Chaotic behavior in externally modulated hydrodynamic systems

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An amplitude equation for a codimension-two bifurcation point is studied in the presence of a periodically modulated Rayleigh number. The boundary limits of the convective state and flow patterns above threshold are calculated. It is found that the system exhibits chaotic behavior close to the codimension-two point. The Lyapunov exponent associated with these trajectories is calculated.

Hydrodynamic systems exhibit a rich variety of phenomena arising from the nonlinear nature of the basic equations which describe them. In particular, they exhibit instabilities which lead to complicated flow patterns. A canonical example of such a phenomenon is the Rayleigh-Benard instability in a layer of fluid heated from below.[1,2] Usually in a single-component fluid this instability leads to a stationary flow pattern. However, theoretical studies of models associated with binary mixtures indicate that these systems are expected to exhibit two kinds of instabilities, leading, respectively, to stationary and oscillatory flow patterns when the two external parameters which control the system, namely, the temperature and concentration gradients ΔT and ΔC, are varied.[3-7] In the experiment, ΔT and ΔC are coupled by the thermal diffusion coefficient, which usually is a function of the average temperature and concentration. Subsequently the two types of flow patterns have been observed experimentally in, for example,[4] water-methanol and water-ethanol mixtures. At the point where the two bifurcation lines intersect, the two types of flow patterns compete, giving rise to interesting nonlinear phenomena. Such a point is called a codimension-two (CT) bifurcation point.[8-12] The amplitude equation associated with this point has previously been derived, and the $(\Delta C, \Delta T)$ phase diagram was analyzed. The phase diagram exhibits two transition lines to the two types of flow pattern, together with other instability lines separating the two convecting regions (see Fig. 1).

In this paper we are concerned with the effect of external modulation (e.g., of the temperature gradient) on the phase diagram. This modulation provides an easily accessible and controllable parameter, and therefore is very attractive experimentally. Moreover, simple models corresponding to these systems may be derived from the hydrodynamic equations. These models may then be analyzed theoretically in a tractable way. Previous theoretical studies based on an amplitude equation and the Lorenz model were mainly con-

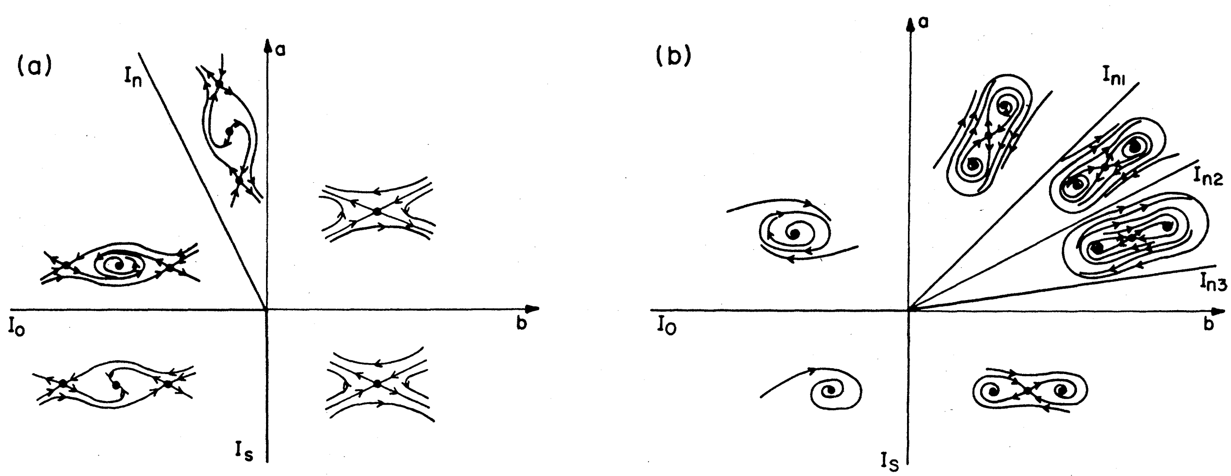

FIG. 1. Phase diagram of Eq. (1) for $f_2 < 0$ and $\epsilon_1 = \epsilon_2 = 0$. (a) $f_1 > 0$. I_0 is a second-order line separating the conductive phase (which exists in the lower left quadrant of the a-b plane) from the oscillatory one. I_s is a first-order line separating the conductive and the stationary phases. The oscillatory and the stationary phases are separated by a nonlinear second-order line I_n. As this line is approached from the oscillatory phase, the frequency of the periodic motion goes to zero. (b) $f_1 < 0$. In this case both I_0 and I_s are second-order lines. The oscillatory and the stationary phases coexist in the region bounded by the lines I_{n1} and I_{n3}. The transition between the two phases is therefore first order. Lines I_{n1} and I_{n3} are stability limits of the stationary and oscillatory phases, respectively. On the line I_{n2} a homoclinic saddle connection of the two unstable limit cycles occurs.

cerned with the effect of time-dependent heating on the stationary instability of the single-component fluid.[13,14] More recently, a generalized Lorenz model for binary mixtures, which includes the effect of external modulation, was considered, and the linear stability analysis was carried out.[15,16] Here we study an amplitude equation of a binary mixture close to the CT point with external modulation. We are mainly interested in the nonlinear behavior of the system, i.e., in studying the flow patterns above threshold. We find that, unlike the relatively simple phase diagram of the unmodulated case (Fig. 1), the phase diagram of the modulated system exhibits a rather complicated pattern of instability lines leading to chaotic behavior in the vicinity of the CT point. Let us consider for definiteness a Rayleigh-Benard system of binary fluid in which the Rayleigh number is given by $R=R_0+R_1\cos(\omega t)$, where R_1 is a small perturbation. The following analysis is quite general, and therefore also applies to other problems such as, for example, magnetoconvection. We describe the system close to the CT point by the following equation:

$$\ddot{x}=[a+\epsilon_1\cos(\omega t)]\dot{x}+[b+\epsilon_2\cos(\omega t)]x+f_1x^3+f_2x^2\dot{x} \ , \tag{1}$$

where x is related to the vertical component of the velocity, and ϵ_1 and ϵ_2 are proportional to R_1. The parameters a and b are functions of the temperature gradient and the concentration. For $R_1=0$ (namely, for $\epsilon_1=\epsilon_2=0$) this equation describes the flow of the fluid near the CT point, given by $a=b=0$.[9,10,12] The linear part of this equation can easily be derived from the linearized hydrodynamic equations which describe the system, considering only linear terms in R_1. We neglect the effect of R_1 on the nonlinear terms f_1 and f_2, since R_1 is assumed to be small while f_1 and f_2 are of $O(1)$. For $R_1=0$ the (a,b) phase diagram has been analyzed in detail.[8,12] In what follows we restrict ourselves to $f_2<0$. The phase diagram is given by Figs. 1(a) or 1(b), depending on whether f_1 is positive or negative, respectively. For $f_1>0$ (corresponding to binary mixtures) there are two transition lines associated with the convective phase: a second-order (forward bifurcation) line I_0 leading to an oscillatory phase, and a first-order (inverse bifurcation) line I_s leading to a stationary phase. The two phases are separated by a second-order line I_n.[10,12] The frequency of oscillation vanishes when this line is approached from the oscillatory phase [Fig. 1(a)]. For $f_1<0$ (corresponding, for example, to some cases of magnetoconvection[10]), both lines I_s and I_0 are second order. However, the two phases are separated by a first-order line. The stability limits of the oscillatory and stationary phases are given, respectively, by the lines I_{n3} and I_{n1}. The two phases coexist between these two lines [Fig. 1(b)].[8] Note that, for $\epsilon_1=\epsilon_2=0$, Eq. (1) is a second-order autonomous equation, and therefore is not expected to yield chaotic behavior. However, when modulation is present, solutions of this equation may exhibit chaotic behavior, resulting in a rather complicated phase diagram. In the following we perform the linear stability analysis of this equation, and study its solutions above threshold. We find that the system does indeed exhibit chaotic behavior close to the CT point.

Linear stability analysis of Eq. (1) near the convective phase $(x=0)$ yields the following expression for the stationary instability surface:

$$b=\tfrac{1}{2}\epsilon_2(\epsilon_2+a\epsilon_1)/(a^2+\omega^2) \ . \tag{2}$$

Near the CT point $(a=b=0)$ this surface is curved towards the convective phase. The behavior of the oscillatory instability surface is more complicated. Let ω_0 be the frequency of the oscillatory phase at threshold (for $\epsilon_1=\epsilon_2=0$ one has $\omega_0=\sqrt{-b}$). For $\omega=2\omega_0$ one finds resonance effects[16] which lead to the following expression for the oscillatory surface:

$$a=-\tfrac{1}{2}(\epsilon_1^2+\epsilon_2^2/|b|)^{1/2} \ , \tag{3}$$

to leading order in ϵ_1 and ϵ_2. For frequency $\omega=2\omega_0/n$, $n>1$, the correction to the expression for the instability surface is of $O(\epsilon^2)$, and the resonance effect shows up at $O(\epsilon^n)$. When the resonance condition is not satisfied, the instability surface stays at $a=0$ for

$$A\equiv\tfrac{1}{16}(\omega^2-4b-1/2\epsilon_1^2)^2+(b\omega^2-1/2\epsilon_2^2)>0 \ . \tag{4}$$

However, when A becomes negative, the surface curves towards the conductive phase, and is given by

$$\tfrac{1}{16}(\omega^2+a^2-4b-1/2\epsilon_1^2)(\omega^2+9a^2-4b-1/2\epsilon_1^2)+2b(\omega^2+a^2)-1/2\epsilon_2(\epsilon_2+a\epsilon_1)=0 \ . \tag{5}$$

For $\epsilon_2=0$ $(f_1<0)$, the behavior of the instability line I_{n1} (given by $a=b$) is very similar to that of the oscillatory instability line $a=0$. Again, if $\omega=2\omega_0$, one finds resonance effects however for the nonresonating ω, the expression for I_{n1} does not change until ϵ_1 reaches the value given by the following expression:

$$\epsilon_1^2=\frac{2\{\omega^4+\omega^2[10(a-b)^2-16b]+[3(a-b)^2+8b]^2\}}{[2\omega^2+10(a-b)+16b]} \ . \tag{6}$$

For greater values of ϵ_1 the stability surface is curved towards the stationary phase. The analysis of this line in the case of $\epsilon_2\neq0$ is more complicated, since the fixed points associated with the stationary phase become limit cycles. We have analyzed the stability of the limit cycles; however, the expression of the resulting critical surface is rather complicated and will be published elsewhere. Consider now the $f_1>0$ case and take $\epsilon_2=0$. By applying the Melnikov method[8] and averaging over the period of the modulation, one can prove that the nonlinear line I_n is not changed.

In order to investigate the flow patterns above threshold, we performed numerical integration of Eq. (1). As already mentioned, the solutions of this equation may exhibit chaotic behavior. If such chaotic regions exist, they should be most easily found close to the I_n line (for $f_1>0$), and between the nonlinear lines I_{n1} and I_{n3} (for $f_1<0$). In these regions the unmodulated system exhibits competition between fixed points and limit cycles. This competition may result in a chaotic behavior when modulation is added. Inspection of the trajectories in these regions shows that this is indeed the case. A single trajectory wanders chaotically between the fixed points and the limit cycle of the unmodulated system. Typical trajectories are shown in Fig. 2. In this figure we consider the $f_1<0$ case and take the system to be close to the I_{n1} line $(\epsilon_1=\epsilon_2=0)$. By increasing ϵ_1 we find a series of period-doubling bifurcations [Figs.

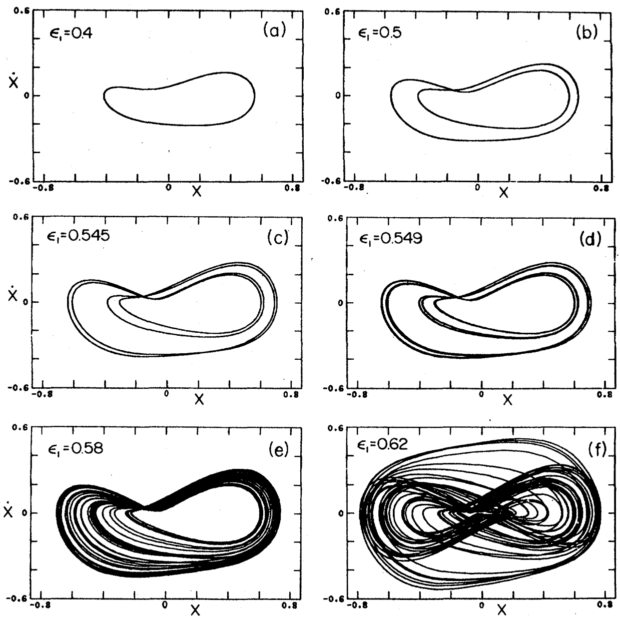

FIG. 2. Phase-space trajectories of Eq. (1) for $f_1 < 0$, $\epsilon_2 = 0$, $a = 0.012$, $b = 0.0125$, $\omega = 0.3123$, and several values of ϵ_1. As one increases ϵ_1 one finds a series of period doublings (a)–(d). For large ϵ_1 (e)–(f) the motion becomes chaotic.

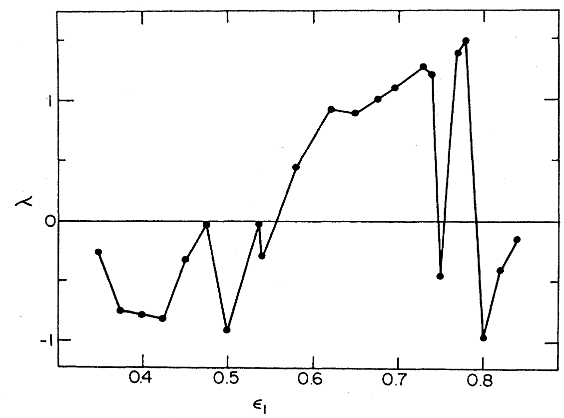

FIG. 3. Plot of the Lyapunov exponent λ as a function of ϵ_1 (for $\epsilon_2 = 0$) corresponding to the sequence given in Fig. 2. The sharp maxima in the region of negative λ indicate the points of period doubling. Positive λ corresponds to chaotic trajectories.

2(a)–2(d)], leading to chaotic trajectories for sufficiently large ϵ_1 [Figs. 2(e) and 2(f)]. For the $f_1 > 0$ case, Eq. (1) does not have a nontrivial ($x \neq 0$) fixed point.[17] We therefore add a term $f_3 x^5$ ($f_3 < 0$) to Eq. (1) and study its trajectories near the I_n line. This coefficient has previously been calculated on the stationary branch of the instability line.[18] We find that like the $f_1 < 0$ case this system exhibits chaotic behavior. Note that the chaotic trajectories are rather easily found. For any point (a,b) close to the nonlinear lines, one seems to be able to vary ϵ_1 and ϵ_2 until chaotic behavior appears.

In order to demonstrate that the observed trajectories are really chaotic, we calculated the Lyapunov exponent, the power spectrum, and the correlation dimension D.[19] It turned out that, indeed, the trajectories which "looked" chaotic had a positive Lyapunov exponent and showed broad bands in the power spectrum. A plot of Lyapunov exponent values for the sequence given in Fig. 2 is shown in Fig. 3. In the region of stable limit cycles the Lyapunov exponent is negative. The sharp maxima in this region correspond to the points of period doubling. This plot is very similar to that obtained for the logistic map.[20] The calculations of the correlation dimension[19] suffered from poor numerical stability. However, they clearly indicate that $D > 2$ for the chaotic trajectories, typically about 2.3–2.8 for different points in the phase diagram. For example, for $\epsilon_1 = 0.62$ in Fig. 2 we find $D = 2.5 \pm 0.2$.

In summary, we have analyzed the onset of convection in some systems in the presence of periodic modulation of the Rayleigh number. By performing linear stability analysis we have calculated the stability boundaries of the convecting phase. We have also found that in such systems it is possible to find chaotic behavior at the onset of convection, that may take place via the period-doubling scenario. We hope that this work will stimulate experimental studies of phase diagrams of modulated systems which are expected to exhibit chaotic behavior near the CT point.

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