Single-mode approximation in laser physics: A critique and a proposed improvement

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We propose the use of a laser model in which the single-mode approximation is introduced for the electric field but not for the atomic variables. We show that our improved single-mode model is in good agreement with the exact steady-state solution for a reasonable range of experimentally relevant parameters well beyond the limits of validity of the mean-field approximation.

I. INTRODUCTION

The single-mode approximation is one of the oldest and most widely used theoretical approaches in the study of laser physics.[1,2] Its strongest appeal derives from its analytic simplicity, in addition to some undeniable successes with the interpretation of numerous basic features of laser operation. Yet, upon close inspection, we find evidence of nontrivial conceptual and practical problems that we wish to open up for discussion, and possibly solve, in the context of this paper.

The conceptual framework of this approximation is rooted in the notion of "cavity mode," a well-defined dynamical entity for a high-Q resonator whose time evolution describes the behavior of the internal field when the laser operates with a single-frequency output. Many laser cavities of practical interest, however, are lossy resonators and their "modes" are no longer sharply recognizable features because of the broadened nature of their resonances. Thus, for example, plane traveling waves cannot be expected to fit the usual ring-cavity boundary conditions in the presence of a finite mirror transmitivity and of the additional losses introduced by transverse mode selectors and other intracavity devices. Yet many lasers operate as single-frequency generators, even under relatively low-Q conditions; this type of situation is incompatible with the notion of spatial field uniformity along the active medium, and the single-mode approximation, at least in the traditional sense, is no longer an accurate description. Attempts to improve the single-mode approximation with the introduction of additional longitudinal modes usually lead to extremely complicated sets of equations.

In this paper we focus on the description of a special type of laser, the unidirectional ring laser in the plane-wave approximation. We take the viewpoint that the conventional Maxwell-Bloch equations with appropriate boundary conditions are an adequate starting point for a rather idealized, but still flexible model: The model is idealized because a plane wave is only a rough approximation for real-life transverse profiles, and because the two-level picture of the active medium may fall short of including all the dynamical effects that can be observed in a laboratory system.

Even with these shortcomings, the Maxwell-Bloch approach imposes no essential restriction on the allowed range of unsaturated gain values, mirror reflectivity, intermode spacing, and detuning parameters. The most obvious obstacle that prevents the representation of the cavity field as a linear superposition of orthogonal modal functions is the form of the boundary conditions which, in the case of a unidirectional ring cavity, involve both a time delay and a scaling of the field amplitude. We overcome this formal difficulty with a suitable transformation of both the space-time coordinates and the dynamical variables, which allows an exact representation of the new variables as linear superpositions of orthogonal cavity functions in the transformed frame.[3,4] We interpret the time-dependent coefficients of these expansions as the ring-laser modes.

The traditional single-mode equations for a ring laser can be derived after imposing a number of restrictive conditions which include the so-called mean-field limit: This term, a bit of a misnomer in view of the possible confusion that can emerge with its homonym from statistical mechanics, is indicative of a situation where the unsaturated laser gain αL and the transmittivity coefficient T of the external mirrors are both vanishingly small, while their ratio $\alpha L/T \equiv 2C$ is an arbitrary finite constant. Most lasers cannot operate under such limiting conditions, while, on the other hand, single-frequency operation is not an uncommon setting, even with relatively low-Q cavities. One is then led to the natural question: Is it possible to incorporate low-Q single-frequency operation in the context of a single-mode theory? We propose that the answer is affirmative if we abandon the common practice of handling both the field and the atomic variables on the same footing.

Our proposal stems from the observation that while the longitudinal profile of the transformed cavity field remains relatively uniform even for parameter values that do not conform with the mean-field requirements, the atomic variables, instead, tend to display a much stronger dependence on the longitudinal coordinate. This implies that the single-mode approximation for the field maintains a high degree of accuracy even when a parallel assumption for the atomic variables can no longer be held. Thus, the atomic equations should be allowed to develop an arbitrary spatial dependence for the polarization and population difference if we want to produce a realistic description of single-frequency operation. Our improved model is designed to provide this extra degree of freedom which is not available in the conventional single-mode model. The resulting equations are necessarily more complicated than those of the traditional single-mode approximation, but they are still considerably simpler, and therefore manageable, than the original Maxwell-Bloch equations.

We have adapted our analysis to cover both homogeneously and inhomogeneously broadened lasers. In the latter case, exact numerical calculations based on the Maxwell-Bloch equations are extremely cumbersome to carry out, so that the improved single-mode model offers significant computational advantages, without excessive loss of accuracy.

This paper is organized as follows. In Sec. II we review the standard Maxwell-Bloch description of a ring laser and introduce a suitable modal decomposition of the equations of motion. The traditional single-mode equations emerge from this infinite set under appropriate conditions. In Sec. III we propose an improved description of a single-frequency operation under less than ideal gain and reflectivity conditions. In Sec. IV we carry out a number of analytical and numerical tests using the steady-state solutions of the exact and approximate equations as elements of comparison. Finally, in Sec. V we conclude with some remarks and with a generalization of our treatment for the case of optical bistability and the laser with an injected signal.

II. MODAL REPRESENTATION OF THE RING-LASER EQUATIONS OF MOTION

The starting point of our analysis is the usual set of Maxwell-Bloch equations for a collection of two-level atoms interacting with a traveling wave whose (scalar) electric field has the form

$$E(z,t)=\tfrac{1}{2}[\varepsilon(z,t)e^{i(k_R z-\omega_R t)}+\text{c.c.}]\ . \tag{2.1}$$

In Eq. (2.1), $\varepsilon(z,t)$ is the slowly varying complex amplitude of the field, ω_R is an arbitrary reference frequency that we can select in the most convenient way, and $k_R=\omega_R/c$. In the following we select ω_R as one of the empty cavity resonances $\omega_{\bar{n}}=2\pi\bar{n}c/\mathscr{L}$, where $\mathscr{L}$ is the length of the ring cavity, and we denote $\omega_{\bar{n}}$ by ω_c.

The equations of motion for a system whose atomic profile is inhomogeneously broadened have the well-known form

$$\frac{\partial\mathscr{F}}{\partial z}(z,t)+\frac{1}{c}\frac{\partial\mathscr{F}}{\partial t}(z,t)=-\alpha\int_{-\infty}^{\infty}d\tilde{\delta}\,g(\tilde{\delta})\mathscr{P}(\tilde{\delta},z,t)\ , \tag{2.2a}$$

$$\frac{\partial\mathscr{P}}{\partial t}(\tilde{\delta},z,t)=\gamma_\perp\{\mathscr{F}\mathscr{D}-[1+i(\tilde{\delta}_{AC}+\tilde{\delta})]\mathscr{P}\}\ , \tag{2.2b}$$

$$\frac{\partial\mathscr{D}}{\partial t}(\tilde{\delta},z,t)=-\gamma_{\parallel}[\tfrac{1}{2}(\mathscr{F}^*\mathscr{P}+\mathscr{F}\mathscr{P}^*)+\mathscr{D}+1]\ , \tag{2.2c}$$

where $\mathscr{F}$ is the scaled Rabi frequency, $\mu\epsilon/(\hbar\sqrt{\gamma_\parallel\gamma_\perp})$, μ is the modulus of the dipole moment, $\mathscr{P}(\tilde{\delta},z,t)$ is the complex polarization per atom, and $\mathscr{D}(\tilde{\delta},z,t)$ is the difference between the ground- and excited-states population probabilities; both $\mathscr{P}$ and $\mathscr{D}$ are related to a particular atomic packet that is detuned away from line center by an amount $\tilde{\delta}$. The symbols $\gamma_\perp$ and $\gamma_\parallel$ denote the decay rates of the polarization and of the atomic inversion, respectively;

$$\alpha=N\mu^2\omega_c/2\hbar\epsilon_0\gamma_\perp c$$

is the unsaturated gain coefficient per unit length, N is the number density of active atoms, and ϵ_0 is the vacuum permittivity. The scaled detuning parameter $\tilde{\delta}_{AC}=(\omega_A-\omega_C)/\gamma_\perp$ measures the separation between the center of the atomic line ω_A and the selected reference frequency ω_C. Finally, $g(\tilde{\delta})$ is the atomic line profile, which reduces to a δ function in the homogeneously broadened limit. Most rates are conveniently scaled to $\gamma_\perp$; the scaled frequencies will be identified with a tilde (i.e., $\tilde{\delta}=\delta/\gamma_\perp$).

The ring-cavity model is supplemented by the boundary conditions

$$\mathscr{F}(0,t)=R\mathscr{F}(L,t-(\mathscr{L}-L)/c)\ , \tag{2.3}$$

where L is the length of the active sample and $R=1-T$ is the reflectivity coefficient of the mirrors. Equation (2.3) differs from the standard periodicity conditions on two accounts: (i) the presence of the scale factor R, which is a consequence of the imperfect nature of the external reflectors, and (ii) the delay $\Delta t\equiv(\mathscr{L}-L)/c$, which is related to the finite size of the cavity.

In steady state ($t\to\infty$) the boundary conditions become isochronous, in effect, but the reflectivity coefficient still prevents them from acquiring the standard periodicity form, except in the extreme case $R\to1$. In general, as we have shown in an earlier publication,[5] the longitudinal profile in the modulus of $\mathscr{F}$ shows large deviations from uniformity, which makes it unreasonable to apply the single-mode approximation in a rigorous way. In fact, even the notion of a modal expansion for the cavity field $\mathscr{F}(z,t)$ seems to be poorly posed, except in the limit $R\to1$.

A convenient alternative description was proposed by Benza and Lugiato[3] in their study of unstable behavior in optically bistable systems. Here we adopt their suggestion and make it our basis for a rigorous model expansion. First, we define a new set of space-time coordinates

$$z'=z\ ,\qquad t'=t+\frac{\mathscr{L}-L}{c}\frac{z}{L} \tag{2.4}$$

and the new field and atomic variables

$$F(z',t')=\mathscr{F}(z',t')e^{(z'/L)\ln R}\ , \tag{2.5a}$$

$$P(\tilde{\delta},z',t')=\mathscr{P}(\tilde{\delta},z',t)e^{(z'/L)\ln R}\ , \tag{2.5b}$$

$$D(\tilde{\delta},z',t')=\mathscr{D}(\tilde{\delta},z',t')\ . \tag{2.5c}$$

The transformation (2.4) makes the boundary conditions (2.3) isochronous in the new reference frame; Eq. (2.5a), instead, removes the multiplicative factor R, so that the new boundary conditions take the standard form

$$F(0,t')=F(L,t') . \tag{2.6}$$

The transformed equations of motion

$$\frac{\partial F}{\partial t'}+\frac{cL}{\mathscr{L}}\frac{\partial F}{\partial z'}=-\kappa\left[F+2C\int_{-\infty}^{+\infty}d\tilde{\delta}\,g(\tilde{\delta})P(\tilde{\delta},z',t')\right], \tag{2.7a}$$

$$\frac{\partial P}{\partial t'}(\tilde{\delta},z',t')=\gamma_\perp\{FD-[1+i(\tilde{\delta}_{AC}+\tilde{\delta})]P\}, \tag{2.7b}$$

$$\frac{\partial D}{\partial t'}(\tilde{\delta},z',t')=-\gamma_\parallel[\tfrac{1}{2}(F^*P+FP^*)e^{-2(z'/L)\ln R}+D+1] \tag{2.7c}$$

differ from the original set (2.2) in two important respects.

(i) The phase velocity of the new field amplitude is $cL/\mathscr{L}$, instead of c; thus the transformation introduces an effective background index of refraction.

(ii) The equations contain an explicit spatial dependence through the exponential factor $\exp[-(2z'/L)\ln R]$.

The symbols C and κ denote $\alpha L/2|\ln R|$ and $c|\ln R|/\mathscr{L}$, respectively.

The main virtue of this approach is that the field amplitude obeys standard periodicity conditions which make it possible to introduce a decomposition of the Fourier type and to identify the natural modal amplitudes of the problem. An additional advantage is that the modulus of the new field amplitude $F(z',t')$ maintains a good degree of uniformity in steady state, even under conditions that are significantly removed from the mean-field limit. This fact, which was exhibited explicitly in Ref. 5 for the case of a homogeneously broadened laser, turns out to be true, even in the presence of inhomogeneous broadening, and to play a central role in the development of an improved single-mode model.

We now introduce the following Fourier decomposition for the transformed variables:[4,6]

$$\begin{bmatrix}F(z',t')\\P(\tilde{\delta},z',t')\end{bmatrix}=e^{-i\delta\Omega t'}\sum_{n=-\infty}^{+\infty}e^{ik_nz'}e^{-i\alpha_nt'}\begin{bmatrix}f_n(t')\\p_n(\tilde{\delta},t')\end{bmatrix}, \tag{2.8a}$$

$$D(\tilde{\delta},z',t')=\sum_{n=-\infty}^{+\infty}e^{ik_nz'}e^{-i\alpha_nt'}d_n(\tilde{\delta},t'), \tag{2.8b}$$

where $\delta\Omega$ is an unknown offset that measures the separation between the carrier frequency of the laser field and the selected cavity resonance. This parameter will be calculated from the steady-state equation. The wave number k_n is selected such that

$$k_n=\frac{2\pi c}{L}n,\quad n=0,\pm1,\pm2,\ldots \tag{2.9}$$

in order to satisfy automatically the boundary conditions (2.6); $\alpha_n=(2\pi c/\mathscr{L})n$ is the frequency of the nth empty cavity resonance. Note that $d_n(\tilde{\delta},t')=d^*_{-n}(\tilde{\delta},t')$ because of the real-valued nature of the population difference.

We identify the expansion amplitudes $f_n(t')$, $p_n(\tilde{\delta},t')$, and $d_n(\tilde{\delta},t')$ as the natural modes of the laser. The associated orthonormal modal functions are

$$u_n(z')=\frac{1}{\sqrt{L}}e^{ik_nz'} \tag{2.10a}$$

with

$$(u_n,u_m)\equiv\int_0^L dz'\,u_n^*(z')u_m(z')=\delta_{m,n} . \tag{2.10b}$$

The infinite set of time-dependent variables f_n, p_n, and d_n obey the following equations of motion:

$$\frac{df_n}{dt'}=i\,\delta\Omega\,f_n-\kappa\left[f_n+2C\int_{-\infty}^{+\infty}d\tilde{\delta}\,g(\tilde{\delta})p_n(\tilde{\delta},t')\right], \tag{2.11a}$$

$$\frac{df_n^*}{dt'}=-i\,\delta\Omega\,f_n^*-\kappa\left[f_n^*+2C\int_{\infty}^{+\infty}d\tilde{\delta}\,g(\tilde{\delta})p_n^*(\tilde{\delta},t')\right], \tag{2.11b}$$

$$\frac{d}{dt'}p_n(\tilde{\delta},t')=\gamma_\perp\left[\sum_{n'}f_{n'}d_{n-n'}(\tilde{\delta},t')-[1+i(\tilde{\delta}_{AC}-\delta\Omega+\tilde{\delta}-\tilde{\alpha}_n)]p_n\right], \tag{2.11c}$$

$$\frac{d}{dt'}p_n^*(\tilde{\delta},t')=\gamma_\perp\left[\sum_{n'}f_{n'}^*d_{n'-n}(\tilde{\delta},t')-[1-i(\tilde{\delta}_{AC}-\delta\Omega+\tilde{\delta}-\tilde{\alpha}_n)]p_n^*\right], \tag{2.11d}$$

$$\frac{d}{dt'}d_n(\tilde{\delta},t')=i\alpha_nd_n-\gamma_\parallel\left[\frac{1}{2}\sum_{n'n''}(f_{n'}^*p_{n''}e^{i\alpha_{n'-n''-n}t'}\Gamma_{n''-n'-n}+f_{n'}p_{n''}^*e^{-i\alpha_{n'-n''+n}t'}\Gamma^*_{n''-n'-n})+d_n+\delta_{n,0}\right], \tag{2.11e}$$

where the mode-mode coupling coefficients Γ are given by

$$\Gamma_p\equiv\frac{1}{L}\int_0^L dz'\,e^{ik_pz'}e^{2(z'/L)|\ln R|}=\frac{1-R^2}{R^2}\,\frac{1}{2|\ln R|+ik_pL},\quad p=0,\pm1,\pm2,\ldots . \tag{2.12}$$

The complexity of the exact modal equations should be enough to discourage attempts at numerical work for the purpose of studying the time-dependent behavior of a ring lasers. In fact, in our own studies of homogeneously broadened lasers, we have chosen to directly integrate the Maxwell-Bloch equations (2.7). On the other hand, Eqs. (2.11) are the most convenient starting points for a derivation and analysis of single-mode equations.

One obvious requirement for this derivation is what might be called the minimum coupling condition

$$\Gamma_p = \delta_{p,0} \tag{2.13}$$

which is satisfied only if the reflectivity is very close to unity. In this limit the parameter C remains bounded only if, at the same time, we require that $\alpha L \rightarrow 0$ with $\alpha L / |\ln R| < \infty$. An important property of Eqs. (2.11), in the approximation (2.13), is that the amplitudes labeled by an index $n \neq 0$ remain equal to zero for all time if they happen to vanish at the start of the evolution. Hence, the modes with $n \neq 0$ can be neglected as long as they are not unstable against small initial fluctuations. This condition is certainly satisfied when the intermode spacing $c/\mathcal{L}$ is sufficiently larger than the power-broadened gain curve. In this case, Eqs. (2.11) reduce to the well-known single-mode mean-field equations

$$\frac{df_0}{dt'} = i\,\delta\Omega f_0 - \kappa\left[f_0 + 2C\int_{-\infty}^{+\infty} d\tilde{\delta}\, g(\tilde{\delta}) p_0(\tilde{\delta},t')\right], \tag{2.14a}$$

$$\frac{dp_0}{dt'} = \gamma_\perp\{f_0 d_0 - [1 + i(\tilde{\delta}_{AC} - \tilde{\delta}\Omega + \tilde{\delta})]p_0\}, \tag{2.14b}$$

$$\frac{dd_0}{dt'} = -\gamma_\parallel[\tfrac{1}{2}(f_0^* p_0 + f_0 p_0^*) + d_0 + 1]. \tag{2.14c}$$

In resonance ($\tilde{\delta}_{AC}=0$, $\tilde{\delta}\Omega=0$) and in the homogeneous broadening limit [$g(\tilde{\delta}) \rightarrow \delta(\tilde{\delta})$], Eqs. (2.14) become the Lorenz equations.

An important point of this derivation is that the mean-field condition is a necessary requirement for the validity of Eqs. (2.14); this is a detail that has been a source of some confusion in the past. It is clear that the single-mode model [Eqs. (2.14)] puts very strong restrictions on the range of variation of the physical parameters. An extension of the traditional single-mode model to experimentally relevant ranges is proposed in Sec. III.

III. AN IMPROVED SINGLE-MODE APPROXIMATION

The steady-state longitudinal profile of the field amplitude $F(z',t')$ tends to remain quite uniform, even for parameter values that are rather widely removed from the ideal mean-field limit. This is not the case for the modulus of the atomic polarization $P(\tilde{\delta},z',t')$ and for the population difference $D(\tilde{\delta},z',t')$, as shown for a typical selection of parameters in Figs. 1(a) and 1(b). Furthermore, we note that in the case of the field variables, the modal decomposition is dictated by the presence of boundary conditions [Eq. (2.6)], that is, by physical reasons, while in the case of the atomic variables, the modal decomposition has only a formal significance. For these reasons a sensible strategy would appear to require a separate handling of the field amplitude and of the atomic variables in the following sense: We maintain the single-mode approximation for the field by setting $f_n=0$ for $n \neq 0$ in Eq. (2.8a), but do not carry out a Fourier expansion of the atomic variables $P(\tilde{\delta},z',t')$ and $D(\tilde{\delta},z',t')$. To

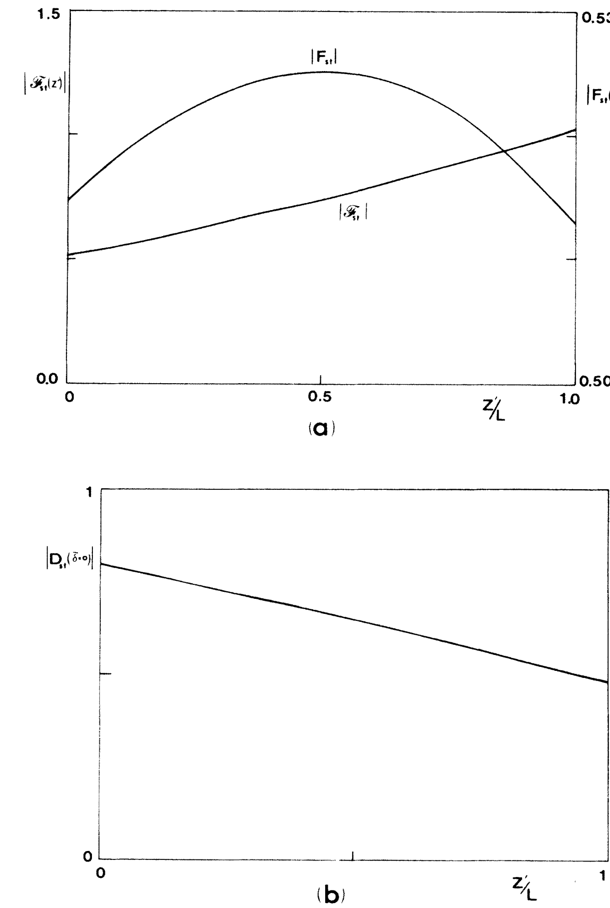

FIG. 1. (a) A comparison between the longitudinal variations of $|\mathcal{F}_{st}|$ [the solution of Eqs. (4.4)] and $|F_{st}|$ [Eq. (2.5a)] for an inhomogeneously broadened laser with $\tilde{\sigma}_D=3$, $R=0.5$, $\tilde{\delta}_{AC}=0$, and $\alpha L=3$. Note that $|\mathcal{F}_{st}|$ and $|F_{st}|$ are plotted with different vertical scales. (b) The longitudinal variation of the population difference corresponding to the parameters of (a).

be more precise, we let

$$P(\tilde{\delta},z',t') = e^{-i\,\delta\Omega t'} p(\tilde{\delta},z',t'), \tag{3.1a}$$

$$D(\tilde{\delta},z',t') = d(\tilde{\delta},z',t') \tag{3.1b}$$

and obtain from the Maxwell-Bloch equations (2.7) the following coupled equations of motion:

$$\frac{df_0}{dt'} = i\,\delta\Omega f_0 - \kappa\left[f_0 + 2C\frac{1}{L}\int_0^L dz' \int_{-\infty}^{+\infty} d\tilde{\delta}\, g(\tilde{\delta}) p(\tilde{\delta},z',t')\right], \tag{3.2a}$$

$$\frac{dp}{dt'}(\tilde{\delta},z',t') = \gamma_\perp\{d f_0 - [1 + i(\tilde{\delta}_{AC} - \tilde{\delta}\Omega + \tilde{\delta})]p\}, \tag{3.2b}$$

$$\frac{d}{dt'} d(\tilde{\delta},z',t') = -\gamma_\parallel[\tfrac{1}{2}(f_0^* p_0 + f_0 p_0^*) e^{2(z'/L)|\ln R|} + d + 1]. \tag{3.2c}$$

The main differences between Eqs. (3.2) and the original Maxwell-Bloch equations (2.7) are evident by inspection.

The longitudinal dependence field has been neglected so that, as a result, the field obeys an ordinary, rather than a partial, difference equation. The spatial dependence of the atomic variables is maintained in the simplified problem, but the z' variable enters only parametrically in the equations of motion. In some way, the role of the spatial dependence is analogous to that of the frequency index that marks the different atomic packets. Each frequency component under the inhomogeneous line evolves independently of the other according to the atomic equations (3.2b) and (3.2c); the same comment applies to the atomic variables that characterize a subsample of the active medium located around the position z'. Of course, these different components interact with one another through the common field f_0 as indicated by Eq. (3.2). From a practical point of view, the numerical solution of Eq. (3.2) is a much less complicated problem than in the case of the full Maxwell-Bloch equations. Of course, it is important to gain some confidence in the reliability of the improved single-mode approximation. In Sec. IV we explore this issue by comparing the steady-state configuration of the exact and approximate equations for both homogeneously and inhomogeneously broadened active media.

IV. COMPARISON BETWEEN EXACT AND APPROXIMATE STEADY STATES

We consider first the exact Maxwell-Bloch equations (2.2). After setting

$$\mathcal{F}(z,t)=\mathcal{F}_{st}(z)e^{-i\delta\Omega t}\,, \tag{4.1a}$$

$$\mathcal{P}(\tilde{\delta},z,t)=\mathcal{P}_{st}(\tilde{\delta},z)e^{-i\delta\Omega t}\,, \tag{4.1b}$$

$$\mathcal{D}(\tilde{\delta},z,t)=\mathcal{D}_{st}(\tilde{\delta},z)\,, \tag{4.1c}$$

the stationary profiles of the atomic variables can be calculated at once with the result

$$\mathcal{P}_{st}(\tilde{\delta},z)=-\mathcal{F}_{st}(z)\frac{1-i(\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta})}{1+(\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta})^2+|\mathcal{F}_{st}(z)|^2}\,, \tag{4.2a}$$

$$\mathcal{D}_{st}(\tilde{\delta},z)=-\frac{1+(\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta})^2}{1+(\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta})^2+|\mathcal{F}_{st}(z)|^2}\,. \tag{4.2b}$$

Next, we set

$$\mathcal{F}_{st}(z)=\rho(z)e^{i\theta(z)} \tag{4.3}$$

and derive the equations that govern the spatial dependence of the field modulus and phase:

$$\frac{d\rho}{dz}=\alpha\int_{-\infty}^{+\infty}d\tilde{\delta}\,g(\tilde{\delta})\frac{1}{1+(\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta})^2+\rho^2}\rho\,, \tag{4.4a}$$

$$\frac{d\theta}{dz}=\frac{\delta\Omega}{c}-\alpha\int_{-\infty}^{+\infty}d\tilde{\delta}\,g(\tilde{\delta})\frac{\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta}}{1+(\tilde{\delta}_{AC}-\tilde{\delta}\Omega+\tilde{\delta})^2+\rho^2}\,. \tag{4.4b}$$

These are to be solved under the constraints imposed by the boundary conditions (2.3) which, in terms of ρ, and θ, take the form

$$\rho(0)=R\rho(L)\,, \tag{4.5a}$$

$$\theta(L)-\theta(0)=-\delta\Omega\frac{\mathcal{L}-L}{c}+2\pi j,\quad j=0,\pm 1,\ldots\,. \tag{4.5b}$$

The index j labels all the possible coexisting steady states that are simultaneously above threshold. Because we are presently concerned with laser configurations for which the single-mode approximation is an adequate one, we can safely assume that only the $j=0$ state will be above threshold.

At this point, it is convenient to discuss separately the steady-state behaviors of homogeneously and inhomogeneously broadened lasers. We begin with the analysis of the homogeneous limit.

A. Homogeneously broadened laser

We set $g(\tilde{\delta})=\delta(\tilde{\delta})$ in Eqs. (4.4) and set $\tilde{\delta}=0$ in the atomic equations. Furthermore, for convenience we introduce the symbol $\tilde{\Delta}=\tilde{\delta}_{AC}-\tilde{\delta}\Omega$. The atomic steady-state profile is given by

$$\mathcal{P}_{st}(z)=-\mathcal{F}_{st}(z)\frac{1-i\tilde{\Delta}}{1+\tilde{\Delta}^2+|\mathcal{F}_{st}(z)|^2}\,, \tag{4.6a}$$

$$\mathcal{D}_{st}(z)=-\frac{1+\tilde{\Delta}^2}{1+\tilde{\Delta}^2+|\mathcal{F}(z)|^2}\,, \tag{4.6b}$$

while the field equations take the form

$$\frac{d\rho}{dz}=\frac{\alpha\rho}{1+\tilde{\Delta}^2+\rho^2}\,, \tag{4.7a}$$

$$\frac{d\theta}{dz}=\frac{\delta\Omega}{c}-\frac{\alpha\tilde{\Delta}}{1+\tilde{\Delta}^2+\rho^2}\,. \tag{4.7b}$$

The field equations can be combined to yield the first integral

$$\ln\left[\frac{\rho(z)}{\rho(0)}\right]=-\frac{1}{\tilde{\Delta}}\left[\theta(z)-\theta(0)-\frac{\delta\Omega}{c}z\right]\,, \tag{4.8a}$$

while Eq. (4.7a) gives immediately

$$(1+\tilde{\Delta}^2)\ln\left[\frac{\rho(z)}{\rho(0)}\right]+\tfrac{1}{2}[\rho^2(z)-\rho^2(0)]=\alpha z\,. \tag{4.8b}$$

After combining Eqs. (4.8) and (4.5) one readily arrives at the result

$$\rho^2(L)=\frac{2}{1-R^2}[\alpha L-(1+\tilde{\Delta}^2)|\ln R|]\,, \tag{4.9a}$$

$$\tilde{\Delta}=\tilde{\delta}_{AC}/(1+\tilde{\kappa}),\quad \tilde{\delta}\Omega=\frac{\tilde{\kappa}}{1+\tilde{\kappa}}\tilde{\delta}_{AC}\,. \tag{4.9b}$$

In addition, if needed, one can calculate the longitudinal profile of the field modulus by solving the transcendental equation (4.8b).

Note that in the homogeneous broadening limit $[g(\tilde{\delta})=\delta(\tilde{\delta})]$, Eq. (2.14) gives

$$|f_0|^2=2C-(1+\tilde{\Delta}^2) \tag{4.10}$$

which coincides with the mean-field limit of Eq. (4.9a); Eq. (4.9b), instead, remains unchanged.

We now consider the homogeneous limit of the single-mode approximation developed in Sec. III. The steady-state values of the atomic variables are given by

$$p_{st} = -f_0 \frac{1-i\tilde{\Delta}}{1+\tilde{\Delta}^2 + |f_0| e^{2(z'/L)|\ln R|}} , \quad (4.11a)$$

$$d_{st} = -\frac{1+\tilde{\Delta}^2}{1+\tilde{\Delta}^2 + |f_0|^2 e^{2(z'/L)|\ln R|}} . \quad (4.11b)$$

The field equation (3.2a) in steady state, with the help of Eq. (4.11a), yields the output intensity $|f_0|^2$ as the solution of

$$1 = 2C\frac{1}{L}\int_0^L dz' \frac{1}{1+\tilde{\Delta}^2 + |f_0|^2 e^{2(z'/L)|\ln R|}} . \quad (4.12)$$

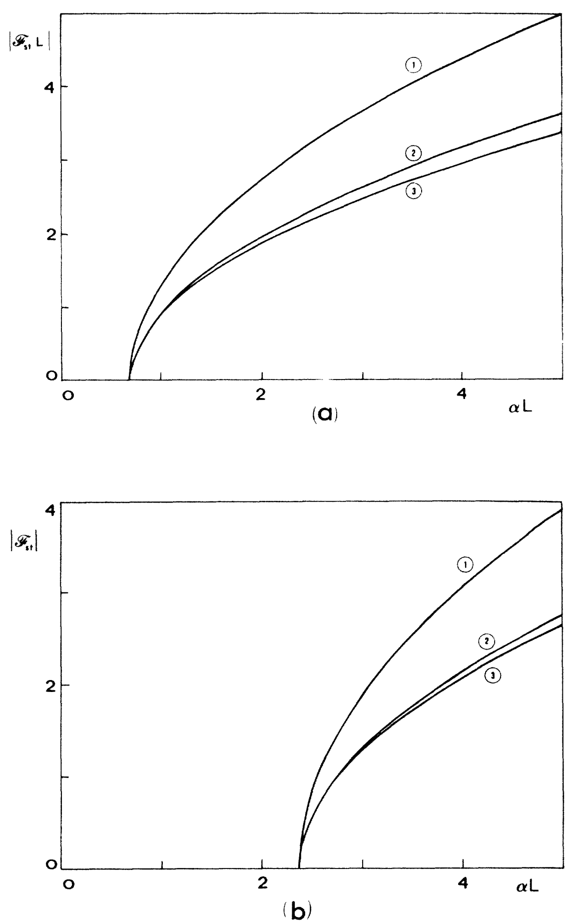

FIG. 2. (a) A comparison between the gain dependence of the output field modulus $|\mathscr{F}_{st}|$ for (1) the mean-field model, (2) the improved single-mode approximation, and (3) the exact solution of the Maxwell-Bloch equations. The parameters used in these calculations are $R=0.5$ and $\tilde{\delta}_{AC}=0$. The value of C for the mean-field calculation is calculated according to the definition $C=\alpha L/2|\ln R|$. (b) A comparison between the gain dependence of the output field modulus $|\mathscr{F}_{st}|$ for (1) the mean-field model, (2) the improved single-mode approximation, and (3) the exact solution of the Maxwell-Bloch equations. The parameters used in these calculations are $R=0.5$, $\tilde{\delta}_{AC}=5.0$, and $\tilde{\alpha}_l=20$. The intermode spacing $\tilde{\alpha}_l$ is necessary to calculate $\tilde{\kappa}$ according to the formula $\tilde{\alpha}_l = 2\pi\tilde{\kappa}/|\ln R|$.

The spatial integral on the right-hand side of Eq. (4.12) can be carried out at once and the required output field intensity takes the form

$$|f_0|^2 = (1+\tilde{\Delta}^2)\frac{1-R^2\exp\left[\frac{|\ln R|}{C}(1+\tilde{\Delta}^2)\right]}{\exp\left[\frac{|\ln R|}{C}(1+\tilde{\Delta}^2)\right]-1} . \quad (4.13)$$

The frequency offset $\tilde{\delta}\Omega$ is still given by the mode-pulling formula [Eq. (4.9b)]. In comparing the results of the exact and the approximate single-mode equations, note that $\rho=(1/R)|f_0|$, as we see from Eqs. (2.5a) and (4.3). The laser threshold $2C=1+\tilde{\Delta}^2$ is the same in the exact equations and in the improved single-mode model.

Figure 2(a) shows a comparison between the modulus of the output field calculated according to the exact equations (4.9a), the improved single-mode approach (4.13), and the mean-field (single-mode) limit (4.10). This calculation corresponds to a resonant situation. A typical example of a detuned configuration is shown in Fig. 2(b).

B. Inhomogeneously broadened laser

The solution of the spatial equations (4.4) for an inhomogeneously broadened active medium is not as simple as in the homogeneous limit. In order to save some numerical labor, we have chosen to analyze only the resonant case which is governed by the steady-state profile $\rho(z)$ solution of the equation

$$\frac{d\rho}{dz} = \alpha \int d\tilde{\delta}\, g(\tilde{\delta}) \frac{1}{1+\tilde{\delta}^2+\rho^2}\rho . \quad (4.14)$$

In this calculation we have chosen

$$g(\tilde{\delta}) = \frac{1}{\sqrt{2\pi\tilde{\sigma}_D^2}} e^{-\tilde{\delta}^2/2\tilde{\sigma}_D^2} , \quad (4.15)$$

where $\sigma_D = \sigma_D/\gamma_\perp$ is the scaled width of the inhomogeneous line. Equation (4.14) must be solved under the condition $\rho(0)=R\rho(L)$. For this purpose, we have adopted the following strategy. For given values of the gain, reflectivity and linewidth, we have selected a sufficiently small value of $\rho(0)$ as "initial condition" for Eq. (4.14). We have solved this integro-differential equation with a proper adaptation of the standard fourth-order Runge-Kutta method and arrived at the appropriate value of $\rho(L)$. Of course, at this point $\rho(0)$ is not equal to $R\rho(L)$. We have constructed the difference $\rho(0)-R\rho(L)$, incremented the initial choice of $\rho(0)$, and repeated this process until the difference $\rho(0)-R\rho(L)$ changed sign. By a combination of linear interpolation techniques and repetitions of the above scheme with finer and finer increments of the initial guess $\rho(0)$, we have produced very accurate solutions $\rho(z)$ in excellent agreement with the constraint $\rho(0)=R\rho(L)$. The mean-field limit and the improved single-mode approximation are easier to handle numerically. The former is given by the solution of the equation

$$1 = 2C\int d\tilde{\delta}\, g(\tilde{\delta})\frac{1}{1+\tilde{\delta}^2+\rho^2} \quad (4.16)$$

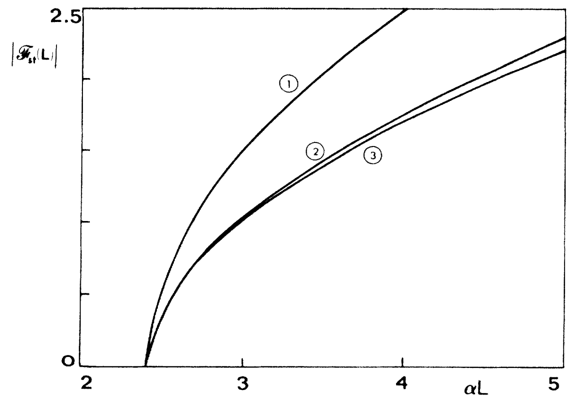

FIG. 3. The same comparison as shown in Fig. 2(a) for an inhomogeneously broadened system with $\tilde{\sigma}_D=3$.

which we had already studied extensively in Ref. 7. The improved single-mode approximation in resonance is described by the state equation

$$1=2C\frac{1}{L}\int_0^L dz'\int_{-\infty}^{+\infty} d\tilde{\delta}\, g(\tilde{\delta})\frac{1}{1+\tilde{\delta}^2+|f_0|^2e^{2(z'/L)|\ln R|}} . \tag{4.17}$$

The spatial integration can be carried out by elementary analytic techniques with the result

$$1=\frac{2C}{|\ln R|}\int d\tilde{\delta}\, g(\tilde{\delta})\frac{1}{1+\delta^2}\ln\left[\frac{1+\tilde{\delta}^2+|f_0|^2}{(1+\tilde{\delta}^2)R^2+|f_0|^2}\right] . \tag{4.18}$$

We have solved both Eqs. (4.16) and (4.18) by a standard Newton-Raphson method and the frequency integrals with a 20-point Gauss quadrature formula.[8] A comparison of the exact and approximate results is shown in Fig. 3.

V. CONCLUDING REMARKS

Figures 2 and 3 show a rather good agreement between the stationary solutions of the improved single-mode approximation and of the exact Maxwell-Bloch equations, even for values of αL of the order of a few units. This is quite unlike the behavior of the mean-field approximation which loses accuracy rather quickly as αL becomes larger than its threshold value. This result indicates clearly the advantage of the improved single-mode model (3.2) over its mean-field counterpart (2.14). The former, in fact, removes the restriction that R be very close to unity. On the other hand, it does not eliminate entirely the restrictions on αL; in fact, the model (3.2) also fails when the gain parameter becomes too large. It is still important to keep in mind that, as shown by Figs. 2 and 3, αL does not have to be small.

Our improved single-mode model can be readily generalized to the case of externally driven systems as optical bistability and the laser with an injected signal. In these systems the reference frequency ω_R should be selected to coincide with the input field frequency, and Eqs. (2.2) take the form

$$\frac{\partial\mathscr{F}}{\partial z}(z,t)+\frac{1}{c}\frac{\partial\mathscr{F}}{\partial t}=\pm\alpha\int_{-\infty}^{+\infty} d\tilde{\delta}\, g(\tilde{\delta})\mathscr{P}(\tilde{\delta},z,t) , \tag{5.1a}$$

$$\frac{\partial\mathscr{P}}{\partial t}(\tilde{\delta},z,t)=\gamma_\perp\{\mathscr{F}\mathscr{D}-[1+i(\tilde{\Delta}+\tilde{\delta})]\mathscr{P}\} , \tag{5.1b}$$

$$\frac{\partial\mathscr{D}}{\partial t}(\tilde{\delta},z,t)=-\gamma_\parallel[\tfrac{1}{2}(\mathscr{F}^*\mathscr{P}+\mathscr{F}\mathscr{P}^*)+\mathscr{D}+1] , \tag{5.1c}$$

where

$$\tilde{\Delta}=(\omega_A-\omega_R)/\gamma_\perp \tag{5.2}$$

and in Eq. (5.1a) we must take the positive sign for optical bistability and the negative sign for the laser with an injected signal. In the case of optical bistability, α represents the unsaturated absorption coefficient per unit length. The boundary conditions are[4]

$$\mathscr{F}(0,t)=TY+Re^{-i\delta_0}\mathscr{F}(L,t-(\mathscr{L}-L)/c) , \tag{5.3}$$

where

$$Y=\mu E_I/\hbar\sqrt{\gamma_\perp\gamma_\parallel T} \tag{5.4}$$

and E_I is the amplitude of the incident field. The cavity detuning parameter δ_0 is given by

$$\delta_0=\frac{\omega_C-\omega_R}{c/\mathscr{L}} , \tag{5.5}$$

where ω_C is the cavity resonance that lies nearest to the frequency of the input field. The transformations (2.5a) and (2.5b) must be generalized as follows,[9]

$$F(z',t')=\mathscr{F}(z',t')\exp\left[\frac{z'}{L}(\ln R-i\delta_0)\right]+TY\frac{z'}{L} , \tag{5.6a}$$

$$P(\tilde{\delta},z',t')=\mathscr{P}(\tilde{\delta},z',t')\exp\left[\frac{z'}{L}(\ln R-i\delta_0)\right] , \tag{5.6b}$$

while Eq. (2.5c) remains unchanged.

Equations (2.4) and (5.6a), applied to the boundary conditions (5.3), yield again Eq. (2.6). Hence, the modal variables can be introduced as done in Sec. II, and the same steps developed in Secs. II and III lead to the mean-field model and to the improved single-mode model for the case $Y\neq 0$.

The success of the improved single-mode model in describing the steady-state behavior suggests that we should now exploit the results of this paper for the analysis of instabilities, following the usual procedure based on the linear stability analysis around the stationary solutions and the subsequent numerical solution of the time-dependent equations in the unstable ranges of parameters. We expect this investigation to be especially interesting in the case of inhomogeneously broadened lasers for which a large amount of experimental data is available.[10] This will be the subject of a future analysis.

We reserve a final comment on a matter of terminology that we would like to adopt in future work. We propose that the set of equations (3.2) be referred to as simply the "single-mode model," and that, in order to avoid confusion, the term "mean-field model" be used for Eqs. (2.14).

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