Second invariant in an excited three-level system

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For a three-level atom excited by two laser fields whose amplitudes are sinusoidally modulated with a phase difference of $\pi/2$, we show that, rather surprisingly, a second (independent) linear constant of motion may be obtained even more directly by the coherence vector method than by the recent density-matrix formulation of Hioe. As a consequence it is found that the two corresponding eigenvectors of the time-independent time-evolution matrix in the degenerate subspace with two zero eigenvalues are orthogonal if a certain relation holds between the laser modulation and Rabi frequencies.

Hioe[1] has presented a new formulation and complete solution of the problem of finding two independent linear constants of motion for a soluble system discovered by Gottlieb[2] in which a three-level atom is excited by two lasers whose fields are sinusoidally amplitude modulated with a phase difference of $\pi/2$. The novelty of Ref. 2 was that the time-evolution matrix is time independent so the equations of motion could be solved in terms of its eigenvalues, and two new linear constants of motion (invariants) could be constructed from the eigenvectors corresponding to its two zero eigenvalues. Reference 2 was itself based on an earlier formulation, involving the coherence vector, of a different problem by Hioe and Eberly.[3] A physical interpretation of the constants of motion similar to that of Hioe[1] has also been given independently by Pegg.[4]

First, we wish to point out that, while Ref. 1 seems to imply that neither of the equations [(17a) or (17b)] there were given for the general case in Ref. 2, this is not really so for the first invariant since in fact, after recasting of variables, one obtains

$$\text{RHS[Eq. (20) of Ref. 2]} = \text{LHS[Eq. (17a) of Ref. 1]} - \Delta/3, \tag{1}$$

where Δ is the detuning at exact two-photon resonance, and may be nonzero. This invariant corresponded to choosing zero sixth component in one eigenvector of the time-evolution matrix L with zero eigenvalue [see Eqs. (8) and (19) of Ref. 2]:

$$\mathbf{x}_1 = \text{col}(A, 0, -\tfrac{1}{2}\Delta, -\omega, 0, 0, 0, -\tfrac{1}{2}\Delta/\sqrt{3}) \;, \tag{2}$$

where ω is the laser-modulation angular frequency and A is the common constant amplitude of the Rabi half-frequencies appearing in Ref. 2.

Hioe[1] also mentioned that it would be cumbersome to arrive at the second constant of motion by our method. If, however, we suppress our mathematical predilection of Ref. 2 for orthogonal eigenvectors in a degenerate subspace, then a second invariant may in fact be found easily. Choosing an independent eigenvector of L (with zero eigenvalue) with zero first component gives

$$\mathbf{x}_3 = \text{col}[0, 0, -\tfrac{1}{2}\sqrt{3}\omega^2, -\sqrt{3}\Delta\omega, 0, \sqrt{3}A\omega, 0, (A^2 - \tfrac{1}{2}\omega^2)] \;. \tag{3}$$

Because the matrix and eigenvectors involved are time independent, the corresponding invariant is still found from the reordered transformed coherence vector $\mathbf{T}'$ given in Ref. 2 even for nonorthogonal eigenvectors in this degenerate subspace, because a constant linear combination of invariants is also invariant. Thus we construct

$$Z_3{}^c \equiv \mathbf{x}_3 \cdot T' \tag{4a}$$

$$= -\tfrac{1}{2}\sqrt{3}[\omega^2 W + 2\Delta\omega v_{13} - 2A\omega\mathscr{V} + (\omega^2 - 2A^2)\mathfrak{M}/\sqrt{3}] \;, \tag{4b}$$

where the new symbols are defined in Ref. 3 and used in Ref. 2. Recasting variables to conform with Ref. 1, we find the second invariant

$$Z_3{}^c = -\sqrt{3}\{\text{LHS[Eq. (17b) of Ref.1]} - 2(A^2 + \omega^2)/3\} \;. \tag{5}$$

Up to overall multiplicative and additive constants, we have easily arrived at the second linear invariant of Ref. 1 in the general case. Thus, our method is not cumbersome: it is merely that, as we noted in Ref. 2, the general expressions themselves which would arise from constructing an invariant from Eq. (4a) with an eigenvector which is orthogonal to (2), are cumbersome. In fact, they will obviously be some linear combination of the invariants given by Eqs. (17a) and (17b) of Ref. 1 which we have just derived above.

One may well ask how it is that the first constant of motion which we found in Eq. (20) of Ref. 2 just happens to coincide [up to an additive constant: see Eq. (1) above] with the first constant of motion which was obtained by Hioe in Eq. (17a) of Ref. 1 via a trace involving the transformed time-independent Hamiltonian. The answer probably lies in the fact that $\mathbf{x}_1$ in Eq. (2) above was actually chosen in Ref. 2 as it seemed to be the simplest solution with components which were linear in the constants A, Δ, ω. This linearity evidently corresponds to the first power of the Hamiltonian, which is linear in these quantities, appearing in the trace in Ref. 1.

It may appear even more serendipitous that the choice of second independent eigenvector, Eq. (3), just happens to yield the second constant of motion appearing in Eq. (17b) of Ref. 1 [see Eq. (5) above]. However, $\mathbf{x}_3$ in Eq. (3) was chosen to be the next simplest solution in mathematical form, being quadratic in the constants A, Δ, ω and containing only one component (the eighth) with more than one term in it. It can now be seen that this choice seems to correspond to the second power of the Hamiltonian appearing in the trace for the evaluation of the second constant of motion in Ref. 1. Actually, the method of Ref. 1 gives as second constant of motion obtained in that manner a linear combination of Eq. (17b) *plus* Δ times (17a) in Ref. 1. Our method, Eqs. (4) and (5) above, has given the second independent invariant directly, unmixed with the first (with the additive constant being only an absolute, nondynamical, constant).

When $\Delta=0$, Eq. (4b) corresponds with Eq. (22a) for the invariant Z_2^c in Ref. 2, as noted in Ref. 1. Indeed, $\mathbf{x}_1$, Eq. (2), is manifestly orthogonal, component by component, with $\mathbf{x}_3$, Eq. (3), if $\Delta=0$. Now, for general parameters,

$$\mathbf{x}_1\cdot\mathbf{x}_3=\Delta(8\omega^2-A^2)/(2\sqrt{3}) \;, \tag{6}$$

confirming the previous statement. But this reveals a further interesting special case even when all parameters are nonzero. If the Rabi and modulation frequencies are such that $8\omega^2-A^2=0$ in (6), i.e.,

$$A=2\sqrt{2}\omega \;, \tag{7}$$

then the two independent constants of motion appearing in (1) and (5) correspond to the eigenvectors

$$\mathbf{x}_1=\mathrm{col}[2\sqrt{2}\omega,0,-\tfrac{1}{2}\Delta,-\omega,0,0,0,-\tfrac{1}{2}\Delta/\sqrt{3}] \;, \tag{8a}$$

$$\mathbf{x}_3=\mathrm{col}[0,0,-\tfrac{1}{2}\sqrt{3}\omega^2,-\sqrt{3}\Delta\omega,0,2\sqrt{6}\omega^2,0,15\omega^2/2] \;, \tag{8b}$$

which are actually *orthogonal* even for detuning $\Delta\neq 0$.

In conclusion, it seems that the two independent invariants for the system under consideration may readily be found by using either of the methods of Ref. 2 (as described above) or Ref. 1. Hioe's approach[1] has the advantage of clarifying the physical basis of the invariants. Finally, our relation (7) between the laser-modulation angular frequency ω and the Rabi half-frequency common amplitude A is an intriguing outcome of the present investigation.

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