Relaxation of a system of charged particles

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(Received 15 November 1984)

The relaxation of an isotropic, approximately monoenergetic system of like particles through small-angle Coulomb collisions is followed numerically. It is found that when energy conservation is ensured, the results are qualitatively different from those of an earlier study [W. M. MacDonald, M. N. Rosenbluth, and Wong Chuck, Phys. Rev. **107**, 350 (1957)], and show a smaller, global relaxation time. A new interpretation is advanced for the earlier results.

I. INTRODUCTION

The relaxation to a Maxwellian of a system of charged particles is of some interest in astrophysics and, latterly, in controlled thermonuclear fusion research. In one of the earliest works on this subject MacDonald, Rosenbluth, and Chuck[1] (referred to hereafter as MRC) solved the time-dependent Fokker-Planck equation numerically for an isotropic system of like particles with an initially Gaussian velocity distribution; the Gaussian distribution was chosen to represent the shape assumed by a delta function at some speed P after a very short time. They showed, after about six Spitzer self-collision times,[2] that while the peak of the Gaussian distribution had quickly relaxed to the equilibrium Maxwellian value, particles with energies several times below and above the average energy still had densities in velocity space significantly different from their expected equilibrium values. The low-energy particles had "overshot" their equilibrium densities while the high-energy particles formed a depleted Maxwellian tail. It was thought that a much longer time was required for these particles to relax to thermal equilibrium.

This study and its conclusions have since been incorporated into several textbooks on plasma physics,[3,4] motivating in part this report. It is shown here that these earlier results could be explained by "numerical cooling," an artifact of the numerical scheme used by MRC. The use of a different, energy-conserving numerical scheme in this study produces qualitatively different results and shows a reduction in the overall relaxation time.

II. NUMERICAL SOLUTION

The evolution of an isotropic distribution of like particles through small-angle Coulomb collisions is described by the Fokker-Planck equation:[5]

$$\frac{\partial f}{\partial t}=\frac{2\pi e^4}{m^2}(\ln\Lambda)\left\{\frac{2}{3}\frac{\partial^2 f}{\partial v^2}\left[\int_v^\infty du\,fu+\frac{1}{v^3}\int_0^v du\,fu^4\right]+\frac{4}{3v}\frac{\partial f}{\partial v}\left[\int_0^\infty du\,fu-\int_0^v du\,fu\left[1-\frac{u}{v}\right]^2\left[1+\frac{u}{2v}\right]\right]+2f^2\right\}, \quad (1)$$

$$\Lambda=(3/2e^3)(k^3T^3/\pi n)^{1/2}\ .$$

The distribution function f has the normalization

$$n=\int_0^\infty dv\,f v^2\ ,$$

where n is the number density and v the particle speed. T is the temperature, k the Boltzmann constant, and m the particle mass.

Following MRC, the following dimensionless quantities are defined:

$$h(\xi,\tau)=f(v,t)(v_0^3/A),\quad \tau=(2\pi e^4/m^2)(A/v_0^3)(\ln\Lambda)t\ ,$$

$$\xi=v/v_0,\quad A=n/I_2,\quad v_0^2=(3kT/m)(I_2/I_4)\ ,$$

$$I_2=\int_0^\infty d\xi\,h\xi^2,\quad I_4=\int_0^\infty d\xi\,h\xi^4\ .$$

The dimensionless time τ corresponding to one Spitzer self-collision time,

$$t_c=m^{1/2}(3kT)^{(3/2)}/(8\times 0.714\pi ne^4\ln\Lambda)\ ,$$

is

$$\tau_c=[4\times 0.714 I_2(I_2/I_4)^{3/2}]^{-1}\ .$$

The integrals I_2 and I_4 are related to the number density and kinetic temperature, respectively, and should be unchanged by self-collisions. Using these dimensionless quantities, Eq. (1) can be transformed:

$$\frac{\partial h}{\partial t}=\frac{2}{3}\frac{\partial^2 h}{\partial\xi^2}\left[\int_{-\xi}^{-\infty}d\eta\,h\eta+\frac{1}{\xi^3}\int_0^\xi d\eta\,h\eta^4\right]+\frac{4}{3\xi}\frac{\partial h}{\partial\xi}\left[\int_0^\infty d\eta\,h\eta-\int_0^\xi d\eta\,h\left[1-\frac{\eta}{\xi}\right]^2\left[1+\frac{\eta}{2\xi}\right]\eta\right]+2h^2\ . \quad (2)$$

MRC integrated this equation directly using the difference equation

$$\left(\frac{\partial h}{\partial t}\right)_j = \frac{2}{3}\left[\frac{h_{j+1}-2h_j+h_{j-1}}{(\Delta\xi)^2}\right]S_J + \frac{4}{3\xi_j}\left[\frac{h_{j+1}-h_{j-1}}{2\Delta\xi}\right](S-R_j)+2h_j^2 \ , \tag{3}$$

where

$$S_j=\frac{1}{\xi_j^3}\int_0^j d\eta\, h\eta^4+\int_j^J d\eta\, h\eta, \quad S=\int_0^J d\eta\, h\eta \ ,$$

$$R_j=\int_0^J d\eta\, h\eta-\frac{3}{2\xi_j}\int_0^j d\eta\, h\eta^2+\frac{1}{2\xi_j^3}\int_0^j d\eta\, h\eta^4 \ ,$$

with subscripts referring to velocity groups. The alternative method used here is based on that of Langdon.[6] Equation (2) may be rearranged into a form which readily exhibits the conservation properties of the Fokker-Planck operator:[3]

$$\frac{\partial h}{\partial t}=\frac{2}{\xi^2}\frac{\partial}{\partial\xi}\left[Eh+D\frac{\partial h}{\partial\xi}\right]=\frac{1}{\xi^2}\frac{\partial}{\partial\xi}F \ , \tag{4}$$

$$E=\int_0^\xi d\eta\, h\eta^2 \ , \tag{4a}$$

$$D=\frac{1}{3\xi}\int_0^\xi d\eta\, h\eta^4+\frac{\xi^2}{3}\int_\xi^\infty d\eta\, h\eta \ . \tag{4b}$$

Equation (4) is finite differenced as

$$\left(\frac{\partial h}{\partial t}\right)_j=\frac{1}{\xi_j^2\Delta\xi}(F_{j+(1/2)}-F_{j-(1/2)}), \quad 1<j<J \ ,$$

$$\left(\frac{\partial h}{\partial t}\right)_1=F_{3/2}/\xi_j^2\Delta\xi \ , \quad \left(\frac{\partial h}{\partial t}\right)_J=-F_{J-(1/2)}/\xi_j^2\Delta\xi \ .$$

It is apparent that number density is conserved exactly with this scheme. The coefficient E is evaluated from Eq. (4a). However, although Eq. (4b) gives an explicit definition of D, it is not used in its numerical evaluation. D is evaluated instead from the relation,

$$\frac{\partial}{\partial\xi}(\xi D)=\xi^2\int_\xi^\infty d\eta\, h\eta \ .$$

This evaluation anticipates the necessary cancellation in the energy moment of the Fokker-Planck operator so that energy is well conserved; a numerical proof is given in Ref. 6.

The difference equation is solved using Crank-Nicholson time integration. Following MRC the initial distribution was chosen to be of the form

$$h(\xi)=0.01\exp\{-10[(\xi-p)/w]^2\} \ ,$$

i.e., a Gaussian centered at p with its width determined by w.

III. RESULTS

MRC used an initial distribution corresponding to $P=0.3$, $w=0.3$, and 24 uniformly spaced velocity groups with $\Delta\xi=0.03$. The results obtained with these parameters using the energy-conserving scheme above are shown in Fig. 1, together with that obtained by MRC. The dimensionless Maxwellian distribution h_m is given by

$$h_m=(4/\pi^{1/2})(3I_2/2I_4)^{3/2}I_2\exp[-(3I_2/2I_4)\xi^2] \ .$$

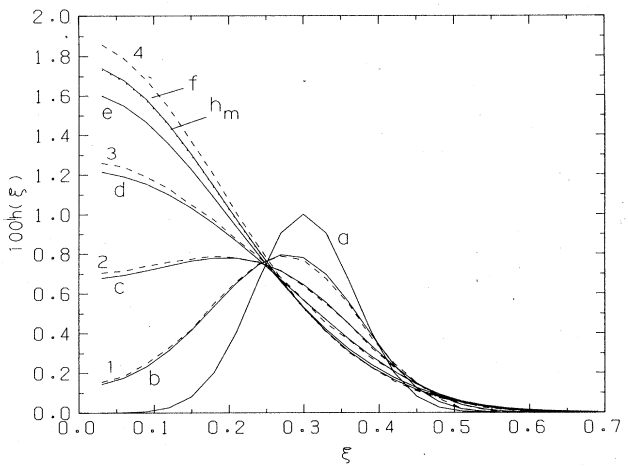


FIG. 1. Evolution of an initially Gaussian distribution. Continuous curves: $h(\xi,\tau)$ calculated with energy-conserving scheme for time τ corresponding to 0.0 (curve a), 10 (b), 30 (c), 70 (d), 200 (e), and 1000 (f). Dashed curves: $h(\xi,\tau)$ from Ref. 1 for τ corresponding to 10.39 (curve 1), 30.31 (2), 72.84 (3), and 484.17 (4). Dotted curve: equilibrium Maxwellian.

The whole range of the distribution has approached to within 10% of the Maxwellian value by about five Spitzer self-collision times ($\tau_c\approx 82$). The results of MRC are shown by the dashed lines. Good agreement between the two sets of results are seen for early time $\tau\lesssim\tau_c$. However, for the MRC's results at $\tau=484.17$, the low-energy part of the curve had overshot its Maxwellian value while the high-energy part formed an underfilled Maxwellian tail. In contrast, with the energy-conserving scheme, the lower-energy particles approached their Maxwellian densities monotonically. The numerical values of the two sets of results at about six Spitzer self-collision times are tabulated in Table I. Note that with the energy-conserving scheme, the Maxwellian tail had filled up by this time.

The finite-difference scheme used by MRC [Eq. (2)] has also been employed here to follow their computation up to $\tau=1000(\approx 12\tau_c)$ (Fig. 2). An apparent explanation for the overshooting by the low-energy particles at $\tau=484.17$ in Ref. 1 is that it was part of a transient oscillatory process. Figure 2 shows this is not the case. It is found that the overshooting and underfilling by the low- and high-energy particles, respectively, increases with time, instead of diminishing. Figure 3 shows the reason for this behavior. The number density and kinetic temperature of the particles are shown as a function of time. It can be seen that while the number density is well conserved, the temperature is dropping rapidly. The Maxwellians with temperatures as calculated at time $\tau=480$ and 1000 agree very well with their respective numerical results for those times (Fig. 2). The overshooting and underfilling of the low- and high-energy particles are seen to be the results of numerical cooling. In comparison with the energy-conserving scheme above, the number density is exactly conserved and the temperature computed at $\tau=1000$ differs from the initial temperature by only about 1%.

Figures 4(a) and 4(b) show the evolution of a more monoenergetic initial distribution, with $P=0.3$ and $w=0.01$ (i.e., a more sharply peaked Gaussian). A total of 224 velo-

TABLE I. h/h_m at $\tau = 480$ for the energy-conserving scheme (column b), and at $\tau = 484.17$ from Ref. 1 (column a).

ξ	a	b
0.03	1.07	0.98
0.06	1.07	0.98
0.09	1.06	0.98
0.12	1.06	0.98
0.15	1.05	0.98
0.18	1.05	0.99
0.21	1.04	0.99
0.24	1.03	0.99
0.27	1.01	1.00
0.30	1.00	1.00
0.33	0.99	1.01
0.36	0.97	1.01
0.39	0.98	1.01
0.42	0.94	1.02
0.45	0.92	1.02
0.48	0.91	1.03
0.51	0.89	1.03
0.54	0.87	1.03
0.57	0.85	1.03
0.60	0.82	1.02
0.63	0.72	1.01
0.66	0.76	1.00
0.69	0.72	0.98
0.72	0.68	0.96

city groups, for $\xi = 0.0$–0.72, were used. The nonuniform velocity mesh was finely resolved around the narrow peak of the initial distribution. The peak collapses very rapidly and the subsequent evolution is qualitatively similar to the previous case, with the low-energy particles approaching their Maxwellian densities monotonically. Again, only about six Spitzer self-collision times are required for the initial distribution to reach thermal equilibrium.

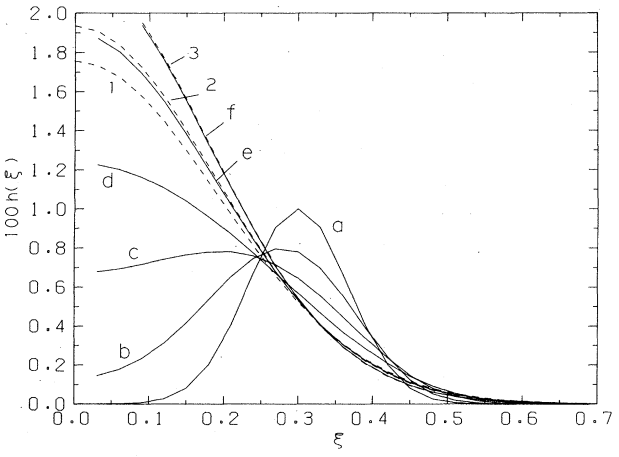


FIG. 2. Evolution of the same initial distribution as in Fig. 1, using MRC's numerical scheme up to $\tau = 1000$. Continuous curves: $h(\xi, \tau)$ for τ corresponding to 0.0 (curve a), 10 (b), 30 (c), 70 (d), 480 (e), and 1000 (f). Dashed curves: Maxwellian distribution with temperature and density as obtained at τ corresponding to 0.0 (curve 1), 480 (2), and 1000 (3).

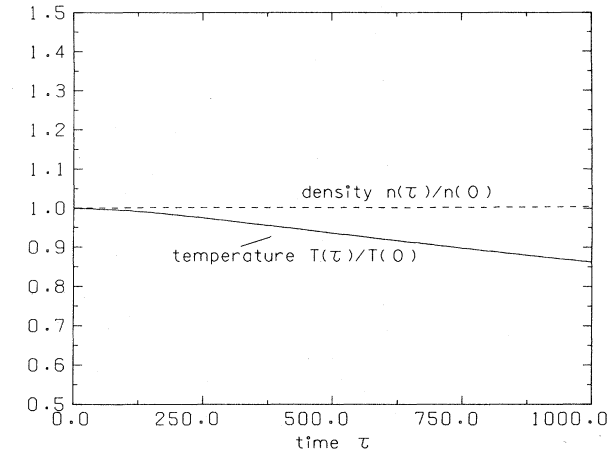
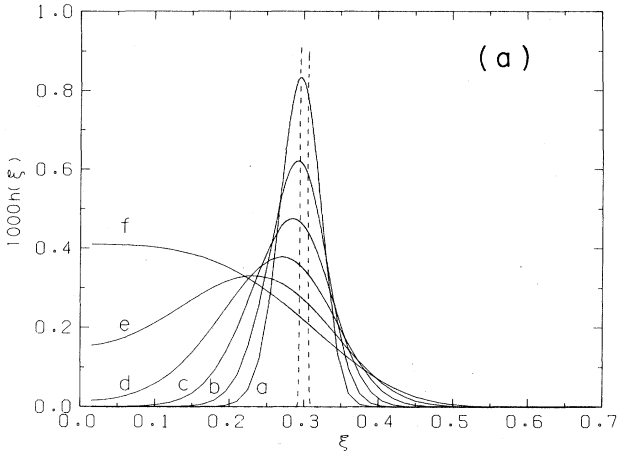


FIG. 3. Computed number density and temperature as a function of time for the MRC numerical scheme.



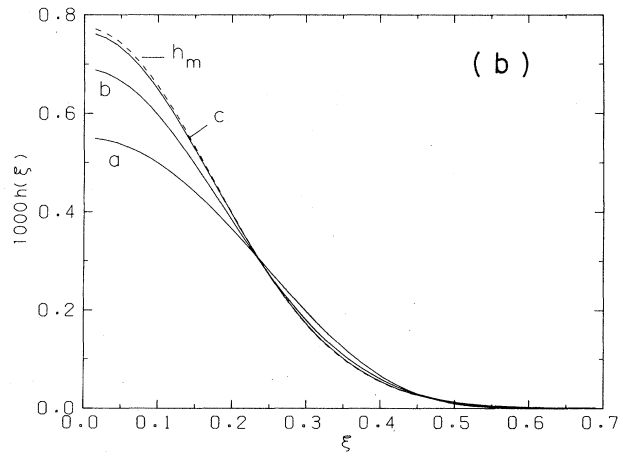


FIG. 4. (a) Evolution of a more monoenergetic initial distribution, indicated by the dashed lines, for the following times (in units of a self-collision time): 0.02 (curve a), 0.04 (b), 0.08 (c), 0.16 (d), 0.32 (e), and 0.64 (f). (b) As in (a), for the following self-collision time(s): 1.0 (curve a), 2.0 (b), and 6.0 (c). Dashed curves indicate the equilibrium Maxwellian.

[1]W. M. MacDonald, M. N. Rosenbluth, and Wong Chuck, Phys. Rev. **107**, 350 (1957).
[2]L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1962), p. 133.
[3]I. P. Shkarofsky, T. W. Johnston, and M. P. Bachynski, *The Particle Kinetics of Plasmas* (Addison-Wesley, London, 1966).
[4]D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964), p. 33; Dwight R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983), p. 67.
[5]M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[6]A. B. Langdon, in Proceedings of the Centre Européen de Calcul Atomique et Moléculaire (CECAM) Workshop on the Flux Limiter and Heat Flow Instabilities in Laser-Plasmas, 14 September–2 October, 1981, Université de Paris–Sud, Orsay, France (unpublished).