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Special relativity: Understanding experimental tests and formulations

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The two major proposed first-order test theories for special relativity are shown to be equivalent. The results of many experimental tests of special relativity are given in terms of the free parameters of this unified test theory.

The special theory of relativity was first formulated by Einstein in 1905.[1] It predicts that the relationship between moving frames in given by the Lorentz transformation,

$$t=\gamma[T-(v/c^2)X]\ ,$$
$$x=\gamma(X-vT)\ ,$$
$$y=Y\ ,$$
$$z=Z\ .$$

Here the capital letters represent measurements in the rest frame, and the small letters represent measurements in a frame moving relative to the rest frame with a velocity $v\hat{\mathbf{x}}$. This convention is used throughout this paper, as well as letting $\beta=v/c$ and $\gamma=(1-\beta^2)^{-1/2}$. Robertson[2] proposed a test theory for special relativity and more recently Mansouri and Sexl[3] proposed another. Test theories are needed to interpret the various experimental tests of special relativity. This paper demonstrates that the two formulations are equivalent and also examines what parameters of this theory are tested by various experiments.

Robertson[2] proposed the following general transformation:

$$T=a_0t+(v/c^2)a_1x\ ,$$
$$X=va_0t+a_1x\ ,$$
$$Y=a_2y\ ,$$
$$Z=a_2z\ .$$

He states that this is the most general linear transformation between two frames based on the following assumptions: (1) Space is isotropic and the spatial coordinates are chosen correctly, (2) the only vector of intrinsic significance is the velocity vector, and (3) clocks are synchronized in all frames using Einstein synchronization.[4]

Mansouri and Sexl[3] give the transformation as

$$t=aT+\boldsymbol{\epsilon}\cdot\mathbf{x}\ ,$$
$$x=b(X-vT)\ ,$$
$$y=(d)Y\ ,$$
$$z=(d)Z\ .$$

They also state that this is also the most general linear transformation, but they use only assumptions (1) and (2) above. The vector $\boldsymbol{\epsilon}$ is dependent on the clock-synchronization condition used. Either formulation reduces to the Lorentz transformation if special relativity is correct. Assuming Einstein clock synchronization as Robertson does, $\boldsymbol{\epsilon}=(\epsilon,0,0)$. Equating the two formulations, one sees by inspection that $d=1/a_2$. A little algebra gives $a=1/a_0$, $b=\gamma^2/a_1$, and $\epsilon=-(a_1/a_0)(v/c^2)$. Thus, using Einstein synchronization the Mansouri and Sexl formulation[3] is equivalent to the Robertson formulation[2] with

$$a=1/a_0\ ,$$
$$b=\gamma^2/a_1\ ,$$
$$d=1/a_2\ ,$$

and

$$\epsilon=\frac{-a_1}{a_0}\frac{v}{c^2}\ .$$

Robertson[2] also expresses a_0, a_1, and a_2 in terms of g_0, g_1, and g_2 as $a_0=\gamma g_0$, $a_1=\gamma g_1$, and $a_2=g_2$. Special relativity represents the limit where $g_0=g_1=g_2=1$. Any test of special relativity can be equated with a measurement of g_0, g_1, or g_2. Mansouri and Sexl[3] express a, b, and d in the low-velocity limit as

$$a = 1 + \alpha v^2/c^2 + \cdots ,$$
$$b = 1 + \beta v^2/c^2 + \cdots ,$$
$$d = 1 + \delta v^2/c^2 + \cdots .$$

There are no odd-order terms in these expansions because a, b, and d are assumed to be independent of the relative direction of motion of the two frames. In the special-relativity limit, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, and $\delta = 0$. A measurement of α is equivalent to a low-velocity measurement of g_0. The ratio g_0/g_1, at low velocity, is equivalent to $\beta - \alpha$ since

$$g_0/g_1 \sim b/a = \frac{1+\beta v^2/c^2}{1+\alpha v^2/c^2} \simeq 1 + (\beta - \alpha) v^2/c^2 .$$

Similarly, g_0/g_2 is related to $\delta - \alpha$ and g_1/g_2 to $\beta - \delta$.

Using the equivalencies derived above, one can compare the results of all the experimental tests of special relativity. Differences of Greek letters are the same as ratios of the g's measured at low velocity. The expansions in terms of α, β, and δ are only valid when $v \ll c$, but the values of g_0, g_1, and g_2 can be measured at any velocity. In Table I the relationships between the various symbols are summarized.

Derivation of a useful result follows. From the generalized transformation, T and X are given by

$$T = a_0 t + (v/c^2) a_1 x$$

and

$$X = v a_0 t + a_1 x .$$

Noting that $a_0/\gamma^2 = g_0/\gamma$, combining these equations results in the expression

$$T = \frac{g_0}{\gamma} t + \frac{v}{c^2} X . \quad (1)$$

First consider tests of special relativity that are sensitive to the time transformation or g_0. The archetypical experiment of this type was performed by Ives and Stilwell.[5] One such test is of the Doppler effect, both the transverse Doppler effect[6-11] and the entire relativistic Doppler effect.[12,13] The relativistic Doppler effect is the more general case. Effects can be observed in two reference frames: the laboratory frame and the center-of-mass or atom's frame. Using Eq. (1), a time difference at a fixed position in space is

$$\Delta T_{\text{atom}} = \frac{g_0}{\gamma} \Delta t_{\text{lab}}$$

or, in terms of frequencies,

$$\nu_{\text{atom}} = \frac{\gamma}{g_0} \nu_{\text{lab}} .$$

Since the nonrelativistic Doppler effect can be written as $E = E_0(1 + \beta \cos\alpha)$,[14] the relativistic Doppler effect is

$$E = E_0 \frac{1}{g_0} \gamma (1 + \beta \cos\alpha) .$$

This result can also be obtained from a purely relativistic derivation. Testing the relativistic Doppler formula is equivalent to measuring the difference between g_0 and 1. The transverse Doppler effect depends on the same formula as the entire relativistic effect, but $\alpha = 90°$, so $E = E_0 \gamma / g_0$. Therefore, the transverse Doppler effect is a measurement of g_0.

A lifetime seen in the laboratory frame is a factor of γ/g_0 longer than the lifetime seen in a moving frame, thus particle lifetime measurements are sensitive to g_0.[15,16] Taking Eq. (1) in the particle's rest frame where $X_1 = X_2$ (X_1 and X_2 are the particle's position at t_1 and t_2)

$$\Delta T = \frac{g_0}{\gamma} \Delta t .$$

Therefore, any lifetime measurement is also sensitive to g_0. A twin-paradox test[17] is a matter of measuring times in two different frames and hence is the same as the lifetime measurements mentioned above. A paradox test is therefore sensitive to g_0 as well.

One could also measure the effective mass of a particle moving at a high velocity[18,19] and see that it matches the form $M = \gamma M_0$. Equivalently, one could measure the energy of a moving particle[20,21] and see that it agrees with the relativistic form $E = \gamma M_0 c^2$. Let τ be defined as the time difference between two events in their rest frame ($\tau = T_1 - T_2$). Then, taking Eq. (1) with $X_1 = X_2$, one obtains $\tau = \Delta T = (g_0/\gamma)\Delta t$. Since $\mathbf{p}$ is defined as $\mathbf{p} = M(d\mathbf{x}/d\tau)$, it follows that $\mathbf{p} = (\gamma/g_0) M_0 \mathbf{v}$. By analogy to the usual case, $M = (\gamma/g_0) M_0$, and, similarly, $E = (\gamma/g_0) M_0 c^2$. Therefore, energy and mass measurements at high velocities are sensitive to g_0.

Another timelike test of special relativity is to measure the velocity of a high-energy electron and compare it with the velocity of a photon, c.[22] An electron's energy is given by

$$E_e = \frac{\gamma}{g_0} M_0 c^2 = \frac{M_0 c^2}{g_0} \left[1 - \frac{V_e^2}{c^2} \right]^{-1/2} .$$

Solving this equation for the electron velocity gives the following result:

$$V_e = c \left[1 - \frac{M_0^2 c^4}{E_e^2 g_0^2} \right]^{1/2} .$$

TABLE I. Conversion factors relating the two test theories I consider.

Term	is equivalent to either	or
a	$1/a_0$	$1+\alpha v^2/c^2 + \cdots$
b	γ^2/a_1	$1+\beta v^2/c^2 + \cdots$
d	$1/a_2$	$1+\delta v^2/c^2 + \cdots$
ϵ	$-\frac{a_1 v}{a_0 c^2}$	
a_0	$1/a$	$g_0 \gamma$
a_1	γ^2/b	$g_1 \gamma$
a_2	$1/d$	g_2

If $\gamma \gg 1$, then $E_e \gg M_0 c^2$, and one can expand the electron's velocity as

$$V_e \simeq c\left[1-\frac{M_0^2 c^4}{2E_e^2 g_0^2}\right],$$

so

$$V_\gamma - V_e \simeq \left[\frac{M_0^2 c^5}{2E_e^2 g_0^2}\right].$$

The difference between the velocity of a high-energy electron and c is thus sensitive to g_0.

Thomas precession, or $g-2$, measurements are also used as a test of special relativity.[23–28] In these experiments γ is compared with $\tilde{\gamma}$ where $\gamma=(1-\beta^2)^{-1/2}$ and $\tilde{\gamma}=(p/M_0)(dp/dE)$.[25] The quantity $g/2-\gamma/\tilde{\gamma}$ is measured for a particle (an electron or a muon) at both high and low velocities. The two results are then compared. From the previous discussion $M=\gamma M_0/g_0$. It follows that $p=\gamma M_0 v/g_0$, and $E=\gamma M_0 c^2/g_0$; therefore, $dp/dv=\gamma^3 M_0/g_0$, and $dE/dv=\gamma^3 M_0 v/g_0$. Given these values, $\tilde{\gamma}=(p/M_0)(dp/dE)=(1/g_0)(\gamma M_0 v/M_0)(1/v)=\gamma/g_0$. The comparison of γ and $\tilde{\gamma}$ is sensitive to the parameter g_0.

A second type of special-relativity test is sensitive to spatial transformations, or g_1 and g_2. This effect was first observed in the Michelson-Morley[29] and Kennedy-Thorndike[30] experiments. One could measure a length in two perpendicular directions[31,32] as Michelson and Morley did. If the two lengths are equal in some preferred or ether frame, then the situation shown in Fig. 1 holds. Using the Pythagorean theorem in the preferred frame,

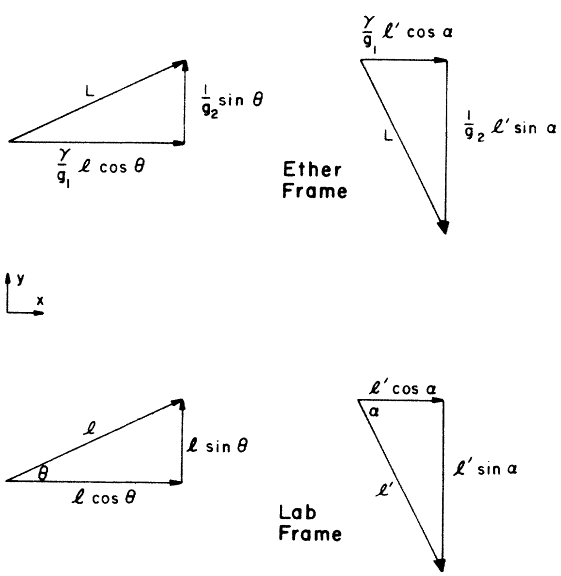

FIG. 1. Comparison of two lengths l and l' in the laboratory frame. These lengths are assumed to be the same in the preferred or ether frame. Here $\gamma=(1-\beta^2)^{-1/2}$ and $\beta=v/c$, and the laboratory frame is moving with velocity $v\hat{\mathbf{x}}$ with respect to the ether frame. Experimentally, $\alpha=\theta-\pi/2$ in most cases.

$$L^2=\left[\frac{1}{g_2}\right]^2 l^2\sin^2\theta+\left[\frac{\gamma}{g_1}\right]^2 l^2\cos^2\theta,$$

and

$$L^2=\left[\frac{1}{g_2}\right]^2 l'^2\sin^2\alpha+\left[\frac{\gamma}{g_1}\right]^2\cos^2\alpha .$$

If $\alpha=\theta-\pi/2$, by equating the expressions for L^2 one derives

$$\left[\left[\frac{l}{l'}\right]^2-\left[\frac{\gamma g_2}{g_1}\right]^2\right]\sin^2\theta-\left[\left[\frac{l'}{l}\right]^2-\left[\frac{\gamma g_2}{g_1}\right]^2\right]\cos^2\theta=0 .$$

The ratio l/l' depends on $\gamma g_2/g_1$ weighted by $\sin^2\theta$ and on $g_1/\gamma g_2$ weighted by $\cos^2\theta$. A few examples may serve to clarify this dependence. If $\theta=0$, then $l/l'=\gamma g_2/g_1$, and if $\theta=\pi/2$, then $l/l'=g_1/\gamma g_2$. For θ near $\pi/4$, there is little correlation between l/l' and g_1/g_2. A length-measuring experiment of this type is sensitive to the ratio of g_1 and g_2.

Measuring the speed of light from a moving frame is also a test of spatial relationships.[33,34] As an example of a moving frame, the speed of the light emitted by a moving particle could be measured. Special relativity predicts that the speed of light is a constant c in all frames. In the rest frame assume a beam of light with velocity c travels at an angle α to the X axis. The X and Y components of the velocity are

$$\dot{X}=\frac{dX}{dT}=c\cos\alpha \quad\text{and}\quad \dot{Y}=\frac{dY}{dT}=c\sin\alpha .$$

From the generalized transformations

$$dt=\frac{\gamma}{g_0}\left[dT-\frac{v}{c^2}dX\right],$$

$$dx=\frac{\gamma}{g_1}(dX-v\,dT),$$

and

$$dy=\frac{1}{g_2}dY,$$

or

$$\dot{x}=\frac{dx}{dt}=\frac{g_0}{g_1}\frac{(\dot{X}-v)}{\left[1-\frac{v}{c^2}\dot{X}\right]}=\frac{g_0}{g_1}c\,\frac{c\cos\alpha-v}{c-v\cos\alpha},$$

$$\dot{y}=\frac{dy}{dt}=\frac{g_0}{g_2}\frac{\dot{Y}}{\gamma(1-(v/c^2)\dot{X})}=\frac{g_0}{g_2}c\,\frac{c\sin\alpha}{\gamma(c-v\cos\alpha)} .$$

In the moving frame

TABLE II. Experimental tests of special relativity. Here R is the sensitivity of the experiment divided by the expected size of the effect. The expected size is 1 in all experiments except the ether drift where it is v/c for the earth moving through the ether or $\simeq 10^{-8}$. (Note that the electron velocity measurement is sensitive to g_0^2 so the result cannot be directly compared with the other g_0 tests.)

Ref.	Test	β	Parameters tested	R
6	Transverse Doppler effect	1×10^{-6}	g_0	3×10^{-2}
7	Transverse Doppler effect	4×10^{-4}	g_0	1×10^{-1}
8	Transverse Doppler effect	7×10^{-7}	g_0	4×10^{-2}
9	Transverse Doppler effect	7×10^{-3}	g_0	5×10^{-2}
10	Transverse Doppler effect	2.3×10^{-3}	g_0	1×10^{-4}
11	Transverse Doppler effect	3.6×10^{-3}	g_0	4×10^{-5}
12	Relativistic Doppler effect	1.0×10^{-2}	g_0	5×10^{-3}
13	Relativistic Doppler effect	0.84	g_0	2.7×10^{-4}
15	Pion lifetime	0.92	g_0	4×10^{-3}
16	Muon lifetime	0.9994	g_0	1×10^{-3}
17	Twin paradox	1×10^{-6}	g_0	3×10^{-2}
18	Mass of a high-velocity proton	0.7	g_0	6×10^{-4}
19	Mass of a high-velocity proton	0.81	g_0	1×10^{-3}
20	Relativistic energy	0.9995	g_0	3×10^{-1}
21	Relativistic energy	0.9988	g_0	1×10^{-2}
22	Velocity of a high-energy electron	$1-(5\times10^{-10})$	g_0	2×10^{-7}
23–26	Muon g factor	0.9994	g_0	2.7×10^{-4}
25–27	Electron g factor	0.57	g_0	3×10^{-6}
28	Electron g factor	0.999 99	g_0	1.7×10^{-2}
31	Ether drift	10^{-4}	g_1/g_2	1×10^{-3}
32	Ether drift	10^{-4}	g_1/g_2	5×10^{-7}
33	Speed of light	10^{-3}	$g_0/g_1, g_0/g_2$	2×10^{-9}
34	Speed of light	0.999 75	$g_0/g_1, g_0/g_2$	1.3×10^{-4}

$$V^2=\dot{x}^2+\dot{y}^2 = c^2\left[\left[\frac{g_0}{g_1}\right]^2\left[\frac{c\cos\alpha - v}{c - v\cos\alpha}\right]^2 + \left[\frac{g_0}{g_2}\right]^2\left[\frac{c\sin\alpha}{\gamma(c-v\cos\alpha)}\right]^2\right].$$

Thus, given α and v, measuring the constancy of the speed of light is equivalent to observing a function of the ratios g_0/g_1 and g_0/g_2.

The results of all experiments can be expressed in terms of one set of parameters. I have chosen g_0, g_1, and g_2, but a_0, a_1, and a_2; a, b, and c; or α, β, and δ could be used just as well. In general, experiments that measure a time are sensitive to the parameter g_0, and experiments that measure lengths are sensitive to g_1 and g_2. The parameter g_0 would be different from 1 if the speed of light were not a constant in all frames. If there were a preferred or ether frame that we were moving through, then $g_1\neq 1$. The parameter g_2, which relates to measurements perpendicular to the motion of the frame, is assumed in most treatments to equal 1. A deviation from 1 in this parameter would not have any physical significance. It should be emphasized that g_0, g_1, and g_2 are all functions of v rather than constants, so the g's should be measured at many different velocities. In Table II, results of many experimental tests of special relativity are listed, along with the velocity at which the experiment was done and the parameters to which the experiment is sensitive. The theory of special-relativity tests has been discussed recently by Maciel and Tiomno.[35]

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[1]A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905).

[2]H. P. Robertson, Rev. Mod. Phys. **21**, 378 (1949).

[3]R. Mansouri and R. U. Sexl, Gen. Relativ. Gravit. **8**, 497 (1977).

[4]Einstein clock synchronization can be defined as follows. Consider two clocks, A and B, at rest with respect to each other. At $t=0$, a light signal is sent from A to B. At $t=t_1$, the signal arrives at B, and a signal is immediately sent back to A, where it arrives at $t=t_2$. Einstein clock synchronization demands that $t_2=2t_1$. Mansouri and Sexl (Ref. 3) give a good discussion of clock-synchronization conditions.

[5]H. E. Ives and G. R. Stilwell, J. Opt. Soc. Am. **28**, 215 (1938).

[6]R. F. C. Vessot and M. W. Levine, Metrologia **6**, 116 (1970).

[7]R. V. Pound and G. A. Rebka, Phys. Rev. Lett. **4**, 274 (1960).

[8]H. Hay, J. Schiffer, T. Cranshaw, and P. Egelstaff, Phys. Rev. Lett. **4**, 165 (1960).

[9]H. I. Mandelberg and L. Witten, J. Opt. Soc. Am. **52**, 529 (1962).

[10]J. J. Syder and J. L. Hall, *Laser Spectroscopy II,* edited by T. W. Hänsch, S. E. Harris, S. Haroche, and J. C. Peblay-Peyroula (Springer, Berlin, 1975), pp. 6—17.

[11]M. Kaivola, O. Poulson, E. Riis, and S. A. Lee, Phys. Rev. Lett. **54**, 255 (1985).

[12]P. Juncar *et al.*, Phys. Rev. Lett. **54**, 11 (1985).

[13]D. W. MacArthur *et al.* (unpublished).

[14]See, e.g., C. Kittel *et al.*, *Mechanics,* Berkeley Physics Course, Vol. 1 (McGraw-Hill, New York, 1965), p. 344.

[15]D. S. Ayres *et al.*, Phys. Rev. D **3**, 1051 (1971).

[16]J. Bailey *et al.*, Nature (London) **268**, 301 (1977).

[17]J. C. Hafele and R. E. Keating, Science **177**, 166 (1972).

[18]D. J. Grove and J. G. Fox, Phys. Rev. **90**, 378 (1953).

[19]V. P. Zrelov, A. A. Tyapkin, and P. S. Farago, Zh. Eksp. Teor. Fiz. **34**, 555 (1958) [Sov. Phys.—JETP **1**, 384 (1958)].

[20]D. R. Walz, H. P. Noyes, and R. L. Carezani, Phys. Rev. A **29**, 2110 (1984).

[21]G. E. Fisher and Y. Murata, Nucl. Instrum. Methods **78**, 25 (1970).

[22]Z. G. T. Guiragossián *et al.*, Phys. Rev. Lett. **34**, 335 (1975).

[23]J. Bailey *et al.*, Phys. Lett. **68B**, 191 (1977).

[24]J. Bailey *et al.*, Nuovo Cimento A **9**, 369 (1972).

[25]D. Newman, G. W. Ford, A. Rich, and E. Sweetman, Phys. Rev. Lett. **40**, 1355 (1978).

[26]R. S. Van Dyke, P. B. Schwinberg, and H. G. Dehmelt, Phys. Rev. Lett. **38**, 310 (1977).

[27]J. C. Wesley and A. Rich, Phys. Rev. A **4**, 1341 (1971).

[28]P. S. Cooper *et al.*, Phys. Rev. Lett. **42**, 1386 (1979).

[29]A. A. Michelson and E. H. Morley, Am. J. Sci. **34**, 333 (1887).

[30]R. J. Kennedy and E. M. Thorndike, Phys. Rev. **42**, 400 (1932).

[31]T. S. Jaseja, A. Javan, J. Murray, and C. H. Townes, Phys. Rev. **133**, A122 (1964).

[32]A. Brillet and J. L. Hall, Phys. Rev. Lett. **42**, 549 (1979).

[33]K. Brecher, Phys. Rev. Lett. **39**, 1051 (1977).

[34]T. Alvager, F. J. M. Farley, J. Kjellman, and I. Wallin, Phys. Lett. **12**, 260 (1974).

[35]A. K. A. Maciel and J. Tiomno, Phys. Rev. Lett. **55**, 143 (1985).