Stimulated recombination: A stochastic approach

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Stimulated recombination is examined as a stochastic process. The master equation, its stationary solution, and the fluctuation around the stationary solution are obtained. The existence of multiple solutions and its implication are discussed.

I. INTRODUCTION

Radiative recombination is a well-studied process for both atomic and molecular systems.[1–4] It is an important phenomenon, and the detailed knowledge of recombination for particular processes is essential for a basic understanding of the dynamics and the morphology of both planetary and stellar atmospheres. Of more recent interest is the possibility of population inversion by means of recombination to some of the excited states of H- and He-like ions, which presumably could be an important factor in producing coherent x-ray radiation. Dynamic molecular plasmas are also quite strongly affected by recombination rates of individual components, some of which are now being studied in isolation.[5] The time dependence and certain aspects of stimulated recombination treated quantum mechanically have been investigated by us.[6,7]

In this work, we have made a theoretical study of the stimulated recombination and the photoionization (or photodissociation) processes in an isolated macroscopic system. The study is based entirely on statistical mechanics of stochastic processes. The assumption of a closed system reduces the problem to the study of a process of a single-component stochastic variable. This in turn allows one to proceed further in analytical terms, without detailed numerical studies. We take full advantage of this and obtain some interesting and useful results.

The plan of this paper is as follows. In Sec. II, we treat the stimulated photorecombination or photoabsorption (or dual process) as a single-step stochastic process for the number of photons, which is its stochastic variable. The master equation is set up here[8] and the assumptions necessary for its derivation are explained. Sections III and IV are devoted to the derivation of a rate equation, along with that for the Fokker-Planck equation. We examine the steady-state solutions of this equation and obtain multiple stationary states. Straightforward stability analysis allows one then to individuate a single stable stationary state. One obtains, therefore, the result that an increase of photons is possible for such systems. Section V addresses the question of fluctuations. A Taylor expansion is made about the stable stationary state, and the relaxation time near the stable stationary state is obtained along with the autocorrelation function. Section VI is devoted to the rate equation (or the macroscopic equation). We also investigate the possibility of calculating fluctuations by means of the equations for the first and the second moments. The coupled equation that can be utilized for such a study is derived, noting the dependence on approximations needed to obtain them. The paper concludes with some remarks on future work as well as a summary of the result of this paper.

We stress the fact that stimulated processes from the continuum are in an early stage of research. The dynamical description by means of detailed knowledge of individual systems in regions where perturbation theory might fail (such as resonances in the continuum along with very high field strengths), is not yet completely understood and therefore cannot obviously be incorporated in a treatment as this one. However, for moderate field strengths, the treatment followed in this paper should be adequate to characterize some of the macroscopic features of the system.

It is perhaps unnecessary to justify the use of methods of statistical mechanics in problems of this kind. The rate of the elementary processes has to be calculated or found by measurements, if possible. The statistical approach can then shed light on some of the macroscopic features or for a large collection of elementary constituents. The techniques employed in this paper are standard and in some form or other have been utilized for a large variety of problems in recent years.

Getting back to the problem of our interest, we specify the processes that are treated in this paper; broadly, they can be of two kinds: atomic or molecular. For the atomic case we consider

$$e^- + X^+ + h\nu \rightarrow X + 2h\nu$$

and (1)

$$X + h\nu \rightarrow e^- + X^+ \ .$$

The first is the stimulated atomic recombination, while the second is simply photoionization.

For the molecular case, we consider

$$X^- + Y^+ + h\nu \rightarrow XY + 2h\nu \ ,$$
$$XY + h\nu \rightarrow X + Y \ , \quad (2)$$

where X and Y are atoms (or molecules) which combine to form a stable molecule XY. Our treatment below will be valid also for

$$X + Y + h\nu \rightarrow XY + 2h\nu \ , \quad (3)$$

i.e., for neutral molecules X and Y, provided one has equal concentration of X and Y molecules. This is because, as we shall see later, we invoke overall charge neutrality for the first two cases. This condition could easily be relaxed, but we limit ourselves to this case for simplicity.

We treat stimulated recombination because it is now possible to envisage powerful lasers with photon frequencies at which single photon ionization (or dissociation) can occur. As examples, we may think of the higher harmonics of ruby and Nd^{2+}-glass lasers, as well as, the excimer and the vacuum ultraviolet (vuv) lasers. Once we have a reasonably large intensity, it is quite possible to have a pulse of such lasers in a small volume of plasma and confine the photons in that volume. The resulting system is what is pertinent to our study.

II. MASTER EQUATION

The four species of matter in which we are interested will be called generically electron, ion, atom, and photon. Processes (1) and (2) or, for that matter, any other pairs of reactions that describe similar processes, are equivalent.

Since we wish to describe the process as a stochastic process, we take advantage of the notion of one-step process (the "generalized random walk"). Before we get involved in the details of this model, first let us enumerate the assumption that we shall make.

(i) Charge neutrality: The system is considered to be neutral in charge. This means that the total number of electrons n_e is equal to that of the ions a^+. The assumption is valid in general for all stable plasma systems, thus, $n_e = a^+$.

(ii) Conservation of particles: There are two conservation laws. They are

$$n_e + a = A \quad (4)$$

and

$$n_e + n = B \ , \quad (5)$$

where n and a are the number of photons and atoms, respectively (A and B are constants). Charge neutrality and the constraints imposed by the number conservation laws thus reduce the problem to that of one species, since the three others can be determined from the knowledge of the one that we strive to determine in the following.

Let us consider the number of photons n to be the undetermined variable. Furthermore, we shall consider this as a stochastic variable dependent on time. The properties of $n(t)$ are to be determined from the master equation. The motivation of utilizing the master equation for processes involving photons are already well known and need not be repeated. Suffice it to say that the processes to consider are single photon process, i.e., a single photon is either absorbed or emitted in an elementary process and therefore this is what is known as a one-step process.

We utilize the notation $p_n(t)$ as the distribution function for n photons, all of the same mode. While in general, we could consider a system in which photons are continuously pumped into the system and in which a loss mechanism exists, we refrain from imposing both of them to keep the problem as transparent as possible (discussion of the general case is given in the Appendix). Thus, the system we consider consists of a pulsed beam of photons, which are introduced and subsequently kept in the system by means of suitable mirrors. The geometry could either be of an optical oscillator (i.e., inside a normal laser system) or a ring laser in which photons are recycled.

In order to write down the master equations for $p_n(t)$, we need the rate of stimulated recombination and that of photoionization. They are given, respectively, as $\lambda(n+1)n_e^2$ and αna, where λ is the recombination coefficient and α is that for photoionization. Notice that for the recombination rate we have utilized the fact that the total number of ions and electrons are equal.

The master equation is now easy to write down. It is given by

$$\frac{dp_n}{dt} = \lambda n n_e^2[n-1]p_{n-1} + \alpha(n+1)a[n+1]p_{n+1} - \lambda(n+1)n_e^2[n]p_n - \alpha n a[n]p_n \ . \quad (6)$$

We utilize the square brackets to signify the functional dependence of n_e and a on the number of photons. Utilizing the conservation laws, Eq. (6) can be rewritten as:

$$\frac{dp_n}{dt} = \lambda n(B-n+1)^2 p_{n-1} + \alpha(n+1)(n+1+A-B)p_{n+1} - \lambda(n+1)(B-n)^2 p_n - \alpha n(n+A-B)p_n \ . \quad (7)$$

Following Van Kampen,[8] we further rewrite Eq. (7) in the following compact form

$$\frac{dp_n}{dt} = (\mathbf{E}-1)r_n p_n + (\mathbf{E}^{-1}-1)g_n p_n \ , \quad (8)$$

where

$$r_n \equiv \alpha n(n+A-B) \ ,$$
$$g_n \equiv \lambda(n+1)(B-n)^2 \ , \quad (9)$$

and $\mathbf{E}$ are operators which have the property

$$\mathbf{E}h_n = h_{n+1}$$

and (10)

$$\mathbf{E}^{-1}h_n = h_{n-1}$$

for an arbitrary function h_n.

The factor $n+1$ in g_n arises from two distinct processes: stimulated and spontaneous emission. Since we consider a relatively large density of photons, a particular mode which is in resonance with the recombining transition, the effect of spontaneous emission is extremely small and can be neglected. We do this simply by dropping 1 from the expression for g_n (the modification that arises from spontaneous emission is in the Appendix) in various practical calculations.

By writing down the master equations above for the variable n, we have automatically made the assumption of a one-step Markov process. It is worthwhile to discuss when this is valid. Of course, the one-step Markov process is a correct description for our stochastic variable if (1) the single-step processes in question can be described by time-independent rates, and (2) if these single-step processes are the dominant ones. Since the system we want to describe is dominated by single photon creation or annihilation processes, we can assume that there exists a time scale where the time-independent rate is a valid description. Furthermore, higher-order processes such as multiphoton creation or annihilation would be of interest only when the field strengths are so high that even when single photon processes exist, the contributions of higher-order terms of the perturbation theory would be of some importance. We eschew such intensities from our consideration.

III. STATIONARY SOLUTION

The coefficients g_n and r_n of the master equation given in Eq. (8), are nonlinear with respect to the photon number n. This means that the usual methods of solving the master equation with, at most, linear coefficients, do not apply. It is always possible to numerically solve the equation by matrix inversion. However, since we wish to analyze the solutions as an explicit function of n, such methods are not very helpful. Thus, we are constrained to make approximations to solve it. Before we embark on attempts to solve the master equation, it is important to obtain the stationary solution as follows.

Consider the expressions for g_n and r_n, i.e., expression (9). These have to be positive or zero. Two cases arise. For $A>B$, one must restrict n between 0 and B. In other words, r_n becomes zero for $n=0$ and the same holds true for g_n for $n=B$. The stochastic variable n obeys natural boundary conditions. The physical meaning is obvious: When the number of photons drops to zero, there can be no further loss, for the first case, while for the second, the limit $n=B$ corresponds to the extinction of the electrons, rendering further recombination impossible.

The stationary solution of Eq. (8) can now be written down. It is

$$p_n^{\rm st}=p_0^s\frac{g_0g_1\cdots g_{n-1}}{r_1r_2\cdots r_n}\ . \tag{11}$$

Utilizing Eq. (9), we have, after a little algebra,

$$p_n^{\rm st}=p_0^s\left[\frac{\lambda}{\alpha}\right]^n\frac{\Gamma(1+A-B)\Gamma^2(B+1)}{\Gamma(n+1+A-B)\Gamma^2(B-n+1)}\ , \tag{12}$$

where we have utilized the notation of gamma functions, and which for all cases of interest, reduces simply to factorials. The constant p_0^s is obtained from normalization, i.e.,

$$\sum_{n=0}^{B}p_n^s=1\ .$$

Now, $p_n^s=0$ for all $n>B$. This allows one to write down p_0^s in a compact form, by performing an infinite summation. One obtains

$$(p_0^s)^{-1}={}_3F_1\left[-B,-B,1;A-B;\frac{\lambda}{\alpha}\right]. \tag{13}$$

Notice that this is a polynomial, since B is a positive integer. Equations (12) and (13), thus, completely determine the stationary distribution of the photon number. Let us now consider the case $B>A$. The range acceptable for p is now changed from $0<n<B$ to $B-A<n<B$. The range $0<n<B-A$ now becomes inaccessable. The stationary solution for Eq. (8) is now different and given by

$$p_n^{\rm st}=p_{B-A}^{\rm st}\frac{g_{B-A}g_{B-A+1}\cdots g_{n-1}}{r_{B-A+1}r_{B-A+2}\cdots r_n}\ . \tag{14}$$

One obtains

$$p_n^{\rm st}=p_{B-A}^{\rm st}\left[\frac{\lambda}{\alpha}\right]^{n+A-B}\frac{(A!)^2}{(n+A-B)![(B-n)!]^2}\ , \tag{15}$$

$$p_{B-A}^{\rm st}=1\Bigg/\left[1+\sum_{n=B-A+1}^{B}\left[\frac{\lambda}{\alpha}\right]^{n+A-B}\frac{(A!)^2}{(n+A-B)![(B-n)!]^2}\right]. \tag{16}$$

Expressions (12), (13), (15), and (16) define the stationary solutions for our problem. With these expressions, one can calculate all the stationary properties of the system, such as the mean photon number, the variance, and the higher moments.

IV. MACROSCOPIC EQUATION AND PHOTON GAIN

We are interested not only in the stationary distribution, but also in the fluctuations. Thus, one needs to actually solve the master equation given by Eq. (8). Since the coefficients are nonlinear with respect to n, this poses a formidable problem. On the other hand, we may only be interested in values of $n\gg 1$, and the coefficients in the master equations are smooth functions of n if we imagine n to be a continuous parameter. In that case, we can transform the function $p_n(t)$ to a function $p(n,t)$. The operators $\mathbf{E}$ and $\mathbf{E}^{-1}$ are expanded in a Taylor series, and we obtain, after a little algebra, the Fokker-Planck equation:

$$\frac{\partial p_n}{\partial t}=\frac{\partial}{\partial n}[\alpha n(n+A-B)-\lambda(n+1)(B-n)^2]p(n,t)+\frac{1}{2}\frac{\partial^2}{\partial n^2}[\alpha n(n+A-B)+\lambda(n+1)(B-n)^2]p(n,t)\ . \tag{17}$$

Clearly, all the higher-order derivates have not been kept in this equation. This equation serves two purposes for us. Firstly, we can write down the macroscopic equation for n:

$$\frac{dn}{dt}=-\alpha n(n+A-B)^2+\lambda(n+1)(B-n)^2\ . \tag{18}$$

This equation is, of course, the same as a phenomenological rate equation. We proceed now to examine the equation closely, which is given by

$$\alpha n_s(n_s+A-B)=\lambda(n_s+1)(B-n_s^2)\ . \tag{19}$$

Setting $\alpha/\lambda=\beta$, one has

$$n_s^3+(1-2B-\beta)n_s^2+[B^2-2B-\beta(A-B)]n_s+B^2=0\ . \tag{20}$$

This is a cubic equation in n_s, having three solutions. Since the term independent of n_s is positive, the nature of three roots can only be the following: (1) *two positive,* one negative; (2) two negative, *one positive;* (3) all negative; or (4) one negative and a pair of complex conjugate roots. Since the last two cases are unphysical, we are left with the first two cases, which we need to examine as a function of the parameters in Eq. (20). Much algebra is avoided, and no essential physics is lost, if we, at this point, make the assumption that $n_s \gg 1$. In any case, this corresponds to neglecting spontaneous emission in the phenomenological rate equation. It also corresponds to the limited range of utility of the Fokker-Planck equation. In that case, one can directly work with Eq. (19) and obtain

$$n_s^0=0\ ,$$
$$n_s^1=B+\frac{\beta}{2}-\left[\frac{\beta^2}{4}+\beta A\right]^{1/2}\ , \tag{21}$$
$$n_s^2=B+\frac{\beta}{2}+\left[\frac{\beta^2}{4}+\beta A\right]^{1/2}\ .$$

Now, the first solution ($n_s^0=0$) is clearly the nonphysical solution. It is the vestige of the negative solution of the cubic equation in the region of the parameters where n_s^1 and n_s^2 are positive. Now, in order for this to be true, one needs

$$B\geq\beta\left[\frac{A}{B}-1\right]\ . \tag{22}$$

When the inequality does not hold, n_s^1 becomes negative, and therefore, is no longer acceptable. Thus we can envision the possibility of a transition of one stationary state to multiple (two) stationary states by varying the parameters of the problem. Varying with A and B fixed, one may plot the solutions n_s^1 and n_s^2 as in Fig. 1(a). Another way of seeing the same solutions is to vary B with A fixed, as shown in Fig. 1(b).

We next consider the stability of the stationary solution. This is obtained by expanding n about the stationary solutions and then examining the growth or decay of the perturbation. A straightforward linear stability analysis shows that n_s^1 is the stable branch. It is to be noticed that n_s^1 exists as a physical solution, only over a fixed interval, i.e., for $0<A<B[(1/\beta)B+1]B$ for a given B. If, however, we consider B as the variable (with A fixed), the domain of existence of n_s^1 as a physical (and stable) stationary solution is the semi-infinite interval $B\geq\beta[(A/B)-1]$. Notice also that $n_s^0(=0)$, and not n_s^2, becomes the stable solution when n_s^1 becomes unphysical. Thus, what we see is that there exists a threshold value of B for which a stable solution appears, which is very similar to a laser threshold. We can define a gain factor

$$g=\frac{n_s^1}{n_0}=1+\frac{n_{e0}}{n_0}+\frac{\beta}{2n_0}-\left[\frac{\beta^2}{4n_0^2}+\frac{\beta(n_{e0}+a_0)}{n_0^2}\right]^{1/2}\ , \tag{23}$$

where n_0, n_{e0}, and a_0 are the initial number of photons, electrons, and neutral atoms, respectively.

There can be two cases of interest. For $n_{e0}\approx a_0$ (approximately half of the atoms are ionized in the plasma), we would have approximately

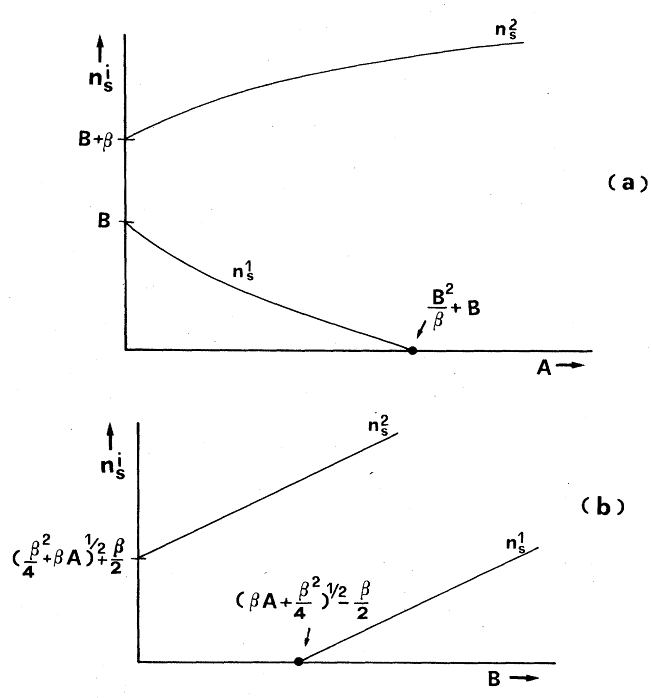

FIG. 1. (a) The two solutions n_s^1 and n_s^2 are shown as a function of A (see the text). At the value $n_s^1=(B^2/\beta)+\beta$, the solution n_s^1 ceases to exist, while n_s^2 continues to grow for all A. (b) The same as in (a) as a function of B. n_s^1 appears at the value of $B=[\beta A+(\beta^2/4)]^{1/2}-\beta/2$. Beyond this point, there exists both the solutions.

$$g \approx 1 + \frac{n_{e0}}{n_0} . \tag{24}$$

The increase in this case will require $n_{e0} \approx n_0$, for a twofold increase, etc.

On the other hand, for $n_{e0} \ll n_0$,

$$g \approx 1 + \frac{1}{n_0}[n_{e0} - (\beta a_0)^{1/2}] . \tag{25}$$

This condition appears to be less favorable than the previous case.

V. FLUCTUATIONS

In the previous section, the condition for a threshold for photon gain and also the value for the stationary state have been found. It is now straightforward to obtain the fluctuations around the stationary state. This is obtained by expanding the Fokker-Planck equation about the stable stationary state. Substituting $n = n_s + x$, one has by Taylor expression

$$\frac{\partial p(x,t)}{\partial t} = [2\lambda B^2 - \alpha n_s - 2\alpha(A-B) - 2\lambda B n_s]\frac{\partial}{\partial x}(xp) + [\alpha n_s(n_s + A - B) + \lambda n_s(B - n_s)^2]\frac{\partial^2 p}{\partial x^2} , \tag{26}$$

where n_s is given by Eq. (21).

Equation (26) may be solved with the initial value $p(x,0) = \delta(x - x_0)$, to give

$$p(x,t) = \left[\frac{2\pi}{L}(1 - e^{-2\Gamma t})\right]^{-1/2} \times \exp\left[\frac{-L(x - x_0 e^{-\Gamma t})}{2(1 - e^{-2\Gamma t})}\right] ,$$

where

$$\Gamma \equiv 2\lambda B^2 - \alpha n_s - 2\alpha(A-B) - 2\lambda B n_s \tag{27}$$

and

$$L \equiv \frac{\Gamma}{\alpha n_s(n_s + A - B) + \lambda n_s(B - n_s^2)} . \tag{28}$$

The expression for Γ has immediate physical significance. This is precisely the inverse of the relaxation time for the system near the stable stationary state. One shows with a little bit of algebra that the autocorrelation function around the stationary state is given by

$$\langle\langle n(t) n(t+\tau) \rangle\rangle = \langle\langle n^2 \rangle\rangle_s e^{-\Gamma t} \tag{29}$$

where $\langle\langle \ \rangle\rangle$ is the symbol for the autocorrelation function. The relaxation time is, therefore, an easily computable quantity given the photoionization and the recombination probabilities of the particular system.

VI. EQUATIONS FOR THE FIRST TWO MOMENTS

The master equation as well as the Fokker-Planck equation allows one to write down the rate equation and also a possible method of calculating the fluctuations. First, consider Eq. (7). One can write down a macroscopic equation simply

$$\frac{d}{dt}\langle n \rangle = \lambda \langle n \rangle^3 - (\alpha + 2B\lambda - \lambda)\langle n \rangle^2 + [\lambda B^2 - \alpha(A-B) - 2B\lambda]\langle n \rangle + \lambda B^2 . \tag{30}$$

This is the rate equation in which all the fluctuations are neglected.

The master equation, on the other hand, leads to

$$\frac{d}{dt}\langle n \rangle = \lambda \langle n^3 \rangle - (\alpha + 2\lambda B - \lambda)\langle n^2 \rangle + [\lambda B^2 - \alpha(A-B) - 2B\lambda]\langle n \rangle + \lambda B^2 \tag{31}$$

and

$$\frac{d}{dt}\langle n^2 \rangle = 2\lambda \langle n^4 \rangle - (4\lambda B + 2\alpha - 3\lambda)\langle n^3 \rangle + [2\lambda B^2 - 6\lambda B - 2\alpha(A-B) + \alpha + \lambda]\langle n^2 \rangle + (3\lambda B^2 - 2\lambda B + \alpha A - \alpha B)\langle n \rangle + \lambda B^2 . \tag{32}$$

Both Eqs. (31) and (32) are exact.

Obviously (31) and (32) form a nonclosed set of equations. If we simply make the approximation $\langle n^3 \rangle = \langle n \rangle^3$ and $\langle n^2 \rangle = \langle n \rangle^2$ in Eq. (31), we obtain the rate equation. A possible method for calculating the variance consists of setting $\langle n^3 \rangle \approx \langle n^2 \rangle \langle n \rangle$ and $\langle n^4 \rangle \approx \langle n^2 \rangle \langle n^2 \rangle$:

$$\frac{d}{dt}\langle n \rangle = \lambda \langle n \rangle \langle n^2 \rangle + \gamma \langle n^2 \rangle + \delta \langle n \rangle + \lambda B^2 , \tag{33}$$

$$\frac{d}{dt}\langle n^2 \rangle = 2\lambda \langle n^2 \rangle^2 + \eta \langle n \rangle \langle n^2 \rangle + \zeta \langle n^2 \rangle + \xi \langle n \rangle + \lambda B^2 . \tag{34}$$

One obtains a set of nonlinear coupled equations which may easily be numerically solved. The approximations for the third and the fourth moments are, in fact, reasonable in the region near the stationary state. It is not possible to solve (33) and (34) analytically. One can, however, write down the exact solution of the macroscopic equation. The decoupling approximations to obtain Eqs. (33) and (34) have, of course, no rigorous justification. They are, however, made in order to avoid a hierarchy of an open set of equations.

VII. CONCLUSION

In this paper, the problem of stimulated recombination (atomic or molecular) and photoionization (or photodissociation) has been studied from a stochastic viewpoint for an isolated system. The statistical point of view allows one to treat some of the macroscopic features of the system.

The master equation of the system has been set up as a one-step process for the photons. Such a creation-annihilation description has previously been employed in other photon processes but our treatment appears to be the first one concerning stimulated recombination and photoionization.

Using some of the standard techniques of statistical mechanics, we derive some results of interest. Firstly, the existence of multiple stationary states is shown and the stable stationary is identified. It is also shown that there is a threshold at which photon gain is possible. Secondly, the stationary probability distribution function is explicitly derived. This allows one to derive all the stationary properties of the system. Thirdly, we have set up the Fokker-Planck equation for the system. This has allowed us to investigate the fluctuations about the stationary state. The autocorrelation function for the photon number near the stationary state has been derived, thereby obtaining the relaxation time for the system.

While the solution of the master equation is a difficult task indeed, we have set up the equation for the generating function. Also, an appendix has been added to outline the exact solution of the macroscopic equation, as well as set up a scheme for calculating the variance and the mean number of photons as a function of time, with a certain approximation of decorrelation.

The paper leaves off with a few unanswered questions. One of the most important is: How to extend the consideration of this work for open systems? One may envisage an open system for either one or more species involved in the processes that have been treated. Some progress in this direction has been made, and is planned to be the subject of a later paper.

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APPENDIX A

In the text, we have studied the master equation for two (or three) different cases, avoiding the case of stimulated recombination for neutral species. In this Appendix, we remedy this by considering

$$X+Y+Nh\nu \rightleftarrows XY+(N+1)h\nu \ . \tag{A1}$$

The essential difference of this case in contrast to those considered in the text is that of the global charge neutrality, a concept not of any use here. The conservation laws here are

$$[X]+n=B \ , \tag{A2}$$

$$[XY]+[X]=A \ , \tag{A3}$$

and

$$[XY]+[Y]=C \ . \tag{A4}$$

The master equation is easily written down, taking into consideration (A2)–(A4). One has

$$\frac{dp_n}{dt}=(\mathbf{E}-1)r_np_n+(\mathbf{E}^{-1}-1)g_np_n \ ,$$

where

$$r_n=\alpha n(n-B+A) \ ,$$
$$g_n=\lambda n(B-n)(n+A-B-C) \ . \tag{A5}$$

The macroscopic equation turns out to be

$$\frac{dn}{dt}=\lambda n(n-B)(n+\beta)-\alpha n(n+\delta) \ , \tag{A6}$$

where

$$\beta\equiv B+C-A; \ \ \gamma\equiv B-A \ .$$

Note β and γ are both positive. The stationary solution, other than the trivial one, is given by

$$n_s^{\pm}=\frac{\delta-2B-\Gamma}{2}\pm\tfrac{1}{2}(\delta^2-2\Gamma\delta+4\Gamma B+4\Gamma\gamma)^{1/2} \ , \tag{A7}$$

where $\Gamma\equiv\alpha/\lambda$. For $\delta=0$, the only admissible solution is

$$n_s^{+}=\frac{-2B-\Gamma}{2}+(\Gamma B+\Gamma\gamma)^{1/2} \ . \tag{A8}$$

Further, to have n_s^+ positive, we need

$$\Gamma[B+n-(XY)]^{1/2}>B+\frac{\gamma}{2} \ . \tag{A9}$$

This is possible only for large values of n.

APPENDIX B

In this Appendix, we return now to the unfinished task of the solution of the master equation. One might, in principle, consider also the solution of the Fokker-Planck equation given by Eq. (14). The later equation is, however, based on approximation based on Taylor series expansion, and therefore it is wiser to focus one's attention on Eq. (6). First, a generating function $F(z,t)$ is utilized, defined by

$$F(z,t)=\sum_{n=0}^{\infty} z^n p(n,t) \ , \tag{B1}$$

where we have utilized the notation $p(n,t)$ for $p_n(t)$. After a little bit of algebra, one finds the equation for $F(z,t)$, which is

$$\begin{aligned}\frac{\partial F(z,t)}{\partial t}&=\lambda z^3(z-1)\frac{\partial^3 F}{\partial z^3}\\ &\quad+[2\lambda z(z-B)-\alpha](z-1)z\frac{\partial^2 F}{\partial z^2}\\ &\quad+(z-1)[\lambda z(B^2-4B+2)\\ &\qquad-\alpha(1+A-B)]\frac{\partial F}{\partial z}+B^2F \ .\end{aligned} \tag{B2}$$

Since this has already been calculated in the text, the advantage of transformation of a difference equation to differential equation is not apparent. One, however, may know the initial photon number. Let us say, $p(n,0)=\delta_{m,n}$. In that case, $F(z,0)=Z^m$. It is then possible to set up an algorithm to numerically compute $F(z,t)$. It is not our intention to display any such computed solution, but only to remark on such a numerical realization of the problem.

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