Semiclassical theory of coupled lasers

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The semiclassical equations of motion for a system of coupled lasers are developed and the frequency locking of the lasers comprising the system is analyzed. It is shown that the frequency-coupling range, in terms of the coupled cavities' mismatch, is proportional to the coupling coefficient. For a system where the cavities are uniformally filled with the active medium, the coupling vanishes regardless of the transmittance of the coupling mirrors. Our theory is valid for all values of coupling and for any number of lasers in the array. It may also be adapted to study different types of coupling arrangements.

I. INTRODUCTION

The use of two consecutive Fabry-Perot interferometers for the purpose of better axial-mode selectivity was suggested by Kleinman and Kisliuk[1] and has since been further investigated by several workers.[2–8] Coupling two or more resonators can lead to desirable properties such as selective mode suppression,[2] enhancement of intracavity laser power,[6] frequency stabilization against cavity-length fluctuations[7] and phased laser arrays.[8]

Spencer and Lamb[7] had previously studied the coupled-laser problem. Their treatment, which was based on expanding the laser field in terms of the individual passive-resonator eigenmodes is limited to the weak-coupling regime. In order to study the strongly-coupled-laser problem, one has to expand the laser field in terms of composite passive-resonator eigenmodes.[5–9] Recently, we have shown that by using the composite resonator eigenmodes, the semiclassical equations for the fields of coupled lasers are very similar in form to those of a single-resonator laser.[5] This is a useful result because the solution of the single-resonator problem is well known.[10] The effects of the coupling between lasers are primarily due to changes in the passive-resonator-mode structure. In particular, the frequency spacings between the modes are no longer uniform, and the cavity losses of different modes are different.

In this paper we derive the semiclassical equations of motion for a system of coupled lasers. We apply these equations to study frequency locking in coupled lasers. The ability to lock the frequencies of the coupled lasers in the presence of resonator-length fluctuations is important for phased laser-array applications. The approach adopted here is general in that it places no restriction on the strength of coupling or the relative lengths of the coupled lasers. It also applies to a variety of geometrical forms of coupling. Throughout this paper, homogeneous broadening is assumed; however, it is a straightforward problem to derive the corresponding equations for an inhomogeneously broadened active medium.

Our theory is based on the knowledge of the composite resonator passive modes. In Sec. II, a simple method is developed for computing the eigenfunctions and eigenfrequencies of these modes. Even though we developed the analysis for resonators coupled in series, the same approach applies to other forms of coupling (Sec. VII). In Sec. III, the results are applied to the case of two coupled resonators. The semiclassical equations of motion are derived in Sec. IV.

Frequency locking is discussed in Sec. V where the coupled resonator field equations are solved numerically for the stable solutions. In Sec. VI the field equations are reduced to equations similar to those encountered in connection with ring lasers. The "decoupled" approximation[10] is used to solve these equations. In Sec. VII the methods of Sec. II are applied to a different form of coupling (star coupling).

II. THE COMPOSITE RESONATOR EIGENMODES

Consider a high-Q composite resonator consisting of M subresonators (see Fig. 1). These subresonators are coupled via transmitting mirrors. These mirrors are characterized by their reflectance amplitudes $r(j)$, $\bar{r}(j)$, and the transmittance amplitude $t(j)$. Here, $r(j)$ $[\bar{r}(j)]$ corresponds to the reflectance amplitudes when a mirror is viewed from the left (right). The transmittance amplitudes as viewed from both sides are taken to be equal since we are assuming that the refractive index of the medium on both sides of the mirrors are the same. The choice of $r(j)$, $\bar{r}(j)$, and $t(j)$ is not completely arbitrary. In the case of a lossless mirror, they must satisfy the relation $r(j)/t(j)=-\bar{r}^*(j)/t^*(j)$.

The intracavity electric field is analyzed in terms of the right and left running waves, $A^+(j)$ and $A^-(j)$, respectively. These running waves have constant, but different amplitudes in each (passive) subresonator. To relate the wave amplitudes of these waves in different sections of the composite resonator, we first relate the waves at both sides of a given mirror, say mirror j which is located at $z=z_j$. The waves to the left are labeled $A^+(j)$ and $A^-(j)$, while the ones on the right are labeled $\bar{A}^+(j)$ and $\bar{A}^-(j)$. Here, $A(j)$ is a shorthand for $A(z_j)$. With the aid of Fig. 2, one can immediately write the following relations:

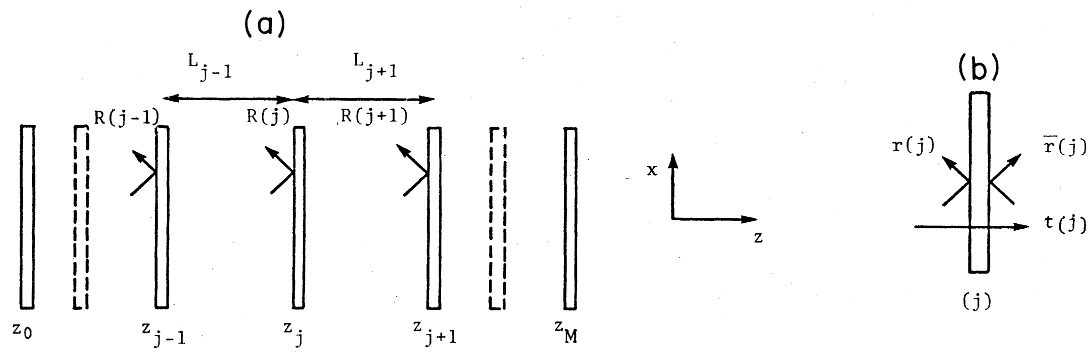

FIG. 1. (a) A coupled array of M resonators. (b) A reflector element with reflectance amplitudes $r(j)$, $\bar{r}(j)$, and transmittance amplitude $t(j)$.

$$\bar{A}^{+}(j)=\bar{r}(j)\bar{A}^{-}(j)+t(j)A^{+}(j) , \tag{1}$$

$$A^{-}(j)=r(j)A^{+}(j)+t(j)\bar{A}^{-}(j) . \tag{2}$$

The waves to the right of mirror j are related to the waves just to the left of mirror $j+1$ through the relations

$$\bar{A}^{+}(j)=A^{+}(j+1)\exp(ikL_{j+1}) , \tag{3}$$

and

$$\bar{A}^{-}(j)=A^{-}(j+1)\exp(-ikL_{j+1}) , \tag{4}$$

where k is the field k vector and $L_{j+1}=z_{j+1}-z_j$ is the length of the subresonator bound by mirrors j and $j+1$. In what follows, we will label this subresonator as the $j+1$ subresonator.

Substituting Eqs. (3) and (4) in Eqs. (1) and (2), and after rearranging terms one obtains the following useful relations:

$$A^{+}(j)=\frac{1}{t(j)}[A^{+}(j+1)\exp(ikL_{j+1}) - A^{-}(j+1)\bar{r}(j)\exp(-ikL_{j+1})] , \tag{5a}$$

$$A^{-}(j)=\frac{1}{t(j)}\{A^{+}(j+1)r(j)\exp(ikL_{j+1}) + A^{-}(j+1)[t^2(j)-r(j)\bar{r}(j)] \times\exp(-ikL_{j+1})\} . \tag{5b}$$

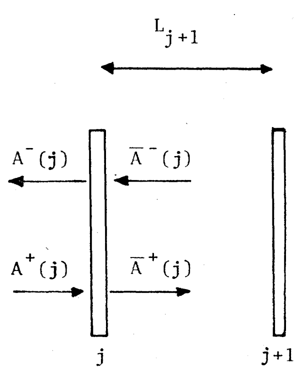

FIG. 2. Counterpropagating waves in the $(j+1)$ cavity.

It is useful to introduce the concept of the effective reflectance amplitude $R(j)=A^{-}(j)/A^{+}(j)$, which is the net reflectance amplitude due to the system of mirrors to the right of the point z_j, including the jth mirror.[11] Dividing Eq. (5b) by (5a), one gets the recursion relation for the effective reflectance at z_j in terms of the effective reflectance at z_{j+1},

$$R(j)=\frac{r(j)+[t^2(j)-r(j)\bar{r}(j)]R(j+1)\exp(-2ikL_{j+1})}{1-\bar{r}(j)R(j+1)\exp(-2ikL_{j+1})} . \tag{6}$$

Since there is no mirror to the right of the point z_M, we have $R(M)=r(M)$, which are the starting values for the recursion relation, Eq. (6). The eigenmodes $U_n(z)$ of the composite resonator corresponding to the eigenfrequencies $\Omega_n=ck_n$, can be derived as follows.

The total field at a point z is the superposition of the two counterpropagating waves at that point. Hence,

$$U_n(z)=A_n^{+}(z)+A_n^{-}(z)=A_n^{+}(j)\exp[-ik_n(z-z_j)] + A_n^{-}(j)\exp[ik_n(z-z_j)] . \tag{7}$$

Since $A_n^{-}(j)=R_n(j)A_n^{+}(j)$, Eq. (7) can be rewritten as

$$U_n(z)=\{\exp[-ik_n(z-z_j)] + R_n(j)\exp[ik_n(z-z_j)]\}A^{+}(j) . \tag{8}$$

Since the terminal mirrors are assumed to be total reflectors with no losses taking place, $R_n(j)$ will have a unity amplitude. Hence, one can write $R_n(j)$ as

$$R_n(j)=-\exp[i\delta_n(j)] , \tag{9}$$

where $\delta_n(j)$ is a phase factor.

Thus, Eq. (8) can be simplified to

$$U_n(z)=B_n(j)\sin[k_n(z-z_j)+\delta_n(j)/2] , \quad z_{j-1}\le z\le z_j \tag{10}$$

where

$$B_n(j)=-2iA_n^{+}(j)\exp[i\delta_n(j)/2] . \tag{11}$$

The amplitude $A_n^{+}(j)$ can be calculated by combining the definition of $R_n(j)$ with Eq. (5a) which results in the following:

$$A_n^+(j)=\frac{1}{t(j)}[\exp(ik_nL_{j+1}) - \bar{r}(j)R_n(j+1)\exp(-ik_nL_{j+1})]A_n^+(j+1) . \quad (12)$$

Equation (12) is a recursion relation for the mode amplitude in different sections of the composite resonator. The starting value for the recursive computation is $A_n^+(M)=C$, an arbitrary constant. To summarize, Eq. (10) is the field distribution in any section of the composite resonator corresponding to an eigenfrequency $\Omega_n=ck_n$. The amplitude $B_n(j)$ of the mode in each subresonator can be calculated by first computing $R_n(j)$ and $A_n^+(j)$ at each mirror by using Eq. (6) and Eq. (12) and then substituting in Eq. (12). As for the eigenfrequencies, we show in Appendix A that they can be calculated as the roots of the following dispersion equation:[12]

$$R_n(j)\bar{R}_n(j)\exp(-2ik_nL_j)=1 , \quad (13)$$

or equivalently,

$$\delta_n(j)+\bar{\delta}_n(j)-2ik_nL_j=-2\pi m,\ \ m=0,1,2,\ldots \quad (14)$$

where $\bar{R}_n(j)$ is the effective reflectance amplitude to the right of mirror [$(j-1)$ when the system is viewed from the right] causing a phase shift of $\bar{\delta}_n(j)$, while $R_n(j)$ is the effective reflectance amplitude at the left side of mirror (j). The physical meaning of Eq. (14) is that for any wave that makes a complete round trip in a subresonator j, the total phase shift should be $2\pi m$, otherwise there will be destructive interference. The waves that satisfy Eq. (14) are the modes of the system. In Appendix A we also show that if Eq. (13) or Eq. (14) is applicable to one of the subresonators comprising the system, then every subresonator will satisfy the same equation. In other words, j is arbitrary and any of the subresonators can be used in Eq. (13).

III. A TWO-RESONATOR SYSTEM

For two coupled resonators such as the one depicted in Fig. 3, the recursion relation (6) gives

$$R_n(1)=\frac{r(1)+[t^2(1)-r(1)\bar{r}(1)]r(2)\exp(-2ik_nL_2)}{1-\bar{r}(1)r(2)\exp(-2ik_nL_2)} , \quad (15a)$$

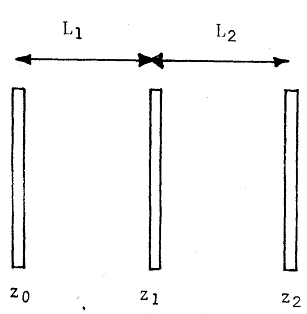

FIG. 3. Two coupled resonators.

and

$$\bar{R}_n(1)=\bar{r}(0) . \quad (15b)$$

Assuming that $r(2)=-1$ and $\bar{r}(1)=-r(1)$, Eq. (15a) simplifies to

$$R_n(1)=-\exp[i\delta_n(1)] , \quad (16)$$

where

$$\delta_n(1)=2\tan^{-1}\left[\frac{r(1)+1}{r(1)-1}\tan(k_nL_2)\right] . \quad (17)$$

Applying Eq. (13) for $j=1$ and using Eqs. (15a) and (15b), we get a transcendental dispersion equation for the eigenfrequencies which can be written as [assuming $\bar{r}(0)=1$],

$$r(1)\cos[k_n(L_1-L_2)]=\cos[k_n(L_1+L_2)] . \quad (18)$$

The eigenmodes can be calculated with the aid of Eq. (12). Setting the arbitrary constant $B_n(2)$ to unity, $B_n(1)$ becomes

$$B_n(1)=[1-2r(1)\cos(2k_nL_2)+r^2(1)]^{1/2}/t(1) . \quad (19)$$

Hence, the eigenmodes take the form

$$U_n(z)=\begin{cases}\sin[k_n(z-z_2)], & z_1\le z\le z_2\\ B_n(1)\sin[k_n(z-z_1)+\delta_n(1)/2], & 0\le z\le z_1\end{cases} \quad (20)$$

where $B_n(1)$ is as given in Eq. (19).

For a given mode, the ratio of the amplitudes in the two subresonators is

$$B_n(1)/B_n(2)=[1-2r(1)\cos(2k_nL_2)+r^2(1)]^{1/2}/t(1) . \quad (21)$$

which has a maximum value of $[1+r(1)]/t(1)$ and a minimum value of $[1-r(1)]/t(1)$.

IV. SEMICLASSICAL EQUATIONS OF MOTION

Instead of representing the laser field in terms of the eigenmodes of one of the subresonators, as in the Spencer-Lamb theory,[7] we represent the electric field by superposition of the normal modes of the whole composite resonator. Hence,

$$E(z,t)=\tfrac{1}{2}\sum_n \mathscr{E}_n(t)e^{-i(\nu_nt+\phi_n)}U_n(z)+\text{c.c.} \quad (22)$$

The polarization can be similarly represented

$$P(z,t)=\tfrac{1}{2}\sum_n \mathscr{P}_n(t)e^{-i(\nu_nt+\phi_n)}U_n(z)+\text{c.c.} \quad (23)$$

By substituting Eqs. (22) and (23) into Maxwell's wave equation, by performing the standard simplifications by adopting the slowly varying phase and amplitude approximation, and by separating the real and imaginary components, one obtains[10]

$$\dot{\mathscr{E}}_n(t)+\Gamma_n\mathscr{E}_n=-\frac{1}{2}\frac{\nu_n}{\epsilon_0}\text{Im}(\mathscr{P}_n) , \quad (24)$$

$$\nu_n+\dot{\phi}_n=\Omega_n-\frac{1}{2}\frac{\nu_n}{\epsilon_0}\mathscr{E}_n^{-1}\mathrm{Re}(\mathscr{P}_n)\ , \quad (25)$$

where Γ_n is the laser-mode bandwidth due to nonresonant losses and the dot represents time derivative. The slowly varying complex polarization, $\mathscr{P}_n(t)$, can be written as[10]

$$\mathscr{P}_n(t)=2\exp[i(\nu_n t+\phi_n)]\frac{\wp}{\mathscr{N}}\sum_{j=1}^{M}\int_{z_{j-1}}^{z_j}dz\, U_n^*(z)\rho_{ab}(z,t)\ , \quad (26)$$

where ρ_{ab} is the off-diagonal matrix element of the polarization of the two-level atomic system, $\wp$ is the dipole moment, and the normalization constant is

$$\mathscr{N}=\sum_{j=1}^{M}\int_0^{L_j}dz\ |U_n(z)|^2\ , \quad (27)$$

where the summation over j corresponds to the contribution of each subresonator.

One can show that under the rate-equation approximation (REA),[10]

$$\rho_{ab}(z,t)=-\frac{\wp}{2\hbar}N(z)\sum_n \mathscr{E}_n(t)\exp[-i(\nu_n t+\phi_n)]U_n(z)\mathscr{D}(\omega-\nu_n)\left[1+\frac{2\gamma_{ab}}{\gamma}\sum_m I_m\,|U_m|^2\mathscr{L}(\omega-\nu_m)\right]\ , \quad (28)$$

where $N(z)$ is the population inversion density, γ is the atomic dipole decay constant, and $\gamma_{ab}=1/2(\gamma_a+\gamma_b)$, the spontaneous emission and inelastic collision contribution to the decay of the atomic dipole. Also

$$\mathscr{L}(\omega-\nu_m)=\gamma^2/[\gamma^2+(\omega-\nu_m)^2]\ , \quad (29a)$$

and

$$\mathscr{D}_x(\omega-\nu_m)=1/[\gamma_x+i(\omega-\nu_m)]\ , \quad (29b)$$

where $x=a,b$ or x may stand for nothing (missing).

Substituting Eq. (28) into Eq. (26), the REA expression for $\mathscr{P}(t)$ becomes

$$\mathscr{P}_n(t)=-i\frac{\wp^2}{\hbar\mathscr{N}}\left[\sum_m \mathscr{D}(\omega-\nu_m)\mathscr{E}_m(t)\exp[i(\nu_n-\nu_m)t+i(\phi_n-\phi_m)]\sum_{j=1}^{M}\int_0^{L_j}dz\frac{U_n^*(z)U_m(z)N(z)}{1+\frac{2\gamma_{ab}}{\gamma}\sum_\sigma I_\sigma\,|U_\sigma|^2\mathscr{L}(\omega-\nu_\sigma)}\right], \quad (30)$$

where I_σ is the dimensionless intensity defined as $I_\sigma=\frac{1}{2}(\wp\epsilon_\sigma/\hbar)^2/(\gamma_a\gamma_b)$. For weak coupling between the subresonators, only one mode will be predominant in each subresonator. This predominance is acquired by the mode with a frequency close to the resonance frequency of the subresonator when isolated (i.e., not coupled). This can be seen clearly from an inspection of Eq. (20). Hence, for single-mode operation, Eq. (30) can be solved and yields

$$\mathscr{P}_n(t)=-\frac{2\wp^2}{\hbar}\bar{N}\frac{(\omega-\nu_n)+i\gamma}{(\omega-\nu_n)^2+\gamma^2}[1-(1+y)^{-1/2}]\mathscr{E}_n/y\ , \quad (31)$$

where

$$y=\frac{2\gamma_{ab}}{\gamma}I_n\mathscr{L}(\omega-\nu_n)$$

and

$$\bar{N}=\frac{1}{\mathscr{N}}\sum_{j=1}^{M}\int_0^{L_j}dz\ |U_n(z)|^2N(z,t)\ .$$

Except for the single-mode case, Eq. (30) is not amenable to analytical manipulation. To overcome this difficulty, we resort to the perturbation theory and evaluate the polarization to third order. To accomplish this, one needs to evaluate the off-diagonal matrix element $\rho_{ab}(z,t)$ to third order. The first- and third-order contributions are derived in several texts[10] and can be written as

$$\rho_{ab}^{(1)}=-\frac{i}{2}\frac{\wp}{\hbar}N(z,t)\sum_m \mathscr{E}_m(t)\exp[-i(\nu_m t+\phi_m)]U_n(z)\mathscr{D}(\omega-\nu_m)\ , \quad (32)$$

and

$$\rho_{ab}^{(3)}=i\left[\frac{\wp}{2\hbar}\right]^3 N(z,t)\sum_\mu\sum_\rho\sum_\sigma \mathscr{E}_\mu\mathscr{E}_\rho\mathscr{E}_\sigma U_\mu U_\rho^* U_\sigma \exp[-i(\nu_\mu-\nu_\rho+\nu_\sigma)t-i(\phi_\mu-\phi_\rho+\phi_\sigma)]$$
$$\times\mathscr{D}(\omega-\nu_\mu+\nu_\rho-\nu_\sigma)[\mathscr{D}_a(\nu_\rho-\nu_\sigma)+\mathscr{D}_b(\nu_\rho-\nu_\sigma)][\mathscr{D}(\omega-\nu_\sigma)+\mathscr{D}(\nu_\rho-\omega)]\ . \quad (33)$$

Substituting Eqs. (32) and (33) in eq. (26), one obtains the required expression for the polarization. In terms of the complex field amplitude $E_n(t)=\mathscr{E}_n \exp[-i\phi_n(t)]$, the equation of motion can be written as

$$\dot{E}_n(t)=i(\nu_n-\Omega_n+i\Gamma_n)E_n(t)+\sum_m M_{nm}E_m(t)\exp[i(\nu_n-\nu_m)t]\sum_\mu\sum_\rho\sum_\sigma T_{n\mu\rho\sigma}E_\mu E_\rho^* E_\sigma \exp[-i(\nu_\mu-\nu_\rho+\nu_\sigma-\nu_n)t] , \tag{34}$$

where

$$M_{nm}=\frac{\nu_n \wp^2}{2\epsilon_0\hbar\mathscr{N}}\mathscr{D}(\omega-\nu_m)W_{nm} , \tag{35}$$

and

$$T_{n\mu\rho\sigma}=\frac{\nu_n\wp^4}{2\epsilon_0\hbar^3\mathscr{N}}\mathscr{D}(\omega-\nu_\mu+\nu_\rho-\nu_\sigma)[\mathscr{D}_a(\nu_\rho-\nu_\sigma)+\mathscr{D}_b(\nu_\rho-\nu_\sigma)][\mathscr{D}(\omega-\nu_\sigma)+\mathscr{D}(\nu_\rho-\omega)]S_{n\mu\rho\sigma} , \tag{36}$$

$$W_{n\sigma}=\tfrac{1}{2}\sum_{j=1}^{M}B_n^*(j)B_\sigma(j)\int_0^{L_j}dz\,N(z)\cos[(k_n-k_\sigma)z+\tfrac{1}{2}\xi_n(j)-\tfrac{1}{2}\xi_\sigma(j)] , \tag{37}$$

and

$$\begin{aligned}S_{n\mu\rho\sigma}=\tfrac{1}{8}\sum_{j=1}^{M}B_n^*(j)B_\mu(j)B_\rho^*(j)B_\sigma(j)\int_0^{L_j}dz\,N(z)&\cos\{(k_n-k_\mu-k_\rho+k_\sigma)z+\tfrac{1}{2}[\xi_n(j)-\xi_\mu(j)-\xi_\rho(j)+\xi_\sigma(j)]\}\\&+\cos\{(k_n-k_\mu+k_\rho-k_\sigma)z+\tfrac{1}{2}[\xi_n(j)-\xi_\mu(j)+\xi_\rho(j)-\xi_\sigma(j)]\}\\&+\cos\{(k_n+k_\mu-k_\rho-k_\sigma)z+\tfrac{1}{2}[\xi_n(j)+\xi_\mu(j)-\xi_\rho(j)-\xi_\sigma(j)]\} .\end{aligned} \tag{38}$$

A comparison of Eq. (34) with the corresponding equation for the standard single-resonator laser reveals the similarity between the two. Equation (34) indicates that the composite laser system has much in common with the single-resonator laser. Of course, the mode structure of the effective laser is different from that of a single cavity. For example, the modes are not equally spaced in the frequency domain and it is possible to have modes that are very close together such that frequency locking can easily take place between these modes.

The coupling coefficient $M_{n\sigma}$ and the saturation coefficient $T_{n\mu\rho\sigma}$ are the sum of the contribution from all subresonators. If any of the subresonators is empty, then there is no contribution from that subresonator since $N(z)=0$ for that section. It is obvious from Eq. (37) and Eq. (38) that the coupling coefficients depend on the population inversion density $N(z)$. This is in contrast to Spencer-Lamb theory of coupled lasers where the coupling coefficients are independent of the population inversion density. It is important to note that according to Eq. (37), the coupling coefficient $M_{n\sigma}$ vanishes if the population inversion density $N(z)$ is uniform and fills the whole composite resonator. This is true since the eigenmodes of the composite resonator are orthogonal.

An important aspect of the composite laser system is the fact that if two or more subresonators have nearly equal lengths, then two or more eigenmodes will be very close to each other in the frequency domain. This implies that it is likely that these modes will lock together and oscillate as a single mode. This phenomenon will be discussed in detail in the next section. Hence, in Eq. (34), terms oscillating at the difference frequency $(\nu_n-\nu_\sigma)$ cannot be neglected as is the usual practice in single-resonator laser theory.[10]

V. TWO-MODE FREQUENCY LOCKING

Assuming that only two modes are oscillating, the equations of motion as represented by Eq. (34) become

$$\frac{d}{d\tau}\widetilde{E}_1(\tau)-i(\nu-\Omega_1+i\Gamma_1)\widetilde{E}_1=\alpha_{11}(1-i\xi)\widetilde{E}_1+\alpha_{12}(1-i\xi)\widetilde{E}_2-\sum_{\mu=1}^{2}\sum_{\rho=1}^{2}\sum_{\sigma=1}^{2}R_{n\mu\rho\sigma}\mathscr{L}(\xi)(1-i\xi)\widetilde{E}_\mu\widetilde{E}_\rho^*\widetilde{E}_\sigma , \tag{39}$$

and

$$\frac{d}{d\tau}\widetilde{E}_2(\tau)-i(\nu-\Omega_2+i\Gamma_2)\widetilde{E}_2=\alpha_{22}(1-i\xi)\widetilde{E}_2+\alpha_{21}(1-i\xi)\widetilde{E}_1-\sum_{\mu=1}^{2}\sum_{\rho=1}^{2}\sum_{\sigma=1}^{2}R_{n\mu\rho\sigma}\mathscr{L}(\xi)(1-i\xi)\widetilde{E}_\mu\widetilde{E}_\rho^*\widetilde{E}_\sigma , \tag{40}$$

where we have assumed that the two modes have close frequencies and replaced ν_1 and ν_2 by ν such that the difference in frequency is absorbed in $\dot{\phi}(t)$. The relative laser frequency detuning $\xi=(\omega-\nu)/\gamma$, where ω is the atomic line center frequency.

Also, we have

$$\alpha_{n\sigma}=\frac{\nu\wp^2}{2\epsilon_0\hbar\gamma}\left[\frac{W_{n\sigma}}{\mathscr{N}}\right]\mathscr{L}(\xi) \quad (41)$$

and

$$R_{n\mu\rho\sigma}=\alpha_{n\sigma}\left[\frac{S_{n\mu\rho\sigma}}{W_{n\sigma}}\right] . \quad (42)$$

For convenience, the electric field, frequency, and time variables have been scaled such that they are dimensionless. The scaling is as follows:

$$\tilde{E}=\frac{\wp}{\hbar(\gamma_a\gamma_b)^{1/2}}E , \quad (43)$$

and

$$\tau=\gamma t . \quad (44)$$

All frequency terms are measured in units of γ.

An inspection of Eqs. (38) and (39) shows that $\alpha_{12}=\alpha_{21}$ and that terms with similar indices are equal regardless of the permutation of the indices. Hence, on separating the real and imaginary terms in Eqs. (39) and (40), we end up with

$$\frac{d\mathscr{E}_1}{d\tau}=(\alpha_{11}-\Gamma_1)\mathscr{E}_1-(R_{1111}\mathscr{E}_1^2+2R_{1122}\mathscr{E}_2^2)\mathscr{L}(\xi)\mathscr{E}_1 +\alpha_{12}(\cos\phi-\xi\sin\phi)\mathscr{E}_2 , \quad (45)$$

$$\frac{d\mathscr{E}_2}{d\tau}=(\alpha_{22}-\Gamma_2)\mathscr{E}_2-(R_{2222}\mathscr{E}_2^2+2R_{1122}\mathscr{E}_1^2)\mathscr{L}(\xi)\mathscr{E}_2 +\alpha_{12}(\cos\phi+\xi\sin\phi)\mathscr{E}_1 , \quad (46)$$

$$\frac{d\phi_1}{d\tau}=\nu-\Omega_1-\xi\alpha_{11}+(R_{1111}\mathscr{E}_1^2+2R_{1122}\mathscr{E}_2^2)\mathscr{L}(\xi)\mathscr{E}_2 -\alpha_{12}(\sin\phi+\xi\cos\phi)\mathscr{E}_2/\mathscr{E}_1 , \quad (47)$$

$$\frac{d}{d\tau}\phi_2=\nu-\Omega_2-\xi\alpha_{22}+(R_{2222}\mathscr{E}_2^2+2R_{1122}\mathscr{E}_1^2)\mathscr{L}(\xi)\xi +\alpha_{12}(\sin\phi-\xi\cos\phi)\mathscr{E}_1/\mathscr{E}_2 , \quad (48)$$

where $\phi=\phi_1-\phi_2$, and we have neglected terms such as $\mathscr{E}_1\mathscr{E}_2^2e^{i\phi}$ since these are small compared to the first-order terms like $\mathscr{E}_1e^{i\phi}$, because we are assuming low-intensity fields in accordance with the assumptions of the third-order theory.

In Eq. (45), for example, $(\alpha_{11}-\Gamma_1)$ is the net gain for the first mode and should be positive for sustained oscillations. R_{1111} is the self-saturation term while R_{1122} is the cross-saturation term. α_{12} is the phase-coupling term that can lead to frequency locking between the two modes.

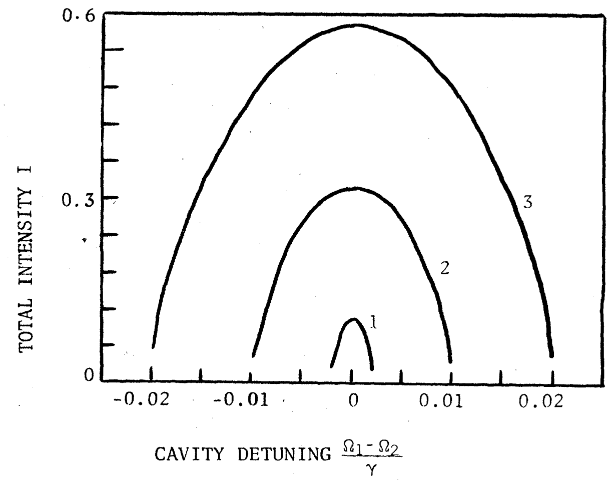

FIG. 4. Total intensity $I=\mathscr{E}_1^2+\mathscr{E}_2^2+2\mathscr{E}_1\mathscr{E}_2\cos\phi$ vs the two cavities mismatch $(\Omega_1-\Omega_2)/\gamma$.

Since we are interested in the steady-state operation of the composite laser system, the time derivatives are set to zero and the four simultaneous nonlinear equations are solved for the unknowns $\mathscr{E}_1$, $\mathscr{E}_2$, ξ, and ϕ. For these solutions, the two modes are locked and oscillate at the common frequency ν or, equivalently, ξ. To solve the nonlinear simultaneous equations we used the Newton-Raphson method for a system of equations. The solutions were also checked for stability using the Hurwitz criterion for stability.[13]

In Fig. 4 we show the total intensity $(\mathscr{E}_1^2+\mathscr{E}_2^2+2\mathscr{E}_1\mathscr{E}_2\cos\phi)$ as a function of the relative detuning of the two passive modes; $(\Omega_1-\Omega_2)/\gamma$. The different curves

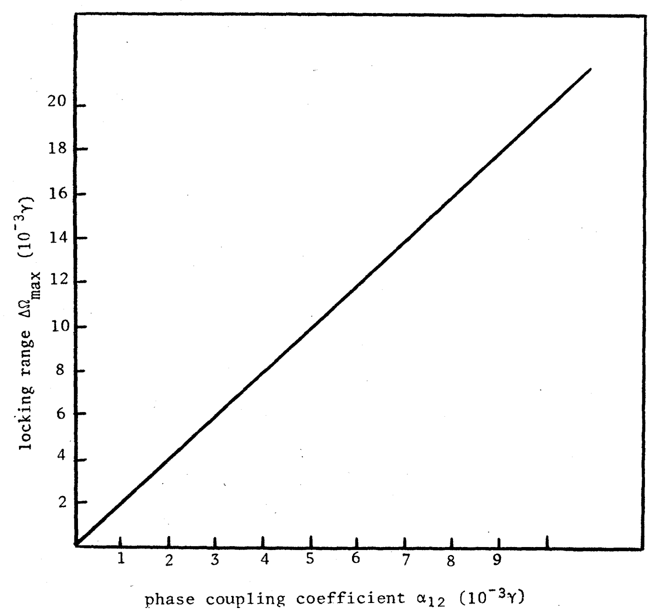

FIG. 5. Maximum locking range $\Delta\Omega_{max}$ vs the coupling coefficient α_{12}.

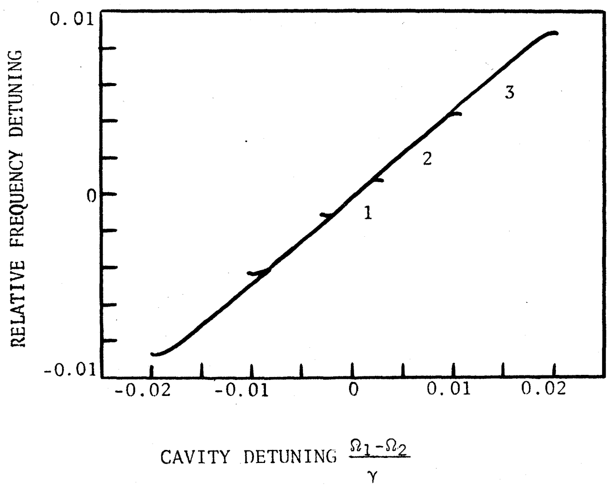

FIG. 6. Relative frequency detuning ξ vs the two cavities mismatch $(\Omega_1-\Omega_2)/\gamma$.

correspond to different values for the phase-coupling coefficient α_{12}. It is apparent that the larger the phase coupling is, the wider the locking range $\Delta\Omega_{\max} = |(\Omega_1-\Omega_2)/\gamma|_{\max}$ is. As a matter of fact, $\Delta\Omega_{\max}$ varies linearly as a function of α_{12}, as indicated in Fig. 5. The frequency of operation for the two locked modes, as a function of the modes mismatch $\Delta\Omega$, is shown in Fig. 6, for the same values of the parameters indicated in Fig. 4. The locked phase difference ϕ is shown in Fig. 7.

The phase-coupling term α_{12}, is the term that may lead to frequency locking of the two modes, as can be seen from inspecting Eqs. (45)–(48). According to Eqs. (41) and (37), the phase-coupling term vanishes if the population inversion density $N(z)$ is uniformally distributed throughout the whole composite system. This follows from the orthogonality of the modes of the composite sys-

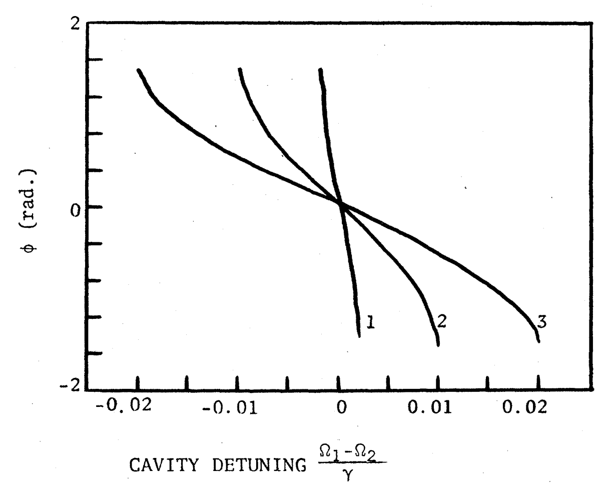

FIG. 7. Relative phase ϕ vs the two cavities mismatch $(\Omega_1-\Omega_2)/\gamma$.

tem. In general, the strength of phase coupling α_{12} depends on the location of the active mediums in the different subresonators for the two-resonator system, it is not difficult to show that the upper limit on the value of α_{12} is given by

$$\alpha_{12} < \frac{\nu \wp^2}{4\epsilon_0 \hbar \gamma} \mathscr{L}(\xi) \left[\bar{N}(2) + \bar{N}(1) \left[\frac{1-r(1)}{1+r(1)} \right] \right] \tag{49}$$

where

$$\bar{N}(j) = \int_0^{L_j} N(z) dz \ .$$

VI. THE DECOUPLED APPROXIMATION

To gain same physical insight into the locking problem we try to cast the equations of motion into a more familiar form. Subtracting Eq. (48) from Eq. (47) the result is

$$\frac{d\phi}{d\tau} = d + l_c \cos\phi + l_s \sin\phi \ , \tag{50}$$

where the unlocked beat frequency

$$d = \Omega_2 - \Omega_1 + \xi(\alpha_{22} - \alpha_{11}) + \xi \mathscr{L}(\xi)(R_{1111}\mathscr{E}_1^2 + 2R_{1122}\mathscr{E}_2^2 - R_{2222}\mathscr{E}_2^2 - 2R_{1122}\mathscr{E}_1^2), \tag{51}$$

and the locking coefficients

$$l_c = \alpha_{12}\xi \left[\frac{\mathscr{E}_1}{\mathscr{E}_2} - \frac{\mathscr{E}_2}{\mathscr{E}_1} \right] , \tag{52}$$

and

$$l_s = -\alpha_{12} \frac{\mathscr{E}_1}{\mathscr{E}_2} + \frac{\mathscr{E}_2}{\mathscr{E}_1} \ . \tag{53}$$

Equation (50) can be rewritten as

$$\frac{d\phi}{d\tau} = d + l \sin(\phi - \phi_0) \ , \tag{54}$$

where

$$l = l_s(1 + l_c^2/l_s^2)^{1/2} \quad \text{and} \quad \phi_0 = \tan^{-1}(l_c/l_s) \ .$$

Equation (54) is identical to the phase-coupling equation of motion for ring lasers.[10] If we assume that d, l_c, and l_s do not change appreciably, they can be considered constants. This amounts to the so called "decoupled approximation," well known in connection with ring lasers.

An inspection of Eq. (54) indicates that there are three possible cases.

(i) If $d > l$, then locking cannot take place and the two modes oscillate at different frequencies. The average beat frequency between the two modes can be shown to be[10]

$$\overline{\Delta\nu} = d(1 - l^2/d^2)^{1/2} \ . \tag{55}$$

(ii) $d = l$. This defines the locking limits. Since

$$l = l_s \left[1 + \xi^2 \frac{\mathscr{E}_2^2 - \mathscr{E}_1^2}{\mathscr{E}_2^2 + \mathscr{E}_1^2} \right] \simeq l_s \ ,$$

then setting $d = l_s$, one obtains the following relation for the maximum locking range in terms of the composite resonator frequency difference,

$$\Delta\Omega_{\max}=\xi(\alpha_{22}-\alpha_{11})-\xi\mathscr{L}(\xi)\mathscr{E}_2^2(R_{2222}-2R_{1122})+\xi\mathscr{L}(\xi)\mathscr{E}_1^2(R_{1111}-2R_{1122})+\alpha_{12}\left[\frac{\mathscr{E}_2}{\mathscr{E}_1}+\frac{\mathscr{E}_1}{\mathscr{E}_2}\right], \quad (56)$$

which shows that $\Delta\Omega_{\max}$ is proportional to the phase coupling coefficient α_{12}, in agreement with the numerical solutions of Eqs. (44)–(47).

(iii) If $d<l$, then the two modes are locked. Solving Eqs. (54) for steady state under the "decoupled approximation," one obtains[10]

$$\phi=\begin{cases}\phi_0-\sin^{-1}(d/l)\ ,\\ \phi_0+\pi+\sin^{-2}(d/l)\ .\end{cases} \quad (57)$$

A stability analysis of Eq. (54) shows that the solution is stable provided that

$$l\cos(\phi-\phi_0)<0\ . \quad (58)$$

VII. ALTERNATIVE TYPES OF COUPLING

The series coupling of resonators is a relatively simple form of coupling resonators. It is convenient for many applications where the laser output power is not of primary concern. However, when one is interested in obtaining high-power radiation by adding the output from a number of lasers, other forms of coupling should be used. A general form of parallel coupling is shown in Fig. 8. Assume that the coupling mirror at the center has a transmittance coefficient t and reflectance coefficient r for reflection from the upper face and $\bar{r}$ for reflection from the bottom face. Following a procedure similar to the one adopted for series coupling, one can relate the counterpropagating waves in the four different resonators and solve the resulting coupled algebraic equations in terms of A^+ and A^-, for example. If one performs the calculations, the effective reflectance of the system just outside the mirror M_4 is

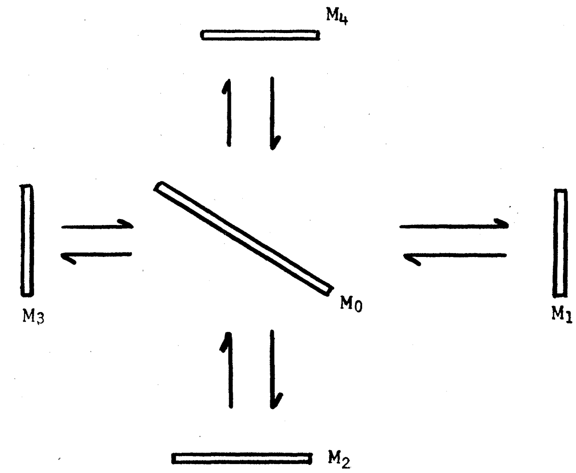

FIG. 8. Four coupled resonators in a star-shaped coupling. The coupling mirror is M_0. Each one of the mirrors can be a system of resonators or an ordinary reflector.

$$R_4=\frac{r(4)+[t^2(4)-r(4)\bar{r}(4)]R\exp(-2ikL_4)}{1-\bar{r}(4)R\exp(-2ikL_4)}\ , \quad (59)$$

where

$$R=r^2(0)r(1)\exp(-2ikL_1)+t^2(0)r(2)\exp(-2ikL_2)+R' \quad (60)$$

and

$$R'=\frac{[t(0)r(0)r(1)\exp(-2ikL_1)+t(0)\bar{r}(0)r(2)\exp(-2ikL_2)]^2}{\frac{1}{r(3)}\exp(2ikL_3)-t^2(0)r(1)\exp(-2ikL_1)-r^2(0)r(2)\exp(-2ikL_2)}\ . \quad (61)$$

The length of the different cavities are measured with respect to the center of the central coupling mirror. Of course, each of the mirrors in the system can be a subsystem of coupled resonators itself, in which case the reflectance coefficients should be replaced by the effective reflectance coefficient of the subsystem. The eigenfrequencies are the roots of denominator in Eq. (59).

APPENDIX

Here we derive the dispersion relation of Eq. (13). Suppose we have calculated the effective reflectance of the system at the point z_0, just to the left of the reflecting surface $r(0)$. According to the recursion relation (6), $R(0)$ is related to $R(1)$ via

$$R(0)=\frac{r(0)+[t^2(0)-r(0)\bar{r}(0)]R(1)\exp(-2ikL_1)}{1-\bar{r}(0)R(1)\exp(-2ikL_1)}\ . \quad (A1)$$

Since we are interested in bound eigenmodes that are localized inside the composite resonator system, we require that the external wave $A^+(z)(z<0)$ should be zero. However, by definition, $R(0)=A^-(0)/A^+(0)$, which means that

$$\bar{r}(0)R(1)\exp(-2ikL_1)=1\ . \quad (A2)$$

However, from Eq. (6)

$$R(1)=\frac{r(1)+[t^2(1)-r(1)\bar{r}(1)]R(2)\exp(-2ikL_2)}{1-\bar{r}(1)R(2)\exp(-2ikL_2)}, \tag{A3}$$

and

$$\bar{R}(1)=\frac{\bar{r}(1)+[t^2(1)-\bar{r}(1)r(1)]\bar{r}(0)\exp(-2ikL_2)}{1-r(1)\bar{r}(0)\exp(-2ikL_1)}. \tag{A4}$$

From Eqs. (A2)—(A4) one can obtain

$$\bar{R}(1)R(2)\exp(-2ikL_2)=1, \tag{A5}$$

where the bar denotes the effective reflectance with respect to the right side. Carrying on the same procedure, one can show by deduction the general dispersion relation

$$\bar{R}(j-1)R(j)\exp(-2ikL_j)=1. \tag{A6}$$

The eigenvalues of the system are those values of k which satisfy Eq. (A6). In the general case where losses are taken into account, k will be a complex quantity where the imaginary component corresponds to the net gain required to balance the net losses in the system. These values are useful in the calculation of the threshold intensity values for oscillation to take place. This can be seen by inspecting Eq. (A6) or Eq. (A2). Representing unity by the equivalent expression of $\exp(-i2\pi n)$, where n is an integer, one can see that Eq. (A6) results in two equations with two unknowns; the resonance wavelength and the overall losses of the system. If each segment of the structure has a different amount of (nonresonant) losses, then these can be incorporated in the analysis by noting that the phase length of each segment (kL_j) must be multiplied by the refractive index of the medium in that segment. Hence, one can introduce the losses by taking the refractive index to be a complex quantity, with the imaginary component representing the losses.

[1]D. A. Kleinman and P. P. Kislink, Bell Syst. Tech. J. **41**, 453 (1962).
[2]See, for example, the review article by P. W. Smith, Proc. IEEE **60**, 422 (1972), and the references mentioned therein.
[3]W. T. Tsang, N. A. Olsson, and R. A. Logan, Appl. Phys. Lett. **42**, 650 (1983).
[4]K. J. Ebeling and L. A. Coldren, J. Appl. Phys. **54**, 2962 (1983).
[5]S. A. Shakir and W. Chow, Opt. Lett. **9**, 202 (1984).
[6]Y. C. See, Shekhar Guha, and Joel Falk, Appl. Opt. **20**, 1211 (1981).
[7]M. B. Spencer and W. E. Lamb, Jr., Phys. Rev. A **5**, 893 (1972).
[8]E. M. Philipp-Rutz, Appl. Phys. Lett. **26**, 475 (1975); C. J. Buczek, R. J. Freiberg, and M. L. Skolnick, Proc. IEEE **61**, 1411 (1973); W. Chow, Opt. Lett. **7**, 417 (1982); G. Palma, in Proceedings of the 13th Colloquium on Quantum Electronics, Snowbird, Utah, 1983 (unpublished).
[9]R. Lang and M. O. Scully, Phys. Rev. A **7**, 1788 (1973).
[10]M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, London, 1974).
[11]P. H. Berning, in *Physics of Thin Films,* edited by G. Hass (Academic, New York, 1963), Vol. 1.
[12]S. A. Shakir and A. F. Turner, Appl. Phys. A **29**, 151 (1982).
[13]H. Haken, *Synergetics, An Introduction,* 2nd ed. (Springer, New York, 1978).