[\frac{1}{N}\right]\right] . \tag{14}$$

III. NUMERICAL RESULTS AND DISCUSSIONS

First, we define some symbols that appear in the text. Assume X is some critical quantity, and x the corresponding critical index. Then,

$$\bar{X}_N \equiv \tfrac{1}{2}(X_N+X_{N-1}) ,$$

$$X(N,M) \equiv \frac{1}{N-M}(NX_N-MX_M) ,$$

$$x_N \equiv N(X_N/X'-1) ,$$

$$X_N \equiv N(X_N/X'-1) ,$$

where X' is the estimated value of X. The definition of the Neville table is as follows:

$$e_N^0 \equiv X_N ,$$

$$e_N^l \equiv \{Ne_N^{l-1}-[N-(N-M)l]e_M^{l-1}\}/(N-M)l ,$$

and $M=N-1$ for the close-packed lattice, $M=N-2$ for the loose-packed ones.

After extensively applying the ERM to various lattice models, we get the following results.

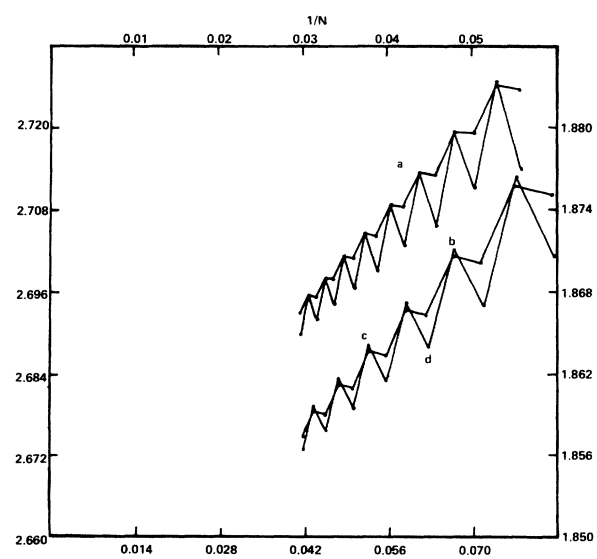

FIG. 1. Consecutive constants vs $1/N$ for a SAW on two lattices. Curves a and b are μ_N^S and μ_N, respectively, for a hexagonal lattice; curves c and d are μ_N^S and μ_N for a square lattice.

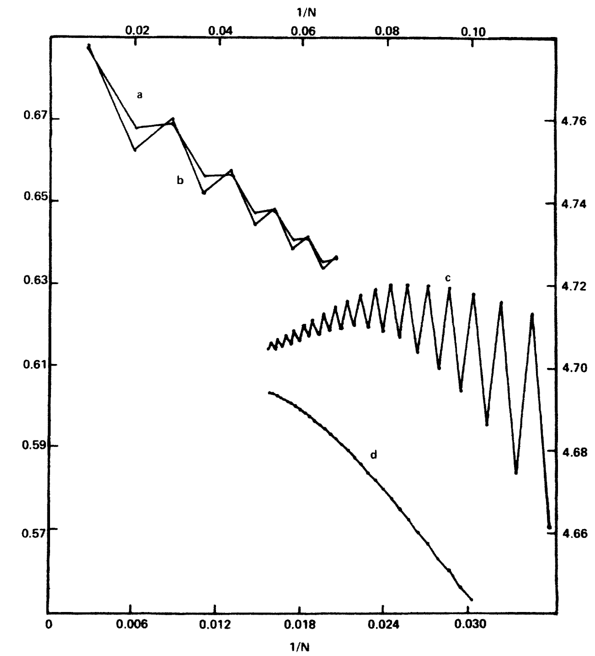

FIG. 2. Consecutive constants and correlation-length exponents vs $1/N$ for two lattice models. Curves a and b are μ_N^S and μ_N, respectively, for the SAW on a sc lattice; curves c and d are ν_N and ν_N^S, respectively, for the spiral SAW on a square lattice.

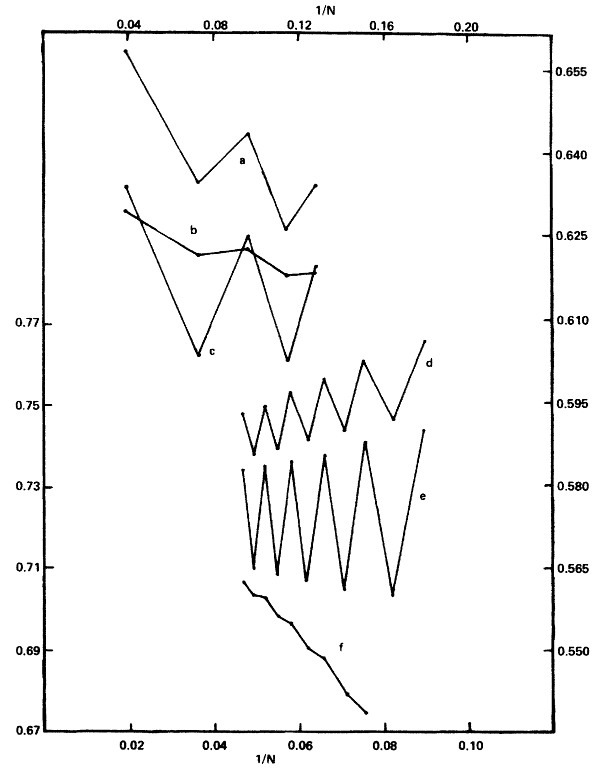

FIG. 3. Correlation-length exponents vs $1/N$ for the SAW on square and sc lattices. Curves a, b, and c are ν_N^{S1}, ν_N^{S2}, and ν, respectively, for the sc lattice; curves d, e, and f are ν_N^{S1}, ν_N^{S2}, and ν for the square lattice.

A. Self-avoiding walk (SAW) (Ref. 6), including some return to the origin SAW, on square, triangular, hexagonal, simple-cubic (sc), and face-centered-cubic (fcc) lattices

One can find from Figs. 1–3 that, as we have expected, the ERM reduces greatly the odd-even oscillations appearing in the loose-packed lattices, and the results of the ν_N^S's are the steadiest ones. From these figures and Tables I–IV, one can find further that the odd-step sequences for μ_N^S's and γ_N^S's are close to those for μ_N's and γ_N's, respectively. The contents in the figures and tables mentioned above strongly support the impression that the results of the ERM are steadier than those of the TRM's. For example, from Table II, the range of variation of γ_N's ($\mu'=4.1517$) is 0.0013 for SAW's on a triangular lattice from $N=14$ to $N=18$, while the corresponding variation for γ_N^S (with the same μ') is only 0.0006.

The sequences provided in the Neville table by the ERM often appear to converge more rapidly and more steadily than those by the traditional one. For example, they provide more reliable results for the correlation-length exponent (≈ 0.75) until $l=6$ (l is the order of the Neville tables) in Table II.

Another even more interesting feature is presented in Table IV(b): when $l \geq 2$, the ν_N's provided by the Neville tables for a SAW on a sc lattice in the TRM increase with N, thus giving an incorrect tendency when compared with the value of 0.5875 in which the confluent correction has been considered. However, a correct tendency of our ν_N^{S2}'s is preserved even in high-order Neville tables.

In Table II, it is also shown that the variations of $\mu(N,M)$ and γ_N with N for a triangular lattice are opposite to those of the ERM quantities. Thus it is helpful for the error estimation for those quantities.

TABLE I. The susceptibility exponents for the SAW problem on a square lattice.

N	γ_N ($\mu'=2.6385$)	$\bar{\gamma}_N$ ($\mu'=2.6385$)	γ_N ($\mu'=2.639$)	$\bar{\gamma}_N$ ($\mu'=2.639$)	γ_N^S ($\mu'=2.6385$)	$\bar{\gamma}_N^S$ ($\mu'=2.6385$)	γ_N^S ($\mu'=2.639$)	$\bar{\gamma}_N^S$ ($\mu'=2.639$)
20	1.3052	1.3318	1.3011	1.3280	1.3305	1.3426	1.3267	1.3388
21	1.3567	1.3309	1.3527	1.3270	1.3531	1.3418	1.3491	1.3379
22	1.3078	1.3323	1.3036	1.3281	1.3309	1.3420	1.3267	1.3379
23	1.3553	1.3315	1.3508	1.3272	1.3518	1.3414	1.3474	1.3371
24	1.3100	1.3326	1.3054	1.3281	1.3313	1.3416	1.3267	1.3370

TABLE II. (a) The linear projections of the connective constant and the susceptibility exponent for the SAW on triangular lattice. (b) The Neville table of correlation-length exponents for the SAW on a triangular lattice.

(a)

N	μ_N	$\mu(N,M)$	γ_N ($\mu'=4.1517$)	γ_N ($\mu'=4.1515$)	μ_N^S	$\mu^S(N,M)$	γ_N^S ($\mu'=4.1517$)	γ_N^S ($\mu'=4.1515$)
14	4.249 00	4.1536	1.3280	1.3287	4.251 34	4.1508	1.3360	1.3367
15	4.242 60	4.1533	1.3284	1.3291	4.244 65	4.1510	1.3358	1.3366
16	4.237 00	4.1531	1.3287	1.3295	4.238 80	4.1511	1.3357	1.3365
17	4.232 05	4.1530	1.3290	1.3299	4.233 64	4.1512	1.3355	1.3364
18	4.227 64	4.1527	1.3293	1.3302	4.229 06	4.1512	1.3354	1.3363

(b)

N	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$
			$e_N^0=\nu_N$		
13	0.7463	0.7439			
14	0.7462	0.7441	0.7455		
15	0.7460	0.7444	0.7459	0.7477	
16	0.7459	0.7444	0.7450	0.7409	0.7205
17	0.7459	0.7448	0.7474	0.7589	0.8177
			$e_N^0=\nu_N^{S2}$		
13	0.7477	0.7443	0.7440		
14	0.7474	0.7443	0.7441	0.7445	
15	0.7472	0.7443	0.7444	0.7455	0.7484
16	0.7470	0.7443	0.7445	0.7454	0.7448
17	0.7469	0.7444	0.7448	0.7464	0.7496

B. Site lattice animal (SLA) on square, triangular, and hexagonal lattices (Ref. 7)

From Tables V—VII, it is seen that in all cases the ERM results are steadier than those of the ratio method. All the ν_N^S's are closer to the well-accepted value of $\nu=1$.

From the tables mentioned above we find further that the two sequences corresponding to two different values of λ have an opposite tendency in variation; one thus has reason to expect that the exact value of λ would have an intermediate value between these two λ's.

Also from the opposite tendencies of $\lambda(N,M)$ and $\lambda^S(N,M)$, one can make an error estimation for λ's.

C. Directed-site lattice animal (DSLA) on square, triangular, and hexagonal lattices (Ref. 8)

The DSLA on a triangular lattice is the only exception for which the TRM provides a better result than that of the ERM even if N is large. The reason is obvious: the ratio method provides an exact solution by accident, since the asymptotic behavior is just an exact relation in this case. In the other two remaining cases (square and hexagonal lattices) the ERM still prevails against the TRM. The DSLA on a triangular lattice is the only exception in our numerical analysis of the various lattice models.

D. Spiral SAW on square lattice (Ref. 9)

Figure 2 shows a reduced odd-even oscillation appearing in the ERM, and Table VIII shows that the sequence in the ERM is steadier.

E. SAPW's on square, triangular, and sc lattices (Ref. 1)

The SAPW problems have attracted much interest recently.[1,10] We have pointed out that errors occurred in

TABLE III. The connective constant and the susceptibility exponent for the SAW on a hexagonal lattice.

N	μ_N	$\mu(N,M)$	γ_N ($\mu'=1.848$)	μ_N^S	$\mu^S(N,M)$	γ_N^S ($\mu'=1.848$)
30	1.866 96	1.8484	1.3078	1.868 90	1.8474	1.3394
31	1.869 23	1.8468	1.3562	1.869 06	1.8467	1.3533
32	1.865 92	1.8504	1.3103	1.867 60	1.8480	1.3394
33	1.867 90	1.8472	1.3554	1.867 74	1.8473	1.3525
34	1.864 90	1.8486	1.3110	1.866 42	1.8476	1.3389

TABLE IV. (a) The susceptibility exponent for the SAW on a simple-cubic lattice. (b) The Neville tables of correlation-length exponent for the SAW on a simple-cubic lattice.

(a)

N	γ_N ($\mu'=4.6835$)	$\bar{\gamma}_N$	γ_N^S ($\mu'=4.6835$)	$\bar{\gamma}_N^S$
15	1.1756	1.1635	1.1746	1.1670
16	1.1530	1.1643	1.1600	1.1673
17	1.1744	1.1637	1.1734	1.1667
18	1.1542	1.1643	1.1604	1.1669
19	1.1734	1.1638	1.1726	1.1665

(b)

N	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$
			$e_N^0=\nu_N$		
5	0.6346	0.6158	0.6521		
6	0.6042	0.6021	0.6062	0.6062	
7	0.6256	0.6032	0.5938	0.5841	0.5974
8	0.6027	0.5894	0.5947	0.5909	0.5909
9	0.6201	0.6007	0.5974	0.5993	0.6012
			$e_N^0=\nu_N^{S2}$		
5	0.6300	0.6085	0.6440		
6	0.6219	0.6110	0.6098	0.6098	
7	0.6229	0.6054	0.6030	0.5962	0.6078
8	0.6180	0.6063	0.6015	0.5988	0.5988
9	0.6185	0.6028	0.5597	0.5981	0.5983

TABLE V. The connective constants and the susceptibility exponents for the SLA on a triangular lattice.

N	λ_N	$\lambda(N,M)$	λ'_N ($\nu'=1$)	ν_N ($\lambda'=5.19$)	ν_N ($\lambda'=5.18$)	λ_N^S	$\lambda^S(N,M)$	λ'^S_N ($\nu'=1$)	ν_N^S ($\lambda'=5.19$)	ν_N^S ($\lambda'=5.18$)
12	4.768 10	5.1690	5.2016	0.9755	0.9542	4.757 68	5.1838	5.1902	0.9996	0.9783
13	4.799 08	5.1709	5.1990	0.9792	0.9560	4.790 38	5.1827	5.1896	1.0010	0.9778
14	4.825 74	5.1724	5.1970	0.9826	0.9575	4.818 36	5.1821	5.1890	1.0025	0.9774
15	4.848 93	5.1736	5.1953	0.9858	0.9587	4.842 59	5.1818	5.1885	1.0041	0.9771
16	4.869 29	5.1746	5.1939	0.9887	0.9597	4.863 77	5.1816	5.1880	1.0057	0.9768

TABLE VI. The connective constants and the susceptibility exponents for the SLA on a square lattice.

N	λ_N	$\lambda(N,M)$	ν_N ($\lambda'=4.06$)	ν_N ($\lambda'=4.07$)	λ_N^S	$\lambda^S(N,M)$	ν_N^S ($\lambda'=4.06$)	ν_N^S ($\lambda'=4.07$)
15	3.802 24	4.0530	0.9523	0.9868	3.795 36	4.0632	0.9777	1.0122
16	3.818 00	4.0540	0.9537	0.9907	3.812 04	4.0624	0.9772	1.0141
17	3.831 96	4.0549	0.9549	0.9943	3.826 73	4.0620	0.9767	1.0161
18	3.844 40	4.0556	0.9559	0.9977	3.839 79	4.0618	0.9763	1.0182
19	3.855 57	4.0563	0.9567	1.0010	3.851 46	4.0616	0.9759	1.0202

TABLE VII. The connective constant and the susceptibility exponent for the SLA on a hexagonal lattice.

N	λ_N	$\lambda(N,M)$	ν_N ($\lambda'=3.04$)	ν_N ($\lambda'=3.03$)	λ_N^S	$\lambda^S(N,M)$	ν_N^S ($\lambda'=3.04$)	ν_N^S ($\lambda'=3.03$)
18	2.873 55	3.0285	0.9855	0.9294	2.868 24	3.0362	1.0170	0.9610
19	2.881 79	3.0294	0.9888	0.9294	2.877 06	3.0359	1.0184	0.9590
20	2.889 25	3.0305	0.9918	0.9291	2.885 01	3.0360	1.0197	0.9570
21	2.896 03	3.0313	0.9946	0.9285	2.892 21	3.0361	1.0209	0.9550
22	2.902 21	3.0318	0.9972	0.9279	2.898 75	3.0362	1.0222	0.9530

TABLE VIII. The Neville table of correlation-length exponent for the spiral SAW on a square lattice.

	$e_N^0=\nu_N$			$e_N^0=\nu_N^{S2}$		
N	$l=0$	$l=1$	$l=2$	$l=0$	$l=1$	$l=2$
60	0.6151			0.6018		
61	0.6166	0.5866		0.6023	0.6324	
62	0.6145	0.5971	0.6306	0.6027	0.6312	0.5880
63	0.6158	0.5918	0.6677	0.6032	0.6299	0.5935
64	0.6141	0.6014	0.6654	0.6036	0.6289	0.5932

TABLE IX. The susceptibility exponent for SAPW's on a square lattice.

N	γ_N ($\mu'=2.719$)	γ_N ($\mu'=2.718$)	γ_N^S ($\mu'=2.719$)	γ_N^S ($\mu'=2.718$)
14	1.3919	1.3872	1.4078	1.4121
15	1.3829	1.3886	1.4022	1.4068
16	1.3910	1.3970	1.4047	1.4095
17	1.3814	1.3877	1.3988	1.4039
18	1.3903	1.3970	1.4018	1.4072

TABLE X. The susceptibility exponent for SAPW's on a simple-cubic lattice.

N	γ_N ($\mu'=4.850$)	γ_N ($\mu'=4.849$)	γ_N^S ($\mu'=4.850$)	γ_N^S ($\mu'=4.849$)
7	1.1207	1.1222	1.1269	1.1284
8	1.1221	1.1238	1.1267	1.1284
9	1.1211	1.1229	1.1254	1.1273
10	1.1228	1.1249	1.1262	1.1283
11	1.1208	1.1231	1.1245	1.1268

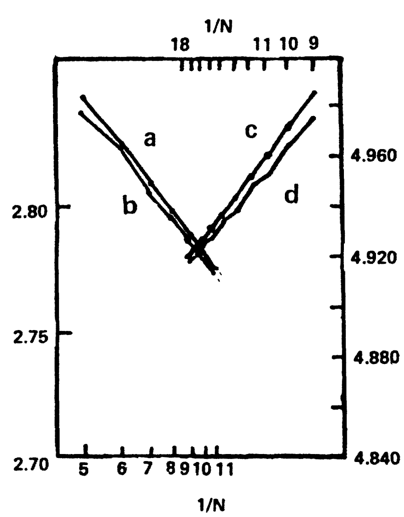

FIG. 4. Consecutive constant vs $1/N$ for the SAPW on square and triangular lattices. Curves a and b are ν_N^S and ν_N for the square lattice; curves c and d are ν_N^S and $\nu(N)$ for the triangular lattice.

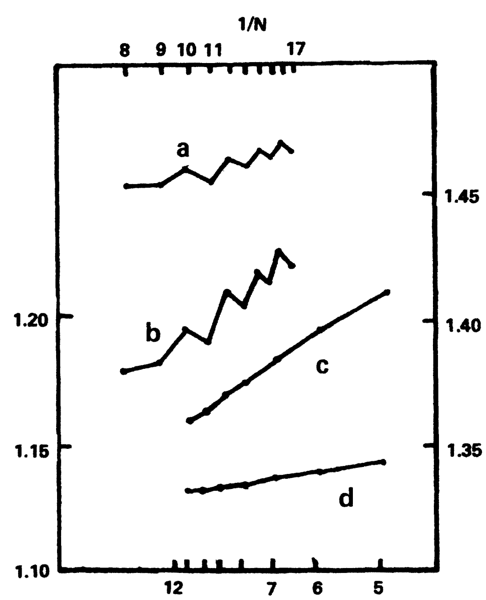

FIG. 5. Correlation exponents vs $1/N$ for the SAPW on square and sc lattices. Curves a and b are $2\nu_N^{S1}$ and $2\nu_N$ for the square lattice; curves c and d are $2\nu_N^{S1}$ and $2\nu_N$ for the sc lattice.

the exact enumeration, the fundamental starting point in series expansion, in the previous works,[11] and after a corrected exact enumeration up to $N=18$, 12, and 11 on square, triangular, and sc lattices, respectively, these authors suggested it could belong to a different universality class. This has attracted renewed interest and effort; it seems the SAPW's still have the same universality class as do SAW's, but the complexity and the necessity for analysis based on the exact enumeration of more steps was previously unexpected.

Nevertheless, our Figs. 4 and 5 and Tables IX and X show obviously the reduction of odd-even oscillation in the ERM results. It is interesting to note that the two results of μ and γ for the ERM and TRM are close when N is even, in contrast with the SAW. It seems this difference between the SAPW and the SAW is not accidental.

F. The spiral-bond animal (SBA) on square lattice

A new interesting lattice model, the SBA, has been proposed and strongly confirmed by these authors to belong to a different universality class compared with the SLA and DSLA.

In this lattice model, the ERM results are comparable to those of the TRM. Perhaps more exact enumeration is needed to show the prevalence of the ERM.

IV. SUMMARY

In this paper we recommend an extended-ratio method as a worthy one in the series-expansion approach. It is valid whenever the singularity lies on the real axis and dominates like the TRM does. Our numerical results for some varieties of the interesting lattice models show that, in almost every case studied, the ERM results are comparable to (often smoother and better than) those of the TRM, while the computer time used is almost the same for these two methods. The ERM often provides a more rapid rate of convergence; the sequences provided in the Neville table by the ERM often appear to converge more rapidly, more steadily, as well as more reliably. One can even continue the Neville table to higher orders. The Neville table for the SAW even gives a correct tendency when one compares it with the values considering the confluent correction, while one cannot obtain these results by the traditional ratio method. The odd-even oscillation, a trouble encountered in the loose-packed lattice, is reduced greatly in nearly every case. The reason for this reduction has been illustrated in the ERM mathematical basis provided in Appendix C. Also one can get some reasonable error estimates of some critical quantities such as the locations of the singularity, the correlation length, and the susceptibility exponents, in the case when the TRM and the ERM give successive ratios with opposite tendency. We pointed out that the ERM can be equally valuable in Padé approximation as well as in the problem of correction to scaling. Some phenomena presented in the ERM still remain to be resolved, especially the theoretical basis for convergence of the ERM. However, we are encouraged by our results of the application to various lattice models presented in this paper and confirm that the ERM is a very simple but rather valuable method for the numerical analysis in the series-expansion method.

APPENDIX A: TWO MATHEMATICAL THEOREMS

The following two mathematical theorems are often used in proving the mathematical basis for the ERM; we repeat them here to remind the reader.

1. Stoltz's theorem (Ref. 3)

If the following conditions are fulfilled for the two sequences $\{X_N\}$ and $\{Y_N\}$,

$$Y_{N+1}>Y_N,\quad \lim_{N\to\infty} X_N=+\infty,\quad \lim_{N\to\infty} Y_N=+\infty\ ,$$

then we have

$$\lim_{N\to\infty}(X_N/Y_N)=A\ ,$$

provided the following limit exists:

$$\lim_{N\to\infty}[(X_N-X_{N-1})/(Y_N-Y_{N-1})]=A\ . \tag{A1}$$

2. Darboux's first theorem (Ref. 4)

If the function $F(z)$ can be represented by

$$F(z)=A(z)(1-\mu z)^{-\gamma}+B(z)=\sum_{N=0}^{\infty} a_N z^N\ , \tag{A2}$$

where $A(z)$ and $B(z)$ are assumed regular in the neighborhood of $z=z_c\equiv 1/\mu$, and if, in addition, both functions are regular in the disk $|z|\le z_c$, then the asymptotic form of the coefficients a_N can be obtained by substituting for the expansion of $F(z)$ that of $A(z)(1-\mu z_c)^{-\gamma}$. Higher-order approximations may be obtained by replacing $F(z)$ by

$$\left[\sum_{l=0}^{m}\frac{(z-z_c)^l}{l!}A^{(l)}(z_c)\right](1-\mu z)^{-\gamma}\ , \tag{A3}$$

the error involved in stopping at the Nth term being always of $O(1/N)$ times this term.

APPENDIX B: PROOF OF EQS. (5)–(7)

To prove Eqs. (5)–(7), one needs to show the following three equations:

$$\sum_{m=0}^{N} C_m=A\frac{\mu}{\mu-1}\mu^N N^{\gamma-1}\left[1-\frac{\gamma-1}{\mu-1}\frac{1}{N}+O\left[\frac{1}{N^2}\right]\right]\ , \tag{B1}$$

$$\sum_{m=0}^{N} C_m\rho_m=A'\frac{\mu}{\mu-1}\mu^N N^{\gamma+2\nu-1}\times\left[1-\frac{\gamma+2\nu-1}{(\mu-1)N}+O\left[\frac{1}{N^2}\right]\right]\ , \tag{B2}$$

$$\sum_{m=0}^{N}\rho_m=B\frac{N^{2\nu+1}}{2\nu+1}\left[1+\frac{2\nu+1}{2N}+O\left[\frac{1}{N^2}\right]\right]\ . \tag{B3}$$

Firstly, using Stoltz's theorem, we have

$$\lim_{N\to\infty}\left[\sum_{m=0}^{N} C_m \Big/ C_N\right]=\lim_{N\to\infty}[C_N/(C_N-C_{N-1})]$$
$$=\frac{\mu}{\mu-1}\ .$$

Put

$$\sum_{m=0}^{N} C_m \equiv \frac{\mu}{\mu-1}C_N - f(N)\ , \tag{B4}$$

thus one has

$$f(N)-f(N-1)$$
$$\approx A\mu^N N^{\gamma-1}\left[1-\left[1-\frac{1}{N}\right]^{\gamma-1}\right](\mu-1)^{-1} \tag{B5}$$

as N is large enough.

Suppose $\gamma>1$ (which is not necessary—a parallel procedure can be performed equally for $\gamma\le 1$); thus $\lim_{N\to\infty}[f(N)-f(N-1)]=+\infty$ which means $\lim_{N\to\infty} f(N)=+\infty$.

Making use of Stolz's theorem once again,

$$\lim_{N\to\infty}\frac{f(N)}{\mu^N N^{\gamma-2}}=\lim_{N\to\infty}\frac{f(N)-f(N-1)}{\mu^N N^{\gamma-2}-\mu^{N-1}(N-1)^{\gamma-2}}$$
$$=A\frac{\mu(\gamma-1)}{(\mu-1)^2}\ .$$

Thus

$$f(N)=A\frac{\mu(\gamma-1)}{(\mu-1)^2}\mu^N N^{\gamma-2}[1+\delta(N)]\ , \tag{B6}$$

where $\lim_{N\to\infty}\delta(N)=0$. We can further prove that $\delta(N)\sim O(1/N)$ for the same reason.

From (B4) and (B6), one gets (B1) straightforwardly. Considering $C_N\rho_N\approx A'\mu^N N^{\gamma+2\nu-1}$, one can prove Eq. (B2) if one repeats the steps above for Eq. (B1). The same is also true for Eq. (B3), if one sets

$$g(N)\equiv\sum_{m=0}^{N}\rho_m - N\rho_N/(2\nu+1)\ .$$

From (B1)—(B3), one gets (5)—(7) at once. Thus the reasoning of the ERM is illustrated by Stolz's theorem.

An alternative way for proving Eqs. (B1)—(B3) is by means of Darboux's first theorem.

Defining the correlation length $\xi(z)$ as usual,

$$\xi(z)=\sum_{N=0}^{\infty} C_N\rho_N z^N/f(z)\approx B'(1-\mu z)^{-2\nu}\ , \tag{B7}$$

in which $f(z)$ was given in Eq. (8) in the text, one thus has

$$g(z)\equiv\sum_{N=0}^{\infty} C_N\rho_N z^N\approx B(1-\mu z)^{-\gamma-2\nu}\ . \tag{B8}$$

From the formula for generation of power series, one has

$$(1-z)^{-1}f(z)=\sum_{N=0}^{\infty}\left[\sum_{m=0}^{N} C_m\right]z^n=\sum_{N=0}^{\infty} C'_N z^N \tag{B9}$$

and

$$(1-z)^{-1}g(z)=\sum_{N=0}^{\infty}\left[\sum_{m=0}^{N} C_m\rho_m\right]z^N\ ; \tag{B10}$$

comparing (B9) with (A2) and according to Darboux's first theorem, one obtains

$$C'_N=\sum_{m=0}^{N} C_m = A\sum_{l=0}^{\infty}(-1)^l\frac{\mu^{-l}}{l!}\left[\frac{d^l}{dz^l}(1-z)^{-1}\right]_{z=z_l}$$
$$\times(-1)^N\left[l-\frac{\gamma}{N}\right]\mu^N$$
$$=\frac{A}{\Gamma(\gamma)}\frac{\mu}{\mu-1}\mu^N\left[1-\frac{\gamma-1}{\mu-1}\frac{1}{N}+\cdots\right]\ .$$

This is just Eq. (B1). One can get Eq. (B2) from Eq. (B10) by the same steps.

APPENDIX C: REDUCTION OF ODD-EVEN OSCILLATION IN THE ERM

As in Appendix B, two alternative approaches are used.

(i) When N is even, setting $A_l\equiv b_{2l}-b_{2l-1}$ it is easy to prove $\lim_{N\to\infty}A_l=+\infty$, and hence one has

$$\lim_{N\to\infty}\frac{\sum^N A_l}{\mu^{2N}(2N)^{\gamma'-1}}$$
$$=\lim_{N\to\infty}\frac{A_N}{\mu^{2N}(2N)^{\gamma'-1}-\mu^{2N-1}(2N-1)^{\gamma'-1}}=\frac{\mu}{\mu+1}$$

by using Stolz's theorem.

Thus,

$$\sum_{l=0}^{N/2}A_l=\sum_{l=0}^{N}(-1)^l b_l=\frac{\mu}{\mu+1}\mu^N N^{\gamma'-1}[1+\delta(N)] \tag{C1}$$

in which $\delta(N)\sim)D(1/N)$.

When N is odd, set $N=2M+1$ and adapt the above procedure; then one gets

$$\sum_{l=0}^{N}(-1)^l b_l=-\frac{\mu}{\mu+1}\mu^N N^{\gamma'-1}[1+\delta(N)]\ . \tag{C2}$$

Combining (C1) and (C2), one gets the Eq. (12) that appeared in the text.

For the sequences $\sum^N\rho_m$ and $\sum^N C_m\rho_m$ one can prove the reduction of odd-even oscillation by just the same steps mentioned above.

(ii) Consider the generating function of susceptibility

$$G(z)=A(1-\mu z)^{-\gamma}+A'(1+\mu z)^{-\gamma'}\ . \tag{C3}$$

Expanding $G(z)$ and $(1-z)^{-1}G(z)$ around the origin,

$$G(z)=\sum_{N=0}^{\infty} C_N z^N$$

and

$$(1-z)^{-1}G(z)=A(1-z)^{-1}(1-\mu z)^{-\gamma}$$
$$+A'(1-z)^{-1}(1+\mu z)^{-\gamma'}$$
$$=\sum_{N=0}^{\infty} C'_N z^N=\sum_{N=0}^{\infty}\left[\sum_{m=0}^{N} C_m\right]z^N\ .$$

Making use of Darboux's first theorem for the last two lines, then

$$C'_N=\sum_{m=0}^{N} C_m$$

$$=A\frac{\mu}{\mu-1}\frac{\mu^N}{\Gamma(\gamma)}N^{\gamma-1}\left[1-\frac{\gamma-1}{\mu-1}\frac{1}{N}+\cdots\right]$$

$$+A'\frac{\mu}{\mu+1}\frac{(-1)^N}{\Gamma(\gamma)}\mu^N N^{\gamma'-1}\left[1+\frac{\gamma'-1}{\mu+1}\frac{1}{N}+\cdots\right],$$

where the second term in the above line corresponds to the oscillating one, hence the oscillation term is reduced in the ERM.

For the sequence of $\sum^N C_m \rho_m$ one can get the same conclusion.

After comparing the two approaches based on Stolz's theorem and Darboux's first theorem, one finds that the exact asymptotic expressions for $\sum_{m=0}^N C_m$ and $\sum^N C_m\rho_m$ can be derived by utilizing Darboux's first theorem. But we can solve neither the problem with a logarithmic singularity nor the problem concerning $\sum^N \rho_m$ by Darboux's first theorem, while it is easy to resolve these problems by Stolz's theorem. Thus the two approaches complement each other.

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