Theory of resonantly activated rate processes

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(Received 18 March 1985)

The very recent experimental results on the new phenomenon of resonant activation in current-biased Josephson junctions [M. H. Devoret, J. M. Martinis, D. Esteve, and J. Clarke, Phys. Rev. Lett. 53, 1260 (1984)] are completely accounted for via a theoretical approach which settles the problem of determining the rate of escape of a highly inertial Brownian particle from a potential well in the presence of a radiation field. To get this very satisfactory agreement with experiment a theory was developed, the main features of which are as follows. (1) The subtle problem of elimination of irrelevant variables is dealt with by devoting special attention to the case where the time scale of the system of interest is not well separated from that of the irrelevant variables. (2) A perturbation approach is used which, in the absence of stochastic force, is proved to coincide with the well-known method of multiple time scales. (3) It is assumed that the process of excitation-relaxation within the well is much faster than the process of escape from the well itself. The theory of this paper predicts analytically the frequency position for the maximum escape rate in terms of a suitable renormalized anharmonicity parameter α. In the conditions of the aforementioned experiment this theory predicts the shift from the natural frequency to be 2α (with an agreement with experiment of $\pm 1\%$). Furthermore, theory predicts that if the friction is lowered, a new phenomenon takes place: the shift of the maximum from the natural frequency is reduced to α. Analytical predictions on the friction region where this transition takes place are made.

I. INTRODUCTION

The generalization of the Kramers theory[1] to radiatively activated rate processes is an intriguing problem which is not completely solved. In the overdamped case it is possible to build up a Fokker-Planck-like equation for the space variable where the influence of the radiation field appears via a suitable diffusion term.[2,3] This theory, however, leaves completely unexplored the field of photoselective chemical reactions,[4] where inertia should play a fundamental role.

On the other hand, we feel challenged to study precisely this region by the results of a recent experiment on a Josephson junction,[5] some relevant features of which are left theoretically unexplained. These are the following. (1) The escape rate reachs its maximum value at a frequency lower than that corresponding to the natural frequency of the junction. (2) The line shape is asymmetric around the maximum and appears broader in the low-frequency region. The theory developed in the present paper will be proven to satisfactorily account for both properties. Moreover, the agreement is also quantitative as theory predicts the frequency position for the maximum escape rate within the accuracy of the experiment itself.

The particular problem treated in this paper touches also a theoretical aspect of more general interest, which has been the subject of many related papers,[6-9] that is, how to avoid the technical difficulties associated with the summation at infinite order. This paper sheds further insight into this issue.

The outline of the paper is as follows. Section II is devoted to deriving the equation for energy diffusion which will be applied to evaluating the rate of escape in the presence of a radiation field. This rate of escape relies on an analytical expression which, in turn, rests on the amount of energy absorbed from the field per unit of time. Section III will aim mainly at determining, via different techniques, this significant information. Section IV will show this new theory at work by comparing the recent experimental results of Ref. 5 with the theoretical predictions. The concluding remarks will be presented in Sec. V.

II. ENERGY DIFFUSION

The system under study in this paper is

$$\dot{x}=v\ ,$$
$$\dot{v}=-\frac{\partial V}{\partial x}-\gamma v+F(t)\ . \tag{2.1}$$

x and v denote the space coordinate and velocity of a Brownian particle which undergoes the influence of a conservative field associated with the potential

$$V(x)=\tfrac{1}{2}\omega_0^2x^2-\tfrac{1}{4}\beta x^4 , \tag{2.2}$$

a friction term $-\gamma v$, and a time-dependent force $F(t)$, which in turn is divided into two parts

$$F(t)=f_S(t)+f_r(t) , \tag{2.3}$$

$f_S(t)$ being a Gaussian white noise defined by

$$\langle f_S(0)f_S(t)\rangle=2D_S\delta(t)=2\gamma k_BT\delta(t) , \tag{2.4}$$

and $f_r(t)$ the force produced by the interaction with a radiation field of frequency ω_R. Throughout this paper we will devote our attention to the extremely underdamped regime ($\gamma\to 0$).

The basic assumptions on which the theory of the present paper rests are the following.

(a) The variable energy

$$E\equiv\tfrac{1}{2}v^2+V(x) \tag{2.5}$$

is much slower than the variables x and v, thereby making it possible to build up a diffusion equation for this variable alone.

(b) The process of escape from the potential well is extremely slow when compared to the motion of the Brownian particle within the well.

These two assumptions will allow us to focus first on the problem of describing the energy diffusion within the well without paying attention to the escape process. Then the escape process will be accounted for via a first-passage time method relying precisely on the energy-diffusion equation determined through the first step.

In principle, the energy-diffusion equation may be arrived at by rewriting Eq. (2.1) in terms of x and E. This provides

$$\dot{x}=\{2[E-V(x)]\}^{1/2} , \tag{2.6a}$$

$$\dot{E}=-2\gamma[E-V(x)]+F(t)\{2[E-V(x)]\}^{1/2} . \tag{2.6b}$$

If the dynamics of the system, the influence of which on E is simulated by the force $F(t)$, is extremely fast, it is safe to define F via

$$\langle F(0)F(t)\rangle=2D_T\delta(t) , \tag{2.7a}$$

where

$$D_T=D_r+D_S , \tag{2.7b}$$

and D_r is a contribution coming from the radiation field. In such a case the Fokker-Planck equation associated with the system of Eqs. (2.6a) and (2.6b) reads

$$\frac{\partial}{\partial t}\rho(x,E,t)=\left[-\frac{\partial}{\partial x}\{2[E-V(x)]\}^{1/2}+2\gamma\frac{\partial}{\partial E}\left[E-V(x)-\frac{D_T}{2\gamma}\right]+2\frac{\partial^2}{\partial E^2}D_T[E-V(x)]\right]\rho(x,E,t) , \tag{2.8}$$

from which, by applying the standard Stratonovich assumption,[10,11]

$$\rho(x,E,t)=W_2(x,t\mid E)\sigma(E,t) , \tag{2.9}$$

with

$$W_2(x,t\mid E)\propto\frac{1}{[E-V(x)]^{1/2}} , \tag{2.10}$$

and contracting over the variable x we get

$$\frac{\partial}{\partial t}\sigma(E,t)=\left[2\gamma\frac{\partial}{\partial E}\left[\frac{\varphi(E)}{2\varphi'(E)}-\frac{D_T}{2\gamma}\right]+2\frac{\partial^2}{\partial E^2}D_T\frac{\varphi(E)}{\varphi'(E)}\right]\sigma(E,t) , \tag{2.11}$$

where

$$\varphi(E)\equiv\int_R dx[E-V(x)]^{1/2} , \tag{2.12}$$

R being the region characterized by $E\geq V(x)$. The major difficulty involved by highly resonant systems is that these are very far from the condition where the picture of the irrelevant system, as given by Eq. (2.7), is reliable. However, we shall show that an equation of the same type as Eq. (2.11) is really allowed provided that D_R is not regarded as completely independent of the dynamics of the system of interest.

A. Correctly eliminating the radiation field freedom degrees

According to the spirit of the "reduced" model theory[12] (RMT) we must detail, via suitable auxiliary variables, the dynamics of the system resulting in the force $F(t)$ of Eq. (2.1). Special care must be devoted to describing the dynamics of the radiation field since this will significantly affect the system of interest via resonance phenomena.

Thus, let us replace Eq. (2.1) with

$$\begin{aligned}\dot{x}&=v ,\\ \dot{v}&=-\frac{\partial V}{\partial x}-\gamma v+f_S(t)+\omega_I^2y ,\\ \dot{y}&=w ,\\ \dot{w}&=-\omega_R^2y-\lambda w+f_R(t) ,\end{aligned} \tag{2.13}$$

where $f_R(t)$ is a Gaussian white noise defined by

$$\langle f_R(0)f_R(t)\rangle=2D_R\delta(t)\equiv 2\lambda\langle w^2\rangle_{eq}\delta(t) . \tag{2.14}$$

The field will produce resonance effects when condition

$$\lambda<2\omega_R \tag{2.15}$$

is fulfilled.

The Fokker-Planck equation corresponding to Eq. (2.13) reads

$$\frac{\partial}{\partial t}\rho(x,v,y,w,t)=\mathscr{L}\rho(x,v,y,w,t)=\left\{\left[-v\frac{\partial}{\partial x}+\left[\frac{\partial V}{\partial x}\right]\frac{\partial}{\partial v}+\gamma\left[\frac{\partial}{\partial v}v+k_BT\frac{\partial^2}{\partial v^2}\right]\right]-\omega_I^2\frac{\partial}{\partial v}y\right.$$
$$\left.+\left[-w\frac{\partial}{\partial y}+\omega_R^2 y\frac{\partial}{\partial w}+\lambda\left[\frac{\partial}{\partial w}w+\langle w^2\rangle_{\rm eq}\frac{\partial^2}{\partial w^2}\right]\right]\right\}\rho(x,v,y,w,t)\ . \tag{2.16}$$

Let us define $1/\tau_R$ as the largest of the two parameters λ and ω_R; that is

$$\frac{1}{\tau_R}\equiv\max(\lambda,\omega_R)\ . \tag{2.17}$$

Let us also assume the anharmonic term of $V(x)$ to be weak enough as to make it possible to characterize the dynamics of the system of interest via the parameters γ and ω_0.[13] We define

$$\frac{1}{\tau_x}\equiv\max(\gamma,\omega_0)\ . \tag{2.18}$$

If

$$\tau_R\ll\tau_x\ , \tag{2.19}$$

we are allowed to choose the latter term between square brackets of Eq. (2.16) as the unperturbed part of the operator $\mathscr{L}$. In the present paper Eq. (2.19) is not fulfilled. We are thus obliged to define the unperturbed part of $\mathscr{L}$ as also including the former term between square brackets, i.e.,

$$\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1=\mathscr{L}_a+\mathscr{L}_b+\mathscr{L}_1\ ,$$
$$\mathscr{L}_a\equiv-v\frac{\partial}{\partial x}+\left[\frac{\partial V}{\partial x}\right]\frac{\partial}{\partial v}+\gamma\left[\frac{\partial}{\partial v}v+k_BT\frac{\partial^2}{\partial v^2}\right]\ ,$$
$$\mathscr{L}_b\equiv-w\frac{\partial}{\partial y}+\omega_R^2 y\frac{\partial}{\partial w}+\lambda\left[\frac{\partial}{\partial w}w+\langle w^2\rangle_{\rm eq}\frac{\partial^2}{\partial w^2}\right]\ , \tag{2.20}$$
$$\mathscr{L}_1\equiv-y\omega_I^2\frac{\partial}{\partial v}\ .$$

As usual,[12,14] we must apply the Zwanzig projection method to the interaction picture:

$$\frac{\partial}{\partial t}\tilde{\rho}=\mathscr{L}_1(t)\tilde{\rho}\ , \tag{2.21}$$

where

$$\tilde{\rho}\equiv e^{-\mathscr{L}_0 t}\rho\ ,$$
$$\mathscr{L}_1(t)\equiv e^{-\mathscr{L}_0 t}\mathscr{L}_1 e^{\mathscr{L}_0 t}\ . \tag{2.22}$$

By making the Born assumption we then obtain

$$\frac{\partial}{\partial t}\sigma(x,v,t)=\mathscr{L}_a\sigma(x,v,t)+\frac{\omega_I^4}{2}\langle y^2\rangle_{\rm eq}\sum_{\mu=\pm1}\int_0^t d\tau\frac{\partial}{\partial v}e^{(\mathscr{L}_a+i\mu\omega_R-\lambda/2)(t-\tau)}\frac{\partial}{\partial v}\sigma(x,v,t)\ . \tag{2.23}$$

For $t\to\infty$ we have

$$\frac{\partial}{\partial t}\sigma(x,v,t)=\mathscr{L}_a\sigma(x,v,t)+\frac{\omega_I^4}{2}\langle y^2\rangle_{\rm eq}\sum_{\mu=\pm1}\frac{\partial}{\partial v}\frac{1}{\frac{\lambda}{2}-i\mu\omega_R-\mathscr{L}_a}\frac{\partial}{\partial v}\sigma(x,v,t)\ . \tag{2.24}$$

It is also possible to write Eq. (2.24) as (see Appendix A)

$$\frac{\partial}{\partial t}\sigma(x,v,t)=\mathscr{L}_a\sigma(x,v,t)+\frac{\omega_I^4}{2}\langle y^2\rangle_{\rm eq}\left[\frac{\partial^2}{\partial v^2}\hat{W}_{vv}+\frac{\partial^2}{\partial x\partial v}\hat{W}_{xv}+\sum_{n+m=2}\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial v^m}\hat{W}_{x^n v^m}\right]\sigma(x,v,t)\ . \tag{2.25}$$

When dealing with a weakly anharmonic system, we are allowed[14] to neglect terms which break the Fokker-Planck structure. Furthermore, we can replace the diffusion operators $\hat{W}_{vv}$ and $\hat{W}_{xv}$ with their mean values, evaluated over the equilibrium distribution in the absence of radiation field. This leads us to

$$\frac{\partial}{\partial t}\sigma(x,v,t)=\mathscr{L}_a\sigma(x,v,t)+\frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2}\left[\langle\hat{W}_{vv}\rangle\frac{\partial^2}{\partial v^2}+\langle\hat{W}_{xv}\rangle\frac{\partial^2}{\partial x\partial v}\right]\sigma(x,v,t)\ . \tag{2.26}$$

As to $\langle\hat{W}_{xv}\rangle$, this will be proven to not contribute to the equation of energy diffusion. As to $\langle\hat{W}_{vv}\rangle$, this in turn can be given a more effective expression. We see, indeed, from Eq. (2.23) that the kinetic energy absorbed from the radiation field per unit of time is

$$P\equiv\left[\frac{d}{dt}\frac{\langle v^2\rangle}{2}\right]_{\rm rad.field}=-\frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2}\sum_{\mu=\pm1}\int_{-\infty}^{+\infty}dx\,dv\,v\int_0^\infty ds\,e^{(\mathscr{L}_a+i\mu\omega_R-\lambda/2)s}\frac{\partial}{\partial v}\sigma(x,v,t)\ . \tag{2.27}$$

If $\sigma(x,v,t)$ on the right-hand side (rhs) of Eq. (2.27) is replaced by $\sigma_{\rm eq}$, defined by

$$\sigma_{\rm eq} \propto e^{(-v^2/2k_BT)} e^{(-V(x)/k_BT)} , \tag{2.28}$$

from Eq. (2.27) we get

$$P=\frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2k_BT}\sum_{\mu=\pm1}\int_0^\infty ds\int_{-\infty}^{+\infty} dx\,dv\,v e^{(\mathcal{L}_a+i\mu\omega_R-\lambda/2)s} v\sigma_{\rm eq}(x,v) . \tag{2.29}$$

On the other hand, from Eq. (2.26) we obtain

$$P=\frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2}\int_{-\infty}^{+\infty} dx\,dv\,\hat{W}_{vv}\sigma_{\rm eq}(x,v) . \tag{2.30}$$

This means that if the mean-field approximation is related to the equilibrium distribution of Eq. (2.28), then for $\lambda\to 0$

$$\langle \hat{W}_{vv}\rangle_{\rm eq}=\sum_{\mu=\pm1}\int_0^\infty \frac{\langle v v(t)\rangle_{\rm eq}}{2k_BT} e^{i\mu\omega_R t} dt , \tag{2.31}$$

which is precisely the result provided by the linear-response theory.[15]

B. The energy-diffusion equation

We are now in a position to properly apply the Stratonovich method outlined at the very beginning of this section. In Sec. III it will appear clear that in the weakly anharmonic case we are allowed to neglect the anharmonic contribution to the total energy. Let us define, therefore,

$$E\equiv\tfrac{1}{2}v^2+\tfrac{1}{2}\omega_0^2x^2 . \tag{2.32}$$

By applying the Stratonovich method we obtain

$$\frac{\partial}{\partial t}\sigma(E,t)=\left[\gamma\frac{\partial}{\partial E}E-\gamma k_BT\frac{\partial}{\partial E}+\gamma k_BT\frac{\partial^2}{\partial E^2}E - \frac{\partial}{\partial E}\frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2}\langle\hat{W}_{vv}\rangle + \frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2}\langle\hat{W}_{vv}\rangle\frac{\partial^2}{\partial E^2}E\right]\sigma(E,t) . \tag{2.33}$$

The remarkable features of this result are the following.

(a) The weakly anharmonic case, which allows Eq. (2.32) to be used, produces a diffusion equation for E which is completely unaffected by the anharmonicity [the term $\beta x^3\partial/\partial v$ appearing in Eq. (2.16) is proven to not contribute at all if the definition of Eq. (2.32) is used] if the radiation field is not present. When the radiation field is turned on, however, the anharmonic term will be proven to exert a significant role through $\langle\hat{W}_{vv}\rangle$.

(b) In the presence of a radiation field, the Fokker-Planck structure of Eq. (2.11) would be completely recovered [if the definition of Eq. (2.5) were used] when giving the contribution D_R of Eq. (2.7) the following form:

$$D_R=\frac{\omega_I^4\langle y^2\rangle_{\rm eq}}{2}\langle\hat{W}_{vv}\rangle . \tag{2.34}$$

This is a pleasant consequence of applying the correct elimination procedure of Sec. II A.

III. THE ROTATING-WAVE APPROXIMATION

This section will be devoted to determining via different techniques the amount of energy absorbed by the system from the radiation field in the unit of time, that is, D_R.

Let us divide the operator $\mathcal{L}$ of Eq. (2.16) into an unperturbed and perturbed part as follows (we assume $\lambda=0$ and $f_R(t)=0$):

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1 , \tag{3.1}$$

$$\mathcal{L}_0=-v\frac{\partial}{\partial x}+\omega_R^2x\frac{\partial}{\partial v}-w\frac{\partial}{\partial y}+\omega_R^2y\frac{\partial}{\partial w} , \tag{3.2}$$

$$\mathcal{L}_1=(\omega_0^2-\omega_R^2)x\frac{\partial}{\partial v}-\beta x^3\frac{\partial}{\partial v}-\omega_I^2y\frac{\partial}{\partial v} +\gamma\left[\frac{\partial}{\partial v}v+k_BT\frac{\partial^2}{\partial v^2}\right] . \tag{3.3}$$

Note that the unperturbed part consists of two exactly degenerate harmonic oscillators. To get this exact degeneracy we have been constrained to include a detuning term [first contribution to $\mathcal{L}_1$ on Eq. (3.3)]. The other contributions to $\mathcal{L}_1$ will be referred to as anharmonic, pumping, and fluctuation-dissipation terms, respectively.

The only basic assumption behind the theory developed in this section is that the absorption linewidth $\Delta\Gamma$ fulfills the condition

$$\Delta\Gamma\ll\omega_0 . \tag{3.4}$$

This justifies indeed the division of $\mathcal{L}$ into $\mathcal{L}_0$ and $\mathcal{L}_1$ as illustrated by Eqs. (3.2) and (3.3). We shall again apply the interaction picture of Eq. (2.21) with $\mathcal{L}_1(t)$, Eq. (2.22), related to the definition of $\mathcal{L}_0$ and $\mathcal{L}_1$ given by Eqs. (3.2) and (3.3).

By making use of the new variables

$$\alpha_\pm\equiv v\pm i\omega_R x , \tag{3.5}$$

$$\beta_\pm\equiv w\pm i\omega_R y , \tag{3.6}$$

it is shown that in the interaction picture $\mathcal{L}_1(t)$ consists of both a time-independent part and a time-dependent part, to which are associated the oscillation frequencies $\pm 2i\omega_R$. The assumption of Eq. (3.4) allows us to neglect these terms.

We then obtain

$$\overline{\mathscr{L}}_1=\frac{\omega_0^2-\omega_R^2}{2i\omega_R}\left[\frac{\partial}{\partial\alpha_+}\alpha_+-\frac{\partial}{\partial\alpha_-}\alpha_-\right]-\frac{3i\beta}{8\omega_R^3}\left[\frac{\partial}{\partial\alpha_-}(\alpha_+\alpha_-^2)-\frac{\partial}{\partial\alpha_+}(\alpha_-\alpha_+^2)\right]$$
$$+\frac{\gamma}{2}\left[\frac{\partial}{\partial\alpha_+}\alpha_+\frac{\partial}{\partial\alpha_-}\alpha_-+4k_BT\frac{\partial}{\partial\alpha_+}\frac{\partial}{\partial\alpha_-}\right]+\frac{i\omega_I^2}{2\omega_R}\left[\frac{\partial}{\partial\alpha_+}\beta_+-\frac{\partial}{\partial\alpha_-}\beta_-\right]. \tag{3.7}$$

We shall come back to this equation later on [see Eq. (3.45)]. When expressed again in terms of the original variables x and v, $\overline{\mathscr{L}}_1$ of Eq. (3.5) reads

$$\overline{\mathscr{L}}_1=\frac{\Delta\omega}{\omega_R}\left[\frac{\partial}{\partial x}v-\omega_R^2\frac{\partial}{\partial v}x\right]-\frac{3\beta}{\omega_R^4}\left[\frac{\partial}{\partial v}(\omega_R^2xv^2+\omega_R^4x^3)-\frac{\partial}{\partial x}(v^3+\omega_R^2x^2v)\right]$$
$$+\frac{\gamma}{2}\left[\frac{\partial}{\partial v}v+\frac{\partial}{\partial x}x+k_BT\frac{\partial^2}{\partial v^2}+\frac{k_BT}{\omega_R^2}\frac{\partial^2}{\partial x^2}\right]+\frac{\omega_I^2}{2\omega_R^2}\left[\frac{\partial}{\partial x}w-\omega_R^2\frac{\partial}{\partial v}y\right],\quad \Delta\omega\equiv\omega_R-\omega_0\,. \tag{3.8}$$

The main advantage of this representation is that the differential operators concerning the radiation field disappeared. The influence of the radiation field depends on the parameters w and y which express the initial conditions of the oscillator $\dot{y}=w$, $\dot{w}=-\omega_R^2y$.

A. The deterministic limit

This limit is reached when $T\to0$; that is, when the stochastic force is completely absent while maintaining the damping. The linear underdamped theory will predict unbounded oscillations when the radiation field is exactly resonant with the system ($\Delta\omega=0$) irrespectively of how small the excitation is. In the actual system these large oscillations are limited by the damping and the nonlinearity; furthermore, we will be able to demonstrate at the end of this section that in the absence of damping, where the deterministic theory[16] predicts an unbounded and infinitely sharp bended peak, the presence of a stochastic force will produce finite amplitude and width as an effect of the interplay between nonlinearity and fluctuations.

When $T\to0$ the system tends to become deterministic and it is therefore more convenient to describe its temporal evolution in terms of amplitude and phase. Starting from Eq. (3.6) we will introduce the new variables

$$x=a\cos\varphi\,,\qquad v=\omega_Ra\sin\varphi\,, \tag{3.9}$$

where a is the amplitude of the response of the system to excitation and φ is the phase. After a straightforward calculation we obtain

$$\frac{\partial}{\partial t}\rho(a,\varphi,t)=\left[-\Delta\omega\frac{\partial}{\partial\varphi}+\frac{\gamma}{2}\frac{\partial}{\partial a}a+\frac{\gamma k_BT}{2\omega_R^2}\frac{\partial^2}{\partial a^2}-\frac{\gamma k_BT}{2\omega_R^2}\frac{\partial}{\partial a}\frac{1}{a}\right.$$
$$\left.+\frac{\gamma k_BT}{2\omega_R^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{a^2}-\frac{3}{8}\frac{\beta}{\omega_R}\frac{\partial}{\partial\varphi}a^2-\frac{\kappa}{2\omega_R}\frac{\partial}{\partial a}\sin\varphi-\frac{\kappa}{2\omega_R}\frac{\partial}{\partial\varphi}\cos\varphi\frac{1}{a}\right]\rho(a,\varphi,t),\quad \kappa\equiv\omega_I^2y\,. \tag{3.10}$$

When $T=0$ we recover the completely deterministic case. The resulting equations of motion for a and φ can be written as

$$\dot{a}=-\frac{\gamma}{2}a+\frac{\kappa}{2\omega_R}\sin\varphi\,,\qquad a\dot{\varphi}=\Delta\omega a+\frac{3}{8}\frac{\beta}{\omega_R}a^3+\frac{\kappa}{2\omega_R}\cos\varphi\,. \tag{3.11}$$

These same equations can be obtained using a completely different theory, namely the theory of multiple time scales.[16] Steady-state motions occur when $\dot{a}=\dot{\varphi}=0$ and correspond to the solution of

$$\frac{\gamma}{2}a=\frac{1}{2}\frac{\kappa}{\omega_R}\sin\varphi\,,\qquad a\Delta\omega+\frac{3}{8}\frac{\beta}{\omega_R}a^3=-\frac{1}{2}\frac{\kappa}{\omega_R}\cos\varphi\,. \tag{3.12}$$

Squaring and adding these equations we obtain

$$\left[\frac{\gamma^2}{4}+\left[\Delta\omega+\frac{3}{8}\frac{\beta}{\omega_R}a^2\right]^2\right]a^2=\frac{\kappa^2}{4\omega_R^2}\,. \tag{3.13}$$

This equation is an implicit equation for the amplitude of the response a as a function of the detuning $\Delta\omega$ and the amplitude of the excitation κ: it is called the "frequency-response" equation. Figure 1 shows this result for different values of the damping γ.

The results illustrated in Fig. 1 show that the limit $\gamma\to0$ should be approached with some caution. We see indeed that for $\gamma\to0$ the spectral separation of the resonance at ω_0 from the null frequency is completely broken. In the presence of a stochastic force, however, the linewidth of the absorption spectrum will be shown to be limited by a residual width which determines an upper bound to the maximum absorption intensity. This will make the theory valid for a wider interval of the friction

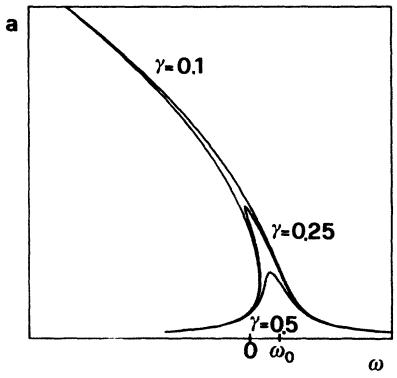

FIG. 1. Effect of damping on the frequency-response curve, Eq. (3.13). ($\beta=1.0$, $\omega_0=1.0$, $k=1.0$).

parameter γ that is $0\leq\gamma<\Delta\Gamma\ll\omega_0$. In other words, it will be shown that the absorption linewidth $\Delta\Gamma$ is finite even in the limit case $\gamma=0$.

B. The general case

To deal with the general case we must face the problem of deciding which is the most suitable basis set to expand the operator $\overline{\mathscr{L}}_1$ of Eq. (3.8). We shall use a basis set of Hermite polynomials and the formalism of creation and destruction operators. We want to introduce creation and destruction operators satisfying the useful properties

$$a_+^\dagger=a_- \ , \qquad a_-^\dagger=a_+ \ . \tag{3.14}$$

To this aim we shall proceed as follows.

(1) First of all we make the transformation

$$\overline{\mathscr{L}}_1\Rightarrow e^{v^2/4k_BT}e^{\omega_R^2x^2/4k_BT}\overline{\mathscr{L}}_1e^{-v^2/4k_BT}e^{-\omega_R^2x^2/4k_BT} \ . \tag{3.15}$$

(2) Also, we use the adimensional variables

$$\xi=\frac{x\omega_R}{(2k_BT)^{1/2}} \ , \tag{3.16}$$

$$\eta=\frac{v}{(2k_BT)^{1/2}} \ . \tag{3.17}$$

The variable ξ is associated with the creation (destruction) operators $a_\pm$ defined by

$$a_\pm=\frac{1}{\sqrt{2}}\left[\xi\mp\frac{\partial}{\partial\xi}\right] , \tag{3.18}$$

which certainly satisfy the properties of Eq. (3.14). The same properties are satisfied by the operators

$$b_\pm=\frac{1}{\sqrt{2}}\left[\eta\mp\frac{\partial}{\partial\eta}\right] , \tag{3.19}$$

associated with the variable η.

The expansion basis set is then given by the direct product

$$|nm\rangle=|n\rangle|m\rangle\equiv N_ne^{-\xi^2/2}H_n(\xi)\cdot N_me^{-\eta^2/2}H_m(\eta) \ , \tag{3.20}$$

where $H_n(\xi)$ and $H_m(\eta)$ denote Hermite polynomials. Note that

$$a_+=\sum_{n=0}^{\infty}|n+1\rangle(n+1)^{1/2}\langle n| \ , \quad a_-=a_+^\dagger \tag{3.21}$$

$$b_+=\sum_{m=0}^{\infty}|m+1\rangle(m+1)^{1/2}\langle m| \ , \quad b_-=b_+^\dagger \ . \tag{3.22}$$

The final result is ($\overline{\Gamma}_1\equiv\overline{\mathscr{L}}_1$)

$$\begin{aligned}\overline{\Gamma}_1=\Delta\omega(a_-b_+-a_+b_-)+\alpha\Big[&\left(a_+b_-b_+b_-+b_-a_+a_-a_+-a_-b_+b_-b_+-b_+a_-a_+a_-\right)\\&+\frac{1}{2}\left(a_+b_-b_+^2+a_+^3b_--a_-b_+^3-b_+a_-a_+^2\right)\\&-\frac{1}{2}\left(a_-b_+b_-^2+a_-^3b_+-a_+b_-^3-b_-a_+a_-^2\right)\Big]-\frac{\gamma}{2}(a_+a_-+b_+b_-)+\frac{\beta_2}{\sqrt{2}}b_+-\frac{\beta_1}{\sqrt{2}}a_+ \ ,\end{aligned} \tag{3.23}$$

where

$$\alpha=-\frac{3}{4}\frac{\beta}{\omega_R^3}k_BT \ , \tag{3.24}$$

$$\beta_1\equiv\frac{\omega_I^2}{2\omega_R}\frac{w}{(2k_BT)^{1/2}} \ , \quad \beta_2\equiv\frac{\omega_I^2}{2\omega_R}\frac{\omega_Ry}{(2k_BT)^{1/2}} \ . \tag{3.25}$$

An aspect of significant relevance of this result is the anharmonic parameter α which expresses the significant interplay between anharmonicity and fluctuation.

We shall call "manifold of νth order" the set of states $|mn\rangle$ satisfying the relationship

$$n+m=\nu \ . \tag{3.26}$$

We see then that the detuning term, the fluctuation-dissipation term, and also the first contribution to the anharmonic term [first contribution between small brackets on the rhs of Eq. (3.23)] cannot produce transitions from one manifold to another, whereas the remaining part of the anharmonic interaction connects a manifold of the

νth order to one of the $(\nu+2)$th order (and vice versa). The radiation term can only provoke transitions from the νth to the $(\nu+1)$th order manifold (the reverse transitions are forbidden).

Let us call

$$\Gamma_d \equiv \Delta\omega(a_-b_+ - a_+b_-) + \alpha(a_+b_-b_+b_- + b_-a_+a_-a_+ - a_-b_+b_-b_+ - b_+a_-a_+a_-) - \frac{\gamma}{2}(a_+a_- + b_+b_-) . \tag{3.27}$$

This is the contribution to $\bar{\Gamma}_1$ which does not provoke transition from one manifold to another. This in turn is divided into antisymmetric Γ_I and symmetric Γ_R parts defined as follows:

$$\Gamma_I = \Delta\omega(a_-b_+ - a_+b_-) + \alpha(a_+b_-b_+b_- + b_-a_+a_-a_+ - a_-b_+b_-b_+ - b_+a_-a_+a_-) , \qquad \Gamma_R = -\frac{\gamma}{2}(a_+a_- + b_+b_-) . \tag{3.28}$$

Let us consider the ideal case where $\gamma=0$. In such a case all the matrix elements on the same manifold vanish when

$$\Delta\omega = \alpha(\nu+1) . \tag{3.29}$$

This suggests that a strong coupling between the 0th-order manifold (ground state) and the νth-order manifold is allowed, thereby leading us to the transition scheme of Fig. 2. The expansion series of Eq. (3.35) would provide a more convincing justification. However, this will be proven to be convergent only in the region $\gamma > 2\alpha$.

We note that to evaluate the amount of energy stored in the system as a function of $\Delta\omega$, we must determine

$$\langle E(\Delta\omega)\rangle \equiv \tfrac{1}{2}\langle \tilde{\pi}_{\rm eq} | (a_+ + a_-)^2 + (b_+ + b_-)^2 | \pi_{\rm eq}\rangle , \tag{3.30}$$

where $|\pi_{\rm eq}\rangle$ and $\langle\tilde{\pi}_{\rm eq}|$ are defined by

$$\bar{\Gamma}_1 |\pi_{\rm eq}\rangle = 0 , \tag{3.31}$$

$$\langle\tilde{\pi}_{\rm eq}|\bar{\Gamma}_1 = 0 . \tag{3.32}$$

It is immediately evident that $\langle\tilde{\pi}_{\rm eq}| = \langle 00|$. As to $|\pi_{\rm eq}\rangle$, we can determine this state via a perturbation approach based on the division of $\bar{\Gamma}_1$ into a perturbed part

$$\Gamma_t = \bar{\Gamma}_1 - \Gamma_d , \tag{3.33}$$

and an unperturbed part Γ_d of Eq. (3.27). Of course this perturbation approach implies a fairly large value of the parameter γ.

From the perturbation theory we get

$$|\pi_{\rm eq}\rangle = \left[1 - \frac{1}{\Gamma_d}\Gamma_t + \frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t - \frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t + \cdots\right]|00\rangle . \tag{3.34}$$

This substituted into Eq. (3.30) produces

$$\langle E(\Delta\omega)\rangle = \tfrac{1}{2}\left\langle s\left|\left[1 - \frac{1}{\Gamma_d}\Gamma_t + \frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t - \cdots\right]\right|00\right\rangle , \tag{3.35}$$

where

$$|s\rangle \equiv \frac{1}{\sqrt{2}}(|20\rangle + |02\rangle) . \tag{3.36}$$

As mentioned before, the condition of Eq. (3.29) produces divergences in the expansion terms of Eq. (3.35), thereby leading to the scheme of Fig. 2.

Note, however, that the expansion series of Eq. (3.35) can only be used when γ is fairly large compared to α. Let us consider the case

$$\gamma \gg \alpha . \tag{3.37}$$

In such a case we can explore physical conditions where

$$\gamma \gg (\beta_1,\beta_2) \gg \alpha , \tag{3.38}$$

while being kept far apart from the saturation region. We can thus neglect the contribution of anharmonicity to Γ_t. This means that $(\beta_2=0)$

$$\langle E(\Delta\omega)\rangle = \frac{1}{2}\left\langle s\left|\frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t\right|00\right\rangle = \frac{\beta_1^2}{(\Delta\omega - 2\alpha)^2 + \frac{\gamma^2}{4}} . \tag{3.39}$$

Note that Eq. (3.35) can also read

$$\langle E(\Delta\omega)\rangle = \left\langle s\left|\frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t - \frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t + \cdots\right|00\right\rangle = \left\langle s\left|\frac{1}{\Gamma_d} - \frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d} + \frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d}\Gamma_t\frac{1}{\Gamma_d} - \cdots\right|s'\right\rangle , \tag{3.40}$$

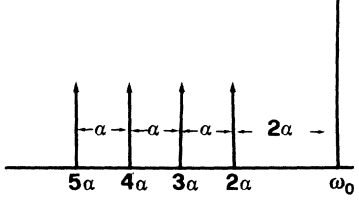

FIG. 2. Schematic representation of the transitions taking place at $\gamma \to 0$ in the hypothetical case where the validity of the expansion series of Eq. (3.35) extends to this limit.

where

$$|s'\rangle=\Gamma_t\frac{1}{\Gamma_d}\Gamma_t|00\rangle\ . \tag{3.41}$$

On the other hand, Eq. (3.40) is equivalent to the Laplace transform at the origin of

$$\langle E(\Delta\omega)\rangle=\int_0^\infty\langle s|e^{-(\Gamma_d+\Gamma_t)t}|s'\rangle dt\ . \tag{3.42}$$

Equation (3.42) is suitable for applying the continued fraction procedure (CFP) of Refs. 12 and 17. Furthermore, the calculation can be made even simpler by considering the case

$$\beta_1,\beta_2\ll\alpha\ . \tag{3.43}$$

After getting the second manifold through the transitions generated by the radiation field, the condition of Eq. (3.43) allows us to assume that the further transitions can be considered as generated by the anharmonicity alone. Thus Eq. (3.42) will be used in the applications of the present paper assuming that only the anharmonicity contributes to Γ_t appearing in the corresponding time evolution, whereas $|s'\rangle$ is generated only by the radiation field.

C. More on the energy-diffusion equation

The major aim of this section is to explore the extremely underdamped region $\gamma<\alpha$. We must get rid of the wrong suggestion from Fig. 2, that in this region many absorption sharp peaks may appear. This is now a settled issue, which is also the subject of an allied paper (concerning the case of the Duffing oscillator where α is positive), the main conclusions of which are supported by analog simulation.[18] We show in this section that the main feature of the extremely underdamped region is the appearance of an asymmetric spectrum peaked at $\Delta\omega=\alpha$ with a linewidth the order of magnitude of which is precisely α when $\gamma\sim0$. In this extreme region Eq. (3.42) cannot be more applied and we must have recourse to the linear-response theory. Although the application of this well-known theory to the extremely low-damping limit can be questioned,[19] the assessment of the existence of the residual linewidth α means that the strength of the stochasticity may be large enough to render the linear-response theory at least a good approximation to the actual absorption spectrum (in the presence of a vanishingly small excitation field).

From Eq. (3.8) we see that in the absence of radiation field the equilibrium distribution attained by the system in the rotating frame of reference is

$$\rho_{\rm eq}(x,v)\propto\exp\left[-\frac{v^2+\omega_R^2x^2}{2k_BT}\right]\ . \tag{3.44}$$

This supports the choice of Sec. II B, Eq. (2.32).

Let us therefore rewrite Eq. (3.8) in terms of the variables α_+ and E, defined via Eq. (2.32). We get

$$\frac{\partial}{\partial t}\rho(\alpha_+,E,t)=\mathscr{L}\rho(\alpha_+,E,t)$$
$$=\left[-i\frac{3}{4}\frac{\beta}{\omega_0^3}\frac{\partial}{\partial\alpha_+}\alpha_+E+\gamma\left[\frac{\partial}{\partial E}E+k_BT\frac{\partial}{\partial E}E\frac{\partial}{\partial E}\right]+\frac{\gamma}{2}\left[\frac{\partial}{\partial\alpha_+}\alpha_++2k_BT\frac{\partial}{\partial\alpha_+}\alpha_+\frac{\partial}{\partial E}\right]\right]\rho(\alpha_+,E,t)\ . \tag{3.45}$$

To apply the linear-response theory we assumed $\beta_1=\beta_2=0$.

From this equation we derive that in the extremely low-friction limit

$$\gamma\ll|\alpha|\ , \tag{3.46}$$

the variable α_+ is much faster than the variable E. α_+ oscillates indeed in time with the mean frequency

$$-\frac{3}{4}\frac{\beta}{\omega_0^3}\langle E\rangle=\alpha\ . \tag{3.47}$$

This means that if the linear-response theory is fulfilled, this regime is characterized by an absorption spectrum peaked precisely at $\Delta\omega=\alpha$.

By applying the linear-response theory we have

$$\langle E(\Delta\omega)\rangle=\omega_R^2\int_0^\infty\langle\alpha_-(0)\alpha_+(t)\rangle_{\rm eq}e^{-i\Delta\omega t}dt\ . \tag{3.48}$$

To determine $\langle\alpha_-(0)\alpha_+(t)\rangle$ we proceed as follows. Let us define

$$\Phi_n(t)=\langle\alpha_-(0)E^n\alpha_+(t)\rangle_{\rm eq}\ , \tag{3.49}$$

$$\Phi_0(t)=\langle\alpha_-(0)\alpha_+(t)\rangle_{\rm eq}\ . \tag{3.50}$$

From Eq. (3.45) we then get

$$\dot\Phi_n(t)=i\alpha\Phi_{n+1}(t)-\gamma(n+\tfrac{1}{2})\Phi_n+\gamma n(n+1)\Phi_{n-1}\ , \tag{3.51}$$

from which

$$\hat\Phi_0(z)=\hat F_0(z)[\langle\varepsilon\rangle+\hat F_1(z)a_0^1[\langle\varepsilon^2\rangle+\hat F_2(z)a_1^2[\langle\varepsilon^3\rangle+\cdots \tag{3.52}$$

with

$$\hat{F}_0(z)=\cfrac{1}{z-\lambda_0-\cfrac{a_0^1 a_1^0}{z-\lambda_1-\cfrac{a_1^2 a_2^1}{z-\lambda_2-\cdots}}},$$

$$\hat{F}_1(z)=\cfrac{1}{z-\lambda_1-\cfrac{a_1^2 a_2^1}{z-\lambda_2-\cdots}}, \tag{3.53}$$

$$a_n^{n-1}=\gamma n(n+1),\quad a_n^{n+1}=i\alpha,\quad \lambda_n=-\gamma(n+\tfrac{1}{2}),$$

and so on.

In Sec. IV we shall use both Eqs. (3.42) and (3.48) supplemented by Eq. (3.52).

When the condition of Eq. (3.46) applies, we can also derive an analytical expression for $\hat{\Phi}_0(z)$. From Eq. (3.45) we indeed get

$$\dot{\alpha}_+(t)=+i\frac{3}{4}\frac{\beta}{\omega_R^3}E\alpha_+(t)-\frac{\gamma}{2}\alpha_+(t), \tag{3.54}$$

from which

$$\alpha_+(t)=\exp\left[i\frac{3}{4}\frac{\beta}{\omega_R^3}Et\right]\exp\left[-\frac{\gamma}{2}t\right]. \tag{3.55}$$

By performing the average over the equilibrium distribution of the variable E we get

$$\begin{aligned}\langle\alpha_-(0)\alpha_+(t)\rangle&=2\int dE\exp\left[i\frac{3}{4}\frac{\beta}{\omega_R^3}Et\right]\exp\left[-\frac{\gamma}{2}t\right]\\&\quad\times\frac{E}{k_BT}\exp\left[\frac{-E}{k_BT}\right]\\&=2k_BT\exp\left[\frac{-\gamma}{2}t\right]\frac{(1+i\alpha t)^2}{(1+\alpha^2t^2)^2}.\end{aligned} \tag{3.56}$$

Equation (3.56) shows clearly that the spectrum for $\gamma\to 0$ is a broad band with linewidth $|\alpha|$ peaked at $\Delta\omega=\alpha$ in full agreement with the Risken theory.[20] This is more clearly detailed in Ref. 18, which is devoted to studying the Duffing oscillator (and, therefore no attention is devoted to the process of escape from a potential well). We would like to remark that a major result of the present paper is that the theory of Sec. IV supplemented by the results of this section allows us to predict a phenomenon seemingly not detected by Devoret *et al.*;[5] that is, the shift of the maximum rate of escape from $\Delta\omega=2\alpha$ to $\Delta\omega=\alpha$ for γ dropping from $\gamma\simeq 7|\alpha|$ (experimental conditions) to $\gamma\ll|\alpha|$.

IV. THEORY VERSUS EXPERIMENT: THE CASE OF RESONANT ACTIVATION FROM THE ZERO-VOLTAGE STATE OF A CURRENT-BIASED JOSEPHSON JUNCTION[5]

In Secs. II and III we studied the case of the potential of Eq. (2.2) for the major purpose of making our arguments more transparent. This also serves the purpose of studying a case closer to that of Ref. 18 (where theory was checked via analog simulation). In this section, with the minor effort detailed in Appendix B, we shall adapt the theory to the study of the recent experimental results by Devoret *et al.*[5] We shall show that the agreement is quite satisfactory (especially as far as the position of the maximum escape rate is concerned) and the possible physical reasons behind the discrepancies will be discussed in the concluding remarks (see Sec. V).

An underdamped Josephson junction with critical current I_0 biased at a constant current I can be modeled as a particle moving in the one-dimensional potential

$$V(\theta)=V_0(s\theta-\cos\theta), \tag{4.1}$$

the corresponding equation of motion being

$$\ddot{\theta}+\gamma\dot{\theta}=-\frac{\partial V}{\partial\theta}+F(t), \tag{4.2}$$

where $F(t)$ is a Gaussian white noise defined by

$$\langle F(0)F(t)\rangle=2\gamma k_BT\delta(t). \tag{4.3}$$

In the presence of a radiation field an additional force will appear and the dynamics of the junction can be correctly represented by the system of Eqs. (2.13). The activation out of the zero-voltage state which corresponds to the particle localized in the potential well (see Fig. 3) by microwaves at a frequency close to the natural (plasma) frequency of the junction, enters in the general case of the escape from a well by a Brownian particle.[1] So in the underdamped regime where energy diffusion is assumed to be the slowest process, we can utilize the theory developed in Secs. II and III suitably adapted to study the effect of a radiation field on the escape time out of the zero-voltage state.

To apply the theory developed in this paper we have to expand the potential $V(\theta)$ around the minimum in order to obtain a potential function which consists of a harmonic contribution and anharmonic corrections. We have that

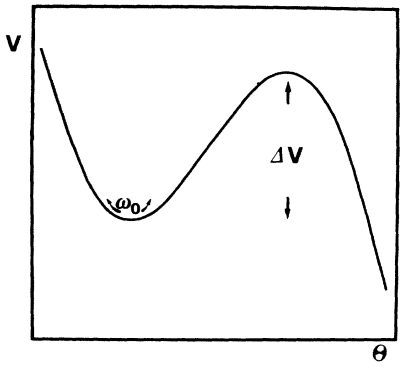

FIG. 3. $V(\theta)$ as given by Eq. (4.1) for a bias current below I_0.

$$V(\theta)-V(\theta_{\min})=\frac{V_0}{2}(1-s^2)^{1/2}(\theta-\theta_{\min})^2 + \frac{1}{6}V_0 s(\theta-\theta_{\min})^3 - \frac{1}{24}V_0(1-s^2)^{1/2}(\theta-\theta_{\min})^4+\cdots . \tag{4.4}$$

Here $V_0=(I_0\Phi_0/2\pi)$, $s=I/I_0$, and $\Phi_0=h/2e$. The natural (plasma) frequency ω_0 corresponds to the harmonic frequency and is given by

$$\omega_0^2\equiv V_0(1-s^2)^{1/2} . \tag{4.5}$$

[In order to obtain frequency in the usual units s^{-1}, we have to multiply Eq. (4.5) by $4\pi^2/C\Phi_0^2$ where C is the shunt capacitance.]

The form of the potential given by Eq. (4.4) is somewhat different from that of Eq. (2.2); however, when we are interested in calculating the energy absorbed in the unit of time, using the rotating-wave approximation, we can show (see Appendix B) that we can still use a potential such as that of Eq. (2.2) provided that we define the effective anharmonic frequency β_{eff}, which in our case is given by (see Appendix B)

$$\beta_{\text{eff}}=\frac{V_0}{6}(1-s^2)^{1/2}+\frac{10}{9}\frac{V_0}{4}\frac{s^2}{(1-s^2)^{1/2}} . \tag{4.6}$$

Using this effective value for the anharmonicity we can calculate the anharmonic parameter α and then, using Eq. (3.42) or Eq. (3.52), we can also calculate the amount of energy absorbed by the system in the unit of time and for a particular value of the detuning $\Delta\omega$. Once this quantity is evaluated, we shall be in a position to use the equation of energy diffusion Eq. (2.33),

$$\frac{\partial}{\partial t}\rho(E,t)=\left[\gamma\frac{\partial}{\partial E}E-\gamma k_BT\frac{\partial}{\partial E}+\gamma k_BT\frac{\partial^2}{\partial E^2}E - P(\Delta\omega)\frac{\partial}{\partial E}+P(\Delta\omega)\frac{\partial^2}{\partial E^2}E\right]\rho(E,t) ,$$

which in turn will be applied to calculate the escape time out of the zero-voltage state.

The escape time τ_P is identified with the time necessary for the system to acquire an amount of energy equal to the height of the potential barrier, and can be determined using the well-known mean first-passage time method[10]

$$\tau_P\equiv\int_{k_BT}^{\Delta V}\frac{\rho_{st}^{-1}(E)}{[\gamma k_BT+P(\Delta\omega)]E}dE\int_0^E\rho_{st}(E')dE' , \tag{4.7}$$

where ΔV is the height of the potential barrier which, for the potential under investigation, reads

$$\Delta V=2V_0[(1-s^2)^{1/2}-s\cos^{-1}s] . \tag{4.8}$$

In the experimental conditions of Ref. 5, $\Delta V\cong 10k_BT$ so that to evaluate τ_P we can make the usual approximations[21]

$$\tau_P\cong\frac{k_BT+\dfrac{P(\Delta\omega)}{\gamma}}{\gamma\Delta V}\exp\left[\frac{\Delta V}{k_BT+\dfrac{P(\Delta\omega)}{\gamma}}\right] . \tag{4.9}$$

As we want to compare theoretical with experimental results rather than looking for τ_P, we will evaluate the natural logarithm of the ratio between the escape time in the presence of the radiation field τ_P and the escape time in its absence τ_0, which is the usual Kramers result for extremely low damping

$$\ln\left[\frac{\tau_P}{\tau_0}\right]=\ln\left[1+\frac{P(\Delta\omega)}{\gamma k_BT}\right]-\frac{\Delta V}{k_BT}\frac{P(\Delta\omega)}{\gamma k_BT+P(\Delta\omega)} . \tag{4.10}$$

When $P(\Delta\omega)$ is small compared to the thermal energy absorbed by the system in the unit of time γk_BT, Eq. (4.10) can be approximated to

$$\ln(\tau_P/\tau_0)\cong-\left[\frac{\Delta V}{k_BT}-1\right]\frac{P(\Delta\omega)}{\gamma k_BT} . \tag{4.11}$$

Figure 4 shows the theoretical predictions based on Eq. (4.11) [or equivalently Eq. (4.10)] compared with the experimental result obtained by Devoret *et al.*[5] Note that the agreement between the two kinds of results is very good, especially for the position of the maximum escape time; we remark that this position was predicted analytically and the agreement with the experimental result is within the experimental errors. Only the friction γ was used as a fitting parameter. However, the value of γ actually used is proven to be well within the uncertainty limits resulting from the experimental errors.[5]

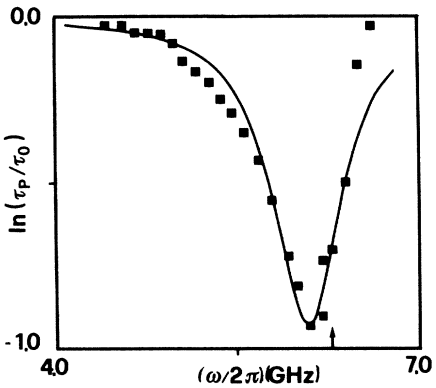

FIG. 4. Theoretical versus experimental results: ——, theoretical result using Eq. (4.11) supplemented by Eq. (3.42) for the calculation of $P(\Delta\omega)$; in this scale the theoretical result provided by Eq. (4.11) when $P(\Delta\omega)$ is calculated via Eq. (3.52) is indistinguishable from that obtained when using Eq. (3.42); ■, experimental points of Ref. 5. The arrow denotes ω_0. The theoretical predictions for ω_0 and α are $\omega_0=6.28$ GHz and $\alpha=0.091$ GHz. Experimental values for T, I, I_0, and C were taken: $T=4.2$ K, $I=3.07$ A, $I_0=4.64$ A, and $C=6.8$ pF. The value of γ used to get the theoretical results was $\gamma=0.64$ GHz.

V. CONCLUDING REMARKS

The problem of activation in extremely underdamped Josephson junction circuits is by itself a problem of remarkable interest.[22] Büttiker, Harris, and Landauer,[23] for example, developed an approach to cover a range of the damping constant wider than the extreme underdamping. The approach used in Sec. IV is based on a theory which in the absence of radiation field coincides with the extreme underdamped regime of Kramers.[1] In principle it would be possible to apply the argument of Buttiker, Harris, and Landauer to the energy-diffusion equation [Eq. (2.33)]. However, it seems to us that the disagreement between the theoretical result and the experimental one could also be ascribed to different causes: (1) The rotating-wave approximation and (2) the experimental errors themselves. Nevertheless, the theoretical predictions on the position of the spectrum peak agree surprisingly well with the experimental results. We are also in a position to predict that the experiment of Ref. 5 would lead to a peak centered at $\omega_0+\alpha$ if a friction $\gamma<\alpha$ were used. An interesting result of this paper is that a transition from a peak centered at $\omega_0+2\alpha$ to a new condition where the peak is centered at $\omega_0+\alpha$ takes place when changing the friction from $\gamma>\alpha$ to $\gamma<\alpha$. This result has been the subject of a preliminary short report[19] and, as the free relaxation spectrum is concerned, has been completely corroborated by the results of analog simulation.[18] This makes the present paper original and timely. Indeed, in addition to Refs. 2 and 3 already commented on in Sec. I, to the best of our knowledge the only previous examples of extension of the Kramers theory to the case of radiative activation are the papers by Ben-Jacob *et al.*[24] These authors, however, confined their attention to the case of a truncated harmonic oscillator, thereby preventing themselves from taking the effects of anharmonicity into account.

As to the more basic issue mentioned in Sec. I and concerning the elimination of irrelevant variables, the analysis of Sec. II A shows in what physical condition the radiation field can be dealt with in the same way as a standard stochastic force characterized by its correlation function [see Eq. (2.7)]. This implies that the time scale of the radiation field is much shorter than the time scale characterizing the system of interest, thereby making it possible to define the operator driving the radiation field as the unperturbed part of the total effective Liouvillian. This division of the total effective Liouvillian leads to the same flaw as that pointed out in Refs. 6 and 7. Hanggi *et al.*[7] showed indeed that the long-time behavior predicted by a standard Markovian assumption[25] is incorrect. We would like to stress that the source of this flaw must be ascribed precisely to this definition of the unperturbed part. The correct approach indicated in Sec. II A, on the contrary, is not fraught with this flaw. When a large time-scale separation between relevant and irrelevant variables is available this approach provides the same results as the standard one, whereas completely different results are obtained when the regime characterized by resonant effects is explored. In the overdamped regime ($\omega_0 \ll \gamma$ and $\omega_R \ll \lambda$) it can be shown that the correct approach of Sec. II A leads to precisely the same Fokker-Planck equation as that recently proposed by Hanggi *et al.*,[7] which in turn correctly describes the long-time behavior.

In the regime explored in this paper, an exact agreement between the results based on the application of the linear-response theory, Eq. (3.52) and Eq. (3.42), has been found. Reference 19 raised the question of whether or not the linear-response theory may be applied in a region of damping weaker than that explored by the experiment of Devoret *et al.*[5] ($\gamma<\alpha$). This is now a well-settled problem which will be the subject of forthcoming papers.[26,27]

ACKNOWLEDGMENT

The work of T. Fonseca was supported by the JNICT (Portugal), Grant No. 49/84. This support is gratefully acknowledged. We warmly thank Dr. M. W. Evans for drawing our attention to Ref. 5.

APPENDIX A

The central task of this appendix is to show how to deal with the term

$$\int_0^t \frac{\partial}{\partial v} e^{(i\mu\omega_R+\mathcal{L}_a)(t-\tau)} \frac{\partial}{\partial v} d\tau \,. \tag{A1}$$

In the harmonic case it is straightforward to show that

$$\mathcal{L}_a^x \left[\frac{\partial}{\partial x}+\varepsilon_\pm \frac{\partial}{\partial v} \right] = \varepsilon_\pm \left[\frac{\partial}{\partial x}+\varepsilon_\pm \frac{\partial}{\partial v} \right] , \tag{A2}$$

where

$$\varepsilon_\pm = -\frac{\gamma}{2} \pm i\omega_0 w \,, \tag{A3}$$

$$w \equiv \left[1-\left[\frac{\gamma}{2\omega_0} \right]^2 \right]^{1/2} \sim 1 \,. \tag{A4}$$

We obtain, therefore,

$$\begin{aligned} \sum_\mu \frac{\omega_I^4 \langle y^2 \rangle_{\rm eq}}{2} \frac{\partial}{\partial v} e^{\mathcal{L}_a s} \frac{\partial}{\partial v} e^{i\mu\omega_R s} &= \frac{\omega_I^4 \langle y^2 \rangle_{\rm eq}}{2} \frac{\partial}{\partial v} \left[\frac{\partial}{\partial x} \left(e^{-(\gamma/2+i\omega_0 w)s} - e^{-(\gamma/2-i\omega_0 w)s} \right) \right. \\ &\quad \left. + \varepsilon_+ \frac{\partial}{\partial v} e^{-(\gamma/2+i\omega_0 w)s} - \varepsilon_- \frac{\partial}{\partial v} e^{-(\gamma/2-i\omega_0 w)s} \right] \left[\sum_{\mu=\pm 1} e^{i\mu\omega_R s} \right] \\ &\cong \left[\frac{\omega_I^4 \langle y^2 \rangle_{\rm eq}}{4i\omega_0} i\omega_0 \frac{\partial^2}{\partial v^2} (e^{\varepsilon_+ s}+e^{\varepsilon_- s}) + \frac{\omega_I^4 \langle y^2 \rangle_{\rm eq}}{4i\omega_0} \frac{\partial}{\partial v} \frac{\partial}{\partial x} (e^{\varepsilon_+ s}-e^{\varepsilon_- s}) \right] \left[\sum_{\mu=\pm 1} e^{i\mu\omega_R s} \right] . \end{aligned} \tag{A5}$$

Via integration on s between 0 and ∞, this precisely reduces to the form of Eq. (2.25) (without the last term between brackets). In the presence of anharmonic interaction via development of $\exp(\mathscr{L}_a t)$ into a Taylor series it is possible to show that derivatives of higher order appear.

APPENDIX B

The main idea is to make a contact with the problem of time-independent perturbation theories[28] by regarding $\Gamma_1(t)$,

$$\Gamma_1(t)=\sum_{\mu=-3}^{+3}\Gamma_\mu e^{i\omega_R\mu t}, \tag{B1}$$

as being a quantum-mechanical operator written in the interaction picture; that is,

$$\Gamma_1(t)\Longrightarrow\sum_{m,m'}|m\rangle\langle m|\Gamma_1|m'\rangle\langle m'|e^{-i(\varepsilon_m-\varepsilon_{m'})t} \tag{B2}$$

where the energies ε_m and $\varepsilon_{m'}$ of the states $|m\rangle$ and $|m'\rangle$ are related to the frequency ω_R via

$$\varepsilon_m-\varepsilon_{m'}=\omega_R(m-m'). \tag{B3}$$

In consequence, the successive perturbation contributions are given by

$$\mathscr{H}_1=\bar{\Gamma}_1=\sum_m|m\rangle\langle m|\Gamma_1|m\rangle\langle m|, \tag{B4}$$

$$\mathscr{H}_2=\sum_m|m\rangle\frac{\langle m|\Gamma_1|m'\rangle\langle m'|\Gamma_1|m\rangle}{(\varepsilon_m-\varepsilon_{m'})}\langle m|, \tag{B5}$$

and so on.

By using this approach we find that for the potential function given by Eq. (4.4), $\mathscr{H}_1$ coincides with Eq. (3.7) while contributions coming from the cubic term of the potential function only appear when a second-order contribution $\mathscr{H}_2$ is considered. Then these are given the form

$$\frac{10}{24}\left[\frac{V_0^2s^2}{4\omega_0^5}\right]\left[\frac{\partial}{\partial\alpha_+}\alpha_+^2\alpha_--\frac{\partial}{\partial\alpha_-}\alpha_+\alpha_-^2\right], \tag{B6}$$

and have precisely the same structure as the first-order contribution of $\mathscr{H}_1$ coming from the quartic term of the potential function. This allows us to replace β on Eq. (3.7) with an effective anharmonic parameter β_{eff}.

$$\beta_{\text{eff}}=\frac{V_0}{6}(1-s^2)^{1/2}+\frac{10}{9}\frac{V_0}{4}\frac{s^2}{(1-s^2)^{1/2}}, \tag{B7}$$

which is the result displayed on Eq. (4.6).

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