Generalizations of the theorem of minimum entropy production to linear systems involving inertia

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The temporal behavior of the excess entropy production P_{ex} is investigated in linear electrical networks and in systems which can be described either by the linearized equations of viscous hydrodynamics or of resistive magnetohydrodynamics. As a result of inertial effects P_{ex} is an oscillatory quantity. A kinetic potential is constructed which contains P_{ex} additively. It is an upper bound of P_{ex} and decreases monotonically in time, enforcing $P_{ex} \to 0$ as $t \to \infty$.

I. INTRODUCTION

For dissipative processes in open systems subject to time-independent nonequilibrium boundary conditions, under rather general circumstances an evolution criterion is valid.[1–3] According to this, a certain contribution to the entropy production decreases until a steady state is reached. Unfortunately, this criterion does, in general, not lead to any conclusions about the temporal behavior of the total entropy production P or about stability. Under much more restricted circumstances the evolution criterion holds for the full entropy production. It is then identical with the theorem of minimum entropy production.[4,5,1–3] Specifically, the latter is valid in the linear range of irreversible thermodynamics if rather mild restrictions are observed. One of these restrictions concerns inertial effects. Thus, the linearized Navier-Stokes equations are not covered by the theorem unless very special conditions are met (see Sec. II F).

The value of the theorem of minimum entropy production lies partly in the fact that it provides an interesting thermodynamic description of dynamic processes outside equilibrium. Furthermore, it reveals an elegant proof of the asymptotic stability of the systems for which it holds. The general evolution criterion may play a similar role if the corresponding fraction of the entropy production can be completed to form a kinetic potential.[2] If possible this is usually achieved by constructing an integrating denominator (p. 107 of Ref. 2).

In this paper, for specific linear dissipative systems involving inertial effects a kinetic potential is constructed in a different way. Generally the excess entropy production P_{ex} is used rather than the full entropy production P, and P_{ex} is supplemented by adding a "dissipative potential" $\bar{P}_{ex}$. The kinetic potential $P_{ex}+\bar{P}_{ex}$ thus obtained is a monotonically decreasing function of time. P_{ex} is a positive quantity which exerts irregular oscillations due to the presence of inertial effects. It is bounded by the kinetic potential, and therefore together with $\bar{P}_{ex}$ the oscillations must vanish asymptotically. What is thus obtained are in fact theorems of minimum excess entropy production which differ from the known theorems also in that P_{ex} may pass through its minimum before the asymptotic state is reached.

In continuous media, the presence of inertial effects allows for the propagation of sound waves or soundlike waves. In view of these the imposition of steady-state boundary conditions as practiced in the known theorems of minimum entropy production would lead to an ill-posed mathematical problem. On the other hand, a comparison between stationary and time-dependent states would become meaningless if there were no requirements of this kind linking both states. How this problem can be settled will be treated in the context of Sec. II, which deals with systems falling under the range of the linearized Navier-Stokes equations. The subject of Sec. III are systems which can be described by the linearized equations of resistive magnetohydrodynamics. Finally, in Sec. IV we consider systems which can be described by linear network equations.

II. SYSTEMS DESCRIBED BY THE LINEARIZED NAVIER-STOKES EQUATIONS

A. Linearization of flow equations and corresponding entropy production

The nonlinear equations which describe the force-free motion of a dissipative fluid are (see, e.g., Ref. 6)

$$\partial_t\rho+\nabla\cdot\rho\mathbf{v}=0\ , \tag{2.1}$$

$$\rho(\partial_t v_i+v_j\partial_j v_i)=-\partial_i p-\partial_j p_{ij}\ , \tag{2.2}$$

$$\rho(\partial_t e+\mathbf{v}\cdot\nabla e)+p\nabla\cdot\mathbf{v}=-\nabla\cdot\mathbf{W}-p_{ij}\partial_j v_i\ , \tag{2.3}$$

where

$$\mathbf{W}=\kappa\nabla T\ , \tag{2.4}$$

$$p_{ij}=-\mu(\partial_i v_j+\partial_j v_i)+(\tfrac{2}{3}\mu-\lambda)\nabla\cdot\mathbf{v}\delta_{ij}\ . \tag{2.5}$$

Here, ρ is the mass density, e the density per mass of the internal energy, $\mathbf{v}$ the fluid velocity, $P_{ij}=p\delta_{ij}+p_{ij}$ is the pressure tensor, and $\mathbf{W}$ the heat flux. κ, λ, and μ are the coefficients of heat conduction, dynamical friction, and bulk viscosity, respectively. ∂_t and ∂_i stand for temporal and spatial derivatives, summation convention being used for indices which appear twice. The evolution of the density per mass s of the entropy is given by

$$Td_t s = d_t e - (p/\rho^2) d_t \rho \; , \tag{2.6}$$

where $d_t = \partial_t + v_i \partial_i$ is the time derivative of a moving fluid element. From Eqs. (2.1), (2.3), and (2.6) one easily obtains (p. 17 of Ref. 2)

$$\partial_t(\rho s) + \nabla\cdot(\rho s \mathbf{v} + \mathbf{W}) = \sigma \; , \tag{2.7}$$

where

$$\sigma = \mathbf{W}\cdot\nabla T^{-1} - T^{-1} p_{ij}\partial_j v_i \tag{2.8}$$

is the local density per volume of the entropy production. After an elementary calculation from Eqs. (2.4), (2.5), and (2.8) one gets

$$\sigma = \frac{\kappa}{T^2}(\nabla T)^2 + \frac{\mu}{T}(\nabla\times\mathbf{v})^2 + \frac{1}{T}(\lambda + \tfrac{4}{3}\mu)(\nabla\cdot\mathbf{v})^2 + \frac{2\mu}{T}[\partial_i(v_j\partial_j v_i) - \partial_j(v_j\partial_i v_i)] \; . \tag{2.9}$$

Now, we linearize Eqs. (2.1)–(2.3) around a homogeneous equilibrium state with temporally and spatially constant values of the state variables ($p_0 \equiv$ const, $\rho_0 \equiv$ const, $\mathbf{v}_0 \equiv 0$) by setting $p \to p_0 + p$, etc. The linearized equations thus obtained are

$$\partial_t \rho + \rho_0 \nabla\cdot\mathbf{v} = 0 \; , \tag{2.10}$$

$$\rho_0 \partial_t \mathbf{v} + \nabla p = (\lambda + \tfrac{4}{3}\mu)\nabla\nabla\cdot\mathbf{v} - \mu\nabla\times(\nabla\times\mathbf{v}) \; , \tag{2.11}$$

$$\partial_t T + \frac{\beta p_0 T_0}{C_v \rho_0}\nabla\cdot\mathbf{v} = \frac{\kappa}{C_v\rho_0}\Delta T \; . \tag{2.12}$$

For all material coefficients appearing in Eqs. (2.10)–(2.12) the constant values of the unperturbed equilibrium state must be taken. The derivation of Eqs. (2.10) and (2.11) from Eqs. (2.1) and (2.2) is immediately evident, Eq. (2.5) having been used in addition. In deriving Eq. (2.12) from Eq. (2.3), use has been made of the well-known thermodynamic relation

$$de = C_v dT - \frac{1}{\rho^2}(\beta p T - p) d\rho \; . \tag{2.13}$$

There, C_v is the specific heat at constant volume and

$$\beta = \frac{1}{p}\left[\frac{\partial p}{\partial T}\right]_\rho \tag{2.14}$$

is the coefficient of thermal stress. In order to linearize Eq. (2.3) one must substitute

$$p_{ij}\partial_j v_i \to 0, \quad p\nabla\cdot\mathbf{v} \to p_0\nabla\cdot\mathbf{v}, \quad \mathbf{v}\cdot\nabla e = 0 \; ,$$

$$\rho\partial_t e \to \rho_0 C_v \partial_t T + (\beta p_0 T_0 - p_0)\nabla\cdot\mathbf{v}$$

and arrives at Eq. (2.12). Equations (2.10)–(2.12) must be supplemented by the linear relation

$$T = \frac{p}{\beta p_0} - \frac{\rho}{\alpha \rho_0} \tag{2.15}$$

which follows from linearization of the equation of state $T = T(p,\rho)$ using the definition (2.14) and

$$\alpha = -\frac{1}{\rho}\left[\frac{\partial \rho}{\partial T}\right]_p \tag{2.16}$$

(α is the coefficient of thermal expansion). Equation (2.15) may be used for eliminating ρ, p, or T from Eqs. (2.10)–(2.12).

Equations (2.10)–(2.12) are the lowest-order approximation to Eqs. (2.1)–(2.3). The corresponding lowest-order approximation to the entropy production (2.9) is

$$\sigma = \frac{\kappa}{T_0^2}(\nabla T)^2 + \frac{\mu}{T_0}(\nabla\times\mathbf{v})^2 + \frac{1}{T_0}(\lambda + \tfrac{4}{3}\mu)(\nabla\cdot\mathbf{v})^2 + \partial_i\left[\frac{2\mu}{T_0}(v_j\partial_j v_i - v_i\partial_j v_j)\right] \tag{2.17}$$

which is second order since $\sigma_0 = 0$. Later on we shall consider stationary solutions $\rho_{10}(\mathbf{r}), p_{10}(\mathbf{r}), \ldots$ of Eqs. (2.10)–(2.12) along with time-dependent perturbations $\rho_1(\mathbf{r},t), p_1(\mathbf{r},t), \ldots$ of these, i.e., in Eqs. (2.10)–(2.12) we set

$$\rho = \rho_{10}(\mathbf{r}) + \rho_1(\mathbf{r},t) \; , \quad \mathbf{v} = \mathbf{v}_{10}(\mathbf{r}) + \mathbf{v}_1(\mathbf{r},t) \; . \tag{2.18}$$

The global second-order excess entropy production caused by the excess quantities $\rho_1, p_1, \ldots$ with respect to the nonequilibrium state $\rho_{10}(\mathbf{r}), p_{10}(\mathbf{r}), \ldots$ is defined by

$$P_{\text{ex}} = \frac{\kappa}{T_0^2}\int(\nabla T_1)^2 d\tau + \frac{\mu}{T_0}\int(\nabla\times\mathbf{v}_1)^2 d\tau + \frac{1}{T_0}(\lambda + \tfrac{4}{3}\mu)\int(\nabla\cdot\mathbf{v}_1)^2 d\tau \; . \tag{2.19}$$

It is obtained from the local entropy production by replacing $T \to T_1$, $\mathbf{v} \to \mathbf{v}_1$ and integrating over the region occupied by the system provided we have $\mathbf{v}_1 = \mathbf{0}$ and $T_1 = 0$ on the boundary (see p. 85 of Ref. 2). Note that it is different from the full excess of the entropy production.

B. Oscillatory character of P and P_{ex}

We shall now exemplify the oscillatory character of P and P_{ex} by considering the special case $\mathbf{v} = v\mathbf{e}_x$, $\kappa = 0$, $\lambda = 0$, $\partial_y = 0$, $\partial_t = 0$. Equations (2.10)–(2.12) reduce to

$$\rho_0\partial_t v + \partial_x p = \tfrac{4}{3}\mu\partial_{xx} v \; , \tag{2.20}$$

$$\partial_t p + \rho_0 c_0^2 \partial_x v = 0 \; , \tag{2.21}$$

where c_0 is the velocity of sound. Equations (2.17) and (2.19) yield

$$P = P_{\text{ex}} = \frac{4}{3}\frac{\mu}{T_0}\int(\partial_x v)^2 d\tau \; . \tag{2.22}$$

Expanding $v = v^0 + \mu v^1 + \cdots$, $p = p^0 + \mu p^1 + \cdots$ to lowest order, sound waves are obtained as solutions of Eqs. (2.20) and (2.21). The corresponding lowest-order entropy production is given by

$$P = \frac{4}{3}\frac{\mu}{T_0}\int(\partial_x v^0)^2 d\tau \; . \tag{2.23}$$

Let us consider equally shaped wave packets which move towards each other (Fig. 1) and let $g(x)$ describe the spatial structure of $\partial_x v^0$ for one of the two packets. At a time t_1 before overlapping occurs we have

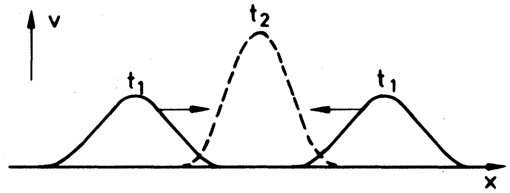

FIG. 1. Collision of sound waves leading to doubled entropy production.

$$P_1 = 2\frac{4}{3}\frac{\mu}{T_0}\int g^2 d\tau \; . \tag{2.24}$$

At the moment t_2 of complete overlapping we have $\partial_x v^0 = 2g$ everywhere and obtain

$$P_2 = \frac{4}{3}\frac{\mu}{T_0}\int (2g)^2 d\tau = 2P_1 \; . \tag{2.25}$$

After the waves have separated P returns to its former value P_1. Higher-order terms involve wave damping and will slightly reduce the values of P obtained in the lowest-order approximation. In the case when both wave packets are reflected by walls, the entire process would now be repeated at a somewhat lower level of the entropy production, and $P(t)$ would exert damped oscillations. A behavior of this kind, corresponding to a rotational (spiral) motion of the system in the space of state variables is known to appear also in systems which do not involve inertia.[7] However, its appearance is then restricted to the nonlinear regime.[8]

A simple mechanical system which exhibits the same qualitative behavior is provided by two suspended balls which bounce against each other inelastically (Fig. 2). During each collision the entropy production rises from zero to some maximum value. This maximum value decreases from period to period. During the time intervals of free motion the dissipative capability of the system is conserved until the way is free for new dissipation during the next collision.

Internal friction and heat conduction lead to dispersion of the sound waves. In consequence sound waves of different wavelength can overtake each other. As with collisions, the entropy production is increased during the overlap period when one wave overtakes another. In a

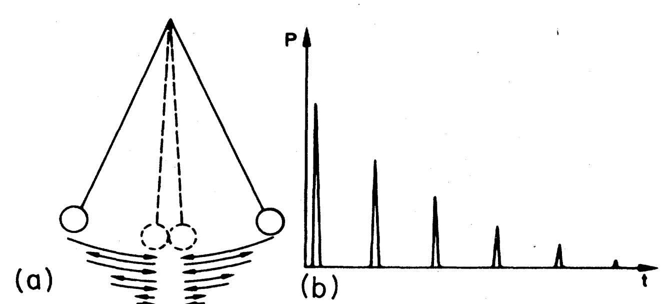

FIG. 2. (a) Suspended balls bouncing against each other. (b) Corresponding entropy production.

three-dimensional flow field there will be many collisions and processes of this kind which occur in an irregular and statistical manner. This will lead to irregular oscillations of the entropy production. Since the whole flow field can be superimposed by damped sinusoidal waves, one would expect that the maxima of the entropy production tend to become smaller as time proceeds. However, if by some favorable interference the whole dissipative capability of the system could unload in one moment, P could become rather large even at later times. These qualitative statements are brought into a quantitative form by the the theorems to be derived.

I briefly comment on a different approach which would appear possible. Consider an ensemble of systems which differ with respect to the wave phases but which can be considered as macroscopically equivalent if an appropriate coarse graining is introduced. Concerning the ensemble average $\langle P \rangle$ of the entropy production I would expect $d\langle P \rangle / dt \leq 0$ since the oscillations of P should average out due to their irregularity. This point of view is confirmed by the theorems to be derived.

C. Boundary conditions

In the known theorems of minimum entropy production the time-dependent states under consideration are exposed to the same time-independent boundary conditions as the steady reference state. It is just this requirement which makes the investigation of $P(t)$ a meaningful problem.

The same requirement would yield an ill-posed problem if it were imposed on the continuous systems considered in Secs. II and III of this paper. The reason is that these systems may exhibit wave solutions (corresponding to hyperbolic characteristics). It can principally not be avoided that these waves cross the boundary of the system and make the boundary state time dependent. There are two ways out of this problem.

1. The steady reference state $\rho_{10}(\mathbf{r}), p_{10}(\mathbf{r}), \ldots$ extends to infinity while, at some initial time t_0, the perturbative excess quantities $\rho_1(\mathbf{r},t), p_1(\mathbf{r},t), \ldots$ vanish everywhere outside some finite region G. All soundlike wave packets would then need an infinite period of time until the infinitely remote boundary of the system is reached. They will, however, be absorbed long before due to dissipation. In fact, there also exist perturbations like heat waves (corresponding to parabolic characteristics) which propagate at infinite speed. Since the amplitude of these waves decays exponentially in space they have no influence on the boundary at infinity. Thus, for all perturbations of the kind described we may assume time-independent boundary conditions at infinity.

2. There exists a possibility for considering also finite systems. Let us suppose that some wave has crossed the boundary of a finite system. We require that it is then absorbed completely outside the system. Intuitively this means that no reflection should occur and that no other waves should be generated which could run back into the system. More rigorously, all manipulations which with the purpose of damping are executed on the flow outside the system (e.g., in wind channels) must amount to the effect that the flow inside the system is exactly as if it were

subject to the conditions of the previous case. Practically it might become difficult to meet this requirement. Theoretically we have obtained a well-defined problem which might be considered as an idealistic model.

D. Extremum properties of the excess entropy production

According to Eqs. (2.10)–(2.12) steady states $\rho_{10}, p_{10}, \ldots$ must satisfy the equations

$$\nabla\cdot\mathbf{v}=0, \quad \nabla p=-\mu\nabla\times(\nabla\times\mathbf{v}), \quad \Delta T=0\,. \tag{2.26}$$

In general, the full entropy production P which is obtained by integrating Eq. (2.17) over the volume of the system does not assume a minimum in steady states. In fact, the Euler equations obtained by minimizing P are

$$(\lambda+\tfrac{4}{3}\mu)\nabla\,\nabla\cdot\mathbf{v}-\mu\nabla\times(\nabla\times\mathbf{v})=0, \quad \Delta T=0\,. \tag{2.27}$$

Inserting Eqs. (2.26) into the first of Eqs. (2.27) leads to the requirement $\nabla p_{10}=0$. Equations (2.26), however, allow for $\nabla p_{10}\neq 0$ provided $\Delta p_{10}=0$.

As for the excess entropy production P_{ex}, the situation is different. In a steady state Eqs. (2.26) are satisfied also by the excess quantities $\rho_1, p_1, \mathbf{v}_1, T_1$. Since these must vanish at infinity, $\Delta T_1=0$ and the condition of integrability $\Delta p_1=0$ leads immediately to $p_1\equiv 0$, $\rho_1\equiv 0$, $T_1\equiv 0$. With this Eqs. (2.26) yield $\Delta\mathbf{v}_1\equiv 0$ and $\mathbf{v}_1\equiv 0$. As expected, all excess quantities vanish in the steady state. Thus P_{ex} assumes a minimum in the steady state since it is quadratic in the excess quantities. According to the theorems of Sec. II E this minimum will be approached in the course of time as $t\to\infty$.

E. Theorem of minimum excess entropy production

According to its definition (2.19) P_{ex} is a non-negative quantity. We define another non-negative quantity $\bar{P}_{\text{ex}}$ by

$$\bar{P}_{\text{ex}}=\frac{\beta p_0}{\alpha\rho_0^3 T_0}\left[\lambda+\frac{4}{3}\mu+\frac{\kappa}{C_v}\right]\int(\nabla\rho_1)^2 d\tau+\frac{C_v}{T_0^2}(\lambda+\tfrac{4}{3}\mu)\int(\nabla T_1)^2 d\tau+\frac{\kappa}{C_v T_0}\int(\nabla\cdot\mathbf{v}_1)^2 d\tau \tag{2.28}$$

which we call dissipative potential for reasons which become obvious later. (Since all coefficients appearing in $\bar{P}_{\text{ex}}$ are positive, $\bar{P}_{\text{ex}}$ is indeed a non-negative quantity.) Under the conditions of the first case considered in Sec. II C the following theorem is valid.

Theorem: The sum $P_{\text{ex}}+\bar{P}_{\text{ex}}$ is a kinetic potential satisfying

$$\begin{aligned}\tfrac{1}{2}d_t(P_{\text{ex}}+\bar{P}_{\text{ex}})
&=-\frac{\kappa}{\rho_0 T_0^2}\left[\lambda+\frac{4}{3}\mu+\frac{\kappa}{C_v}\right]\int(\Delta T_1)^2 d\tau\\
&\quad-\frac{\mu^2}{\rho_0 T_0}\int[\nabla\times(\nabla\times\mathbf{v}_1)]^2 d\tau\\
&\quad-\frac{1}{\rho_0 T_0}(\lambda+\tfrac{4}{3}\mu)\left[\lambda+\frac{4}{3}\mu+\frac{\kappa}{C_v}\right]\int(\nabla\,\nabla\cdot\mathbf{v}_1)^2 d\tau\,.\end{aligned} \tag{2.29}$$

Proof: Let us temporarily forget about Eq. (2.28) and put

$$\bar{P}_{\text{ex}}=a\int(\nabla\rho_1)^2 d\tau+b\int(\nabla T_1)^2 d\tau+c\int(\nabla\cdot\mathbf{v}_1)^2 d\tau\,, \tag{2.30}$$

where a, b, and c are coefficients to be determined. The first step of the proof consists in calculating the time derivatives of the integrals contained in P_{ex} and $\bar{P}_{\text{ex}}$. Note that also the excess quantities must satisfy Eqs. (2.10)–(2.12) and (2.15). One obtains

$$\begin{aligned}\tfrac{1}{2}d_t\int(\nabla T_1)^2 d\tau
&=\int\nabla\cdot[(\partial_t T_1)\nabla T_1]d\tau-\int(\partial_t T_1)\Delta T_1 d\tau\\
&=-\frac{\kappa}{C_v\rho_0}\int(\Delta T_1)^2 d\tau+\frac{\beta p_0 T_0}{C_v\rho_0}\int(\Delta T_1)\nabla\cdot\mathbf{v}_1 d\tau\,.\end{aligned} \tag{2.31}$$

In deriving the final result, $\partial_t T_1$ was expressed by means of Eq. (2.12) and the integral over $\nabla\cdot[(\partial_t T_1)\nabla T_1]$ was transformed into a surface integral by using Gauss's theorem. Since $T_1\equiv 0$ outside some properly chosen sphere, T_1 vanishes at infinity together with all of its derivatives and thus the surface integral disappears. Quite similarly one obtains

$$\tfrac{1}{2}d_t\int(\nabla\times\mathbf{v}_1)^2 d\tau=-\frac{\mu}{\rho_0}\int[\nabla\times(\nabla\times\mathbf{v}_1)]^2 d\tau\,, \tag{2.32}$$

$$\tfrac{1}{2}d_t\int(\nabla\cdot\mathbf{v}_1)^2 d\tau=-\frac{1}{\rho_0}(\lambda+\tfrac{4}{3}\mu)\int(\nabla\,\nabla\cdot\mathbf{v}_1)^2 d\tau-\frac{1}{\rho_0}\int(\Delta p_1)\nabla\cdot\mathbf{v}_1 d\tau\,, \tag{2.33}$$

$$\tfrac{1}{2}d_t\int(\nabla\rho_1)^2 d\tau=\rho_0\int(\Delta\rho_1)\nabla\cdot\mathbf{v}_1 d\tau\,, \tag{2.34}$$

where again some surface integrals vanished. From Eqs. (2.19), (2.30), and (2.31)–(2.34) one obtains

$$\begin{aligned}&\tfrac{1}{2}d_t(P_{\text{ex}}+\bar{P}_{\text{ex}})+\frac{\kappa}{C_v\rho_0 T_0^2}(\kappa+T_0^2 b)\int(\Delta T_1)^2 d\tau+\frac{\mu^2}{\rho_0 T_0}\int[\nabla\times(\nabla\times\mathbf{v}_1)]^2 d\tau+\frac{1}{\rho_0}(\lambda+\tfrac{4}{3}\mu)\left[\frac{1}{T_0}(\lambda+\tfrac{4}{3}\mu)+c\right]\int(\nabla\,\nabla\cdot\mathbf{v}_1)^2 d\tau\\
&=\frac{1}{\rho_0 T_0}\left[\frac{1}{C_v}(\kappa+T_0^2 b)-(\lambda+\tfrac{4}{3}\mu+T_0 c)\right]\int(\Delta p_1)\nabla\cdot\mathbf{v}_1 d\tau+\left[a\rho_0-\frac{\beta p_0}{C_v\alpha\rho_0^2 T_0}(\kappa+T_0^2 b)\right]\int(\Delta\rho_1)\nabla\cdot\mathbf{v}_1 d\tau\,,\end{aligned} \tag{2.35}$$

where on the right-hand side Eq. (2.15) was used for eliminating T_1. Putting

$$a=\frac{\beta p_0}{C_v \alpha \rho_0^3 T_0}(\kappa+T_0^2 b) , \tag{2.36}$$

$$b=\frac{c_v}{T_0^2}(\lambda+\tfrac{4}{3}\mu)+\frac{1}{T_0^2}(C_v T_0 c-\kappa) \tag{2.37}$$

the coefficients a and b are now chosen such that the right-hand side of Eq. (2.35) becomes zero. The coefficient c in Eq. (2.37) is still free and is now chosen such that $\overline{P}_{\rm ex}$ becomes a non-negative quantity. One possible choice is

$$\begin{aligned} a&=\frac{\beta p_0}{\alpha \rho_0^3 T_0}\left[\lambda+\frac{4}{3}\mu+\frac{\kappa}{C_v}\right] , \\ b&=\frac{C_v}{T_0^2}(\lambda+\tfrac{4}{3}\mu) , \\ c&=\frac{\kappa}{C_v T_0} . \end{aligned} \tag{2.38}$$

For this choice Eqs. (2.30) and (2.35) go over into the Eqs. (2.28) and (2.29). □

Since

$$0 \le P_{\rm ex} \le P_{\rm ex}+\overline{P}_{\rm ex} ,$$

the kinetic potential $P_{\rm ex}+\overline{P}_{\rm ex}$ is an upper bound of $P_{\rm ex}$ which decreases monotonically with time according to Eq. (2.29). As was shown in Sec. II B, $P_{\rm ex}$ is an irregularly oscillating function of time. It can become as large as the upper bound since there exist nontrivial states ($p_1=0$, $\rho_1=0$, $T_1=0$, $\mathbf{v}_1=\nabla\times\mathbf{A}_1$) for which $\overline{P}_{\rm ex}=0$. It can also just momentarily touch its lower bound since there exist nontrivial states ($T_1=0$, $\mathbf{v}_1=\mathbf{0}$, $p_1\neq 0$, $\rho_1\neq 0$, $\overline{P}_{\rm ex}\neq 0$) for which $P_{\rm ex}=0$. [Due to $\partial_t \nabla\cdot\mathbf{v}_1=(\Delta p_1)/\rho_0$ it will become nonzero afterwards.] Both kinds of states can be prescribed as transient states at any given time t_0. [Technically, they provide initial conditions if, starting with t_0, one solves the equations of motion (2.10)–(2.12) either forwards or backwards in time.] In particular, a time t_0, at which $P_{\rm ex}(t_0)$ is zero or far below maximum, may be followed by a time t at which $P_{\rm ex}(t)$ is maximum and $\overline{P}_{\rm ex}(t)=0$. From

$$P_{\rm ex}(t_0)+\overline{P}_{\rm ex}(t_0) \ge P_{\rm ex}(t)+\overline{P}_{\rm ex}(t)=P_{\rm ex}(t)$$

we get

$$P_{\rm ex}(t)-P_{\rm ex}(t_0) \le \overline{P}_{\rm ex}(t_0) . \tag{2.39}$$

This means that at any time t_0, $\overline{P}_{\rm ex}(t_0)$ provides an upper bound for the possible increase of $P_{\rm ex}$ at later times. Furthermore, according to Eq. (2.29) $P_{\rm ex}$ decreases at least at the rate at which $\overline{P}_{\rm ex}$ increases. Thus, the physical significance of $P_{\rm ex}$ is that of a dissipative potential in which the capability of the system for later dissipation is stored.

In the asymptotic state towards which the system is driven by Eq. (2.29) the right-hand side must become zero identically in t. In order that this happens the equations

$$\Delta T_1 \equiv 0, \quad \nabla\times(\nabla\times\mathbf{v}_1)\equiv\mathbf{0}, \quad \nabla\nabla\cdot\mathbf{v}_1\equiv\mathbf{0} \tag{2.40}$$

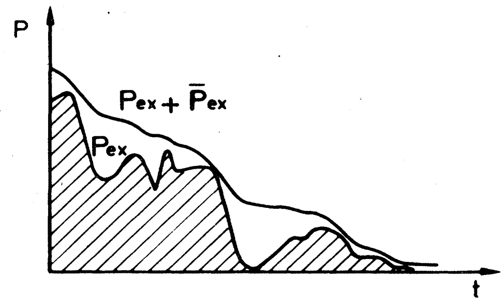

FIG. 3. Time evolution of excess entropy production $P_{\rm ex}$ and kinetic potential $P_{\rm ex}+\overline{P}_{\rm ex}$ in a dissipative hydrodynamic system.

must become valid. Subtracting the last one from the second yields

$$\Delta \mathbf{v}_1 \equiv \mathbf{0} . \tag{2.41}$$

In view of the boundary conditions $T_1=0$ and $\mathbf{v}_1=\mathbf{0}$ at infinity the only solution is $T_1\equiv 0$ and $\mathbf{v}_1\equiv\mathbf{0}$ both in space and time. With this we get from Eqs. (2.11) and (2.15)

$$\nabla p_1 \equiv \mathbf{0}, \quad \nabla \rho_1 \equiv \mathbf{0} , \tag{2.42}$$

and $p_1\equiv 0$, $\rho_1\equiv 0$ in addition. Thus the system tends towards the steady state with $P_{\rm ex}=\overline{P}_{\rm ex}=0$.

In summary the following result is obtained: The kinetic potential $P_{\rm ex}+\overline{P}_{\rm ex}$ decreases monotonically until it disappears. When this happens a steady state is reached. The oscillations of $P_{\rm ex}$ are bounded by the kinetic potential from above and by zero from below. Asymptotically $P_{\rm ex}$ disappears (Fig. 3).

The second case considered in Sec. II A can be reduced to the first one. Let G denote the region which lies inside the boundaries of the finite system under consideration. According to our assumptions about wave damping the flow field in G is identical with that of an infinite system. Hence, the excess entropy production $P_{\rm ex}^G$ inside the system is identical with the contribution of G to the infinite system. The total excess entropy production $P_{\rm ex}$ of the infinite system is subject to the theorem derived and to all of its consequences. Since it contains $P_{\rm ex}^G$ it is an (oscilla-

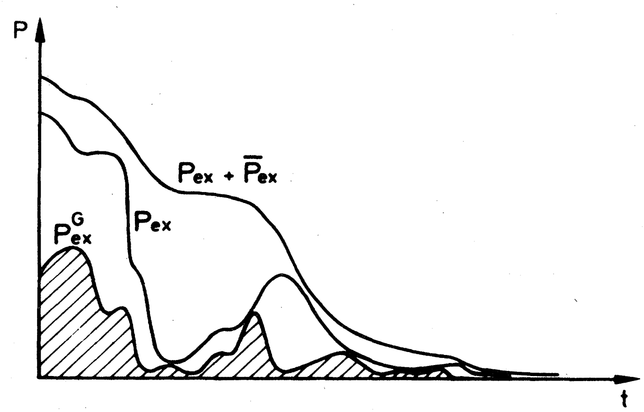

FIG. 4. Time evolution of $P_{\rm ex}^G$, $P_{\rm ex}$, and $P_{\rm ex}+\overline{P}_{\rm ex}$ in a finite dissipative hydrodynamic system.

tory) upper bound of $P_{\rm ex}^G$. $P_{\rm ex}$ is itself bounded by $P_{\rm ex}+\bar{P}_{\rm ex}$ and goes asymptotically towards zero. Hence, also $P_{\rm ex}^G$ must tend towards zero (Fig. 4).

F. Well-known special cases

Under special assumptions Eq. (2.29) reduces to well-known theorems of minimum entropy production. Putting $\lambda=\mu=0$, $\beta=0$, and $\mathbf{v}=\mathbf{v}_{10}+\mathbf{v}_1=\mathbf{0}$, Eq. (2.12) without the velocity term is the only relevant equation which remains from the system (2.10)–(2.12). Equations (2.19) and (2.29) reduce to

$$\tfrac{1}{2}d_t P_{\rm ex}=\tfrac{1}{2}d_t\frac{\kappa}{T_0^2}\int(\nabla T_1)^2 d\tau$$
$$=-\frac{\kappa^2}{C_v\rho_0 T_0^2}\int(\Delta T_1)^2 d\tau_1\ . \qquad (2.43)$$

The same result holds if we replace $T_1\to T_{10}+T_1$ with $P_{\rm ex}\to P$ as a consequence. Thus Eq. (2.43) is the well-known theorem of minimum entropy production for heat conduction.[2]

Setting $C_v=0$, $\beta=0$, $\kappa=0$, $p=\text{const}$, $\nabla\cdot\mathbf{v}=\nabla\cdot(\mathbf{v}_{10}+\mathbf{v}_1)=0$ the system of Eqs. (2.10)–(2.12) reduces to a single diffusion equation for $\mathbf{v}$. In this case, Eqs. (2.19) and (2.29) yield

$$\tfrac{1}{2}d_t P_{\rm ex}=\tfrac{1}{2}d_t\frac{\mu}{T_0}\int(\nabla\times\mathbf{v}_1)^2 d\tau$$
$$=-\frac{\mu^2}{\rho_0 T_0}\int[\nabla\times(\nabla\times\mathbf{v}_1)]^2 d\tau\ . \qquad (2.44)$$

Again the replacement $\mathbf{v}_1\to\mathbf{v}_{10}+\mathbf{v}_1$ is possible with $P_{\rm ex}\to P$, and Eq. (2.44) turns out to be the well-known theorem of minimum energy dissipation by Helmholtz (see, e.g., Ref. 2, p. 79).

III. SYSTEMS DESCRIBED BY THE LINEAR EQUATIONS OF RESISTIVE MAGNETOHYDRODYNAMICS

The nonlinear equations of resistive magnetohydrodynamics contain the magnetic field $\mathbf{B}$ in addition to the hydrodynamic quantities $\rho,p,\mathbf{v},e$ and are given by[9]

$$\partial_t\rho+\nabla\cdot\rho\mathbf{v}=0\ , \qquad (3.1)$$

$$\rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})=-\nabla p+\mathbf{j}\times\mathbf{B},\quad \mathbf{j}=\nabla\times\mathbf{B} \qquad (3.2)$$

$$\rho(\partial_t e+\mathbf{v}\cdot\nabla e)+p\nabla\cdot\mathbf{v}=\eta j^2\ , \qquad (3.3)$$

$$\partial_t\mathbf{B}=\nabla\times(\mathbf{v}\times\mathbf{B})-\nabla\times(\eta\mathbf{j})\ , \qquad (3.4)$$

where $\eta=\eta(T)$ is the specific resistivity; $\nabla\cdot\mathbf{B}=0$ must be imposed as an initial condition. The specific entropy production is again defined by Eq. (2.7) (now $\mathbf{W}=\mathbf{0}$). Quite like Eq. (2.9) one obtains now

$$\sigma=\frac{\eta}{T}j^2\ . \qquad (3.5)$$

We linearize Eqs. (3.1)–(3.4) around a homogeneous equilibrium state with temporally and spatially constant values $\rho_0\equiv\text{const}$, $p_0\equiv\text{const}$, $\mathbf{v}_0\equiv0$, $\mathbf{B}_0=B_0\mathbf{e}_z$, $\mathbf{j}_0\equiv0$ and obtain

$$\partial_t\rho+\rho_0\nabla\cdot\mathbf{v}=0\ , \qquad (3.6)$$

$$\rho_0\partial_t\mathbf{v}=-\nabla p+\mathbf{j}\times B_0\mathbf{e}_z,\quad \mathbf{j}=\nabla\times\mathbf{B} \qquad (3.7)$$

$$\partial_t p+\tfrac{5}{3}p_0\nabla\cdot\mathbf{v}=0\ , \qquad (3.8)$$

$$\partial_t\mathbf{B}=\nabla\times(\mathbf{v}\times B_0\mathbf{e}_z)-\eta_0\nabla\times\mathbf{j}\ . \qquad (3.9)$$

In Eq. (3.8) the assumption $e=\frac{3}{2}p/\rho$ usual in plasma physics was used. [In a general treatment Eqs. (2.13) and (2.15) would have to be used, which would provide no difficulty.] The corresponding approximation of the entropy production is

$$P=\frac{\eta_0}{T_0}\int j^2 d\tau\ . \qquad (3.10)$$

Since Eqs. (3.6)–(3.9) contain damped sound waves and damped Alfvén waves as solutions, P is again an oscillatory function of time. The most general steady-state solution of the Eqs. (3.6)–(3.9) is given by

$$p_{10}=p_{10}(x,y),\quad \rho_{10}=\rho_{10}(x,y)\ ,$$
$$\mathbf{v}_{10}=\frac{\eta_0}{B_0}\left[\frac{\Delta p_{10}}{B_0}\mathbf{e}_z+\nabla\alpha\times\mathbf{e}_z\right]-\frac{\eta_0}{B_0^2}\nabla p_{10}+\nabla\times\mathbf{A}\ , \qquad (3.11)$$
$$\mathbf{j}_{10}=\alpha\mathbf{e}_z+\frac{1}{B_0}\mathbf{e}_z\times\nabla p_{10}\ ,$$

where $p_{10}(x,y)$, $\rho_{10}(x,y)$, $\alpha=\alpha(x,y)$, and $\mathbf{A}=\mathbf{A}(x,y)$ are arbitrary integration functions. As in the hydrodynamic case of Sec. II, P does, in general, not assume a minimum in the steady state. In fact, minimization of P with respect to $\mathbf{B}$ yields the Euler equation

$$\nabla\times\mathbf{j}=\mathbf{0} \qquad (3.12)$$

which is violated by most of the solutions (3.11).

In order to investigate time-dependent solutions, as in Sec. II we add excess quantities to the steady-state quantities

$$p=p_{10}+p_1,\ \ldots,\quad \mathbf{B}=\mathbf{B}_{10}+\mathbf{B}_1\ . \qquad (3.13)$$

p_1, ρ_1, $\mathbf{v}_1$, $\mathbf{j}_1$, and $\mathbf{B}_1$ must again satisfy Eqs. (3.6)–(3.9). Concerning the boundary conditions the same possibilities exist as for pure hydrodynamic flows. For convenience we restrict our consideration to infinite systems (case 1 of Sec. II C) and require that at some initial time t_0 all excess quantities vanish outside some finite region. With this assumption we may impose the boundary condition at infinity that all excess quantities vanish together with their derivatives. For the excess entropy production from Eq. (3.10) we obtain

$$P_{\rm ex}=\frac{\eta_0}{T_0}\int j_1^2 d\tau\ . \qquad (3.14)$$

$P_{\rm ex}$ is quadratic in the excess quantities and has therefore a minimum in the steady state where all excess quantities vanish.

It can be shown again that $P_{\rm ex}$ tends asymptotically towards zero under the burden of a monotonically decreasing upper bound. Specifically, we have the following.

Theorem: Let $\bar{P}_{\rm ex}$ be given by

$$\bar{P}_{ex}=\frac{\eta_0\rho_0}{T_0}\int[(\nabla\cdot\mathbf{v}_1)^2+(\nabla\times\mathbf{v}_1)^2]d\tau+\frac{3\eta_0}{5p_0T_0}\int(\nabla p_1)^2d\tau\ . \tag{3.15}$$

Then $P_{ex}+\bar{P}_{ex}$ is a kinetic potential which satisfies

$$\tfrac{1}{2}d_t(P_{ex}+\bar{P}_{ex})=-\frac{\eta_0^2}{T_0}\int(\nabla\times\mathbf{j}_1)^2d\tau\ . \tag{3.16}$$

Proof: Using Eq. (3.9) to eliminate $\partial_t\mathbf{B}_1$ one obtains

$$\begin{aligned}\tfrac{1}{2}d_tP_{ex}&=\frac{\eta_0}{T_0}\int(\nabla\times\mathbf{B}_1)\cdot(\nabla\times\partial_t\mathbf{B}_1)d\tau\\&=-\frac{\eta_0^2}{T_0}\int[\nabla\times(\nabla\times\mathbf{B}_1)]^2d\tau-\frac{\eta_0B_0}{T_0}\int\mathbf{v}_1\cdot(\mathbf{e}_z\times\Delta\mathbf{j}_1)d\tau\ ,\end{aligned} \tag{3.17}$$

where Gauss's theorem was used for partial integration and surface integrals disappeared due to the boundary conditions at infinity. Using Eqs. (3.7) and (3.8) for eliminating $\partial_t\mathbf{v}_1$ and ∂_tp_1, quite similarly one obtains

$$\begin{aligned}\tfrac{1}{2}d_t\int(\nabla\cdot\mathbf{v}_1)^2d\tau=&-\frac{1}{\rho_0}\int(\Delta p_1)\nabla\cdot\mathbf{v}_1d\tau\\&-\frac{B_0}{\rho_0}\int\mathbf{v}_1\cdot\nabla[\nabla\cdot(\mathbf{j}_1\times\mathbf{e}_z)]d\tau\ ,\end{aligned} \tag{3.18}$$

$$\tfrac{1}{2}d_t\int(\nabla\times\mathbf{v}_1)^2d\tau=\frac{B_0}{\rho_0}\int\mathbf{v}_1\cdot\nabla\times[\nabla\times(\mathbf{j}_1\times\mathbf{e}_z)]d\tau\ , \tag{3.19}$$

$$\tfrac{1}{2}d_t\int(\nabla p_1)^2d\tau=\tfrac{5}{3}p_0\int(\Delta p_1)\nabla\cdot\mathbf{v}_1d\tau\ . \tag{3.20}$$

From Eqs. (3.17)—(3.20) we get

$$\begin{aligned}\tfrac{1}{2}d_t(P_{ex}+\bar{P}_{ex})+\frac{\eta_0^2}{T_0}\int(\nabla\times\mathbf{j}_1)^2d\tau\\=\frac{\eta_0B_0}{T_0}\int\mathbf{v}_1\cdot\{-\mathbf{e}_z\times\Delta\mathbf{j}_1-\nabla[\nabla\cdot(\mathbf{j}_1\times\mathbf{e}_z)]\\+\nabla\times[\nabla\times(\mathbf{j}_1\times\mathbf{e}_z)]\}d\tau\ .\end{aligned} \tag{3.21}$$

Now we have

$$\nabla\times[\nabla\times(\mathbf{j}_1\times\mathbf{e}_z)]-\nabla[\nabla\cdot(\mathbf{j}_1\times\mathbf{e}_z)]=-\Delta(\mathbf{j}_1\times\mathbf{e}_z)=\mathbf{e}_z\times\Delta\mathbf{j}_1\ .$$

Therefore, the right-hand side of Eq. (3.21) vanishes and Eq. (3.16) is proven. □

The kinetic potential $P_{ex}+\bar{P}_{ex}$ is a monotonically decreasing upper bound of P_{ex}. It can be touched by P_{ex} since there exist nontrivial states ($\mathbf{v}_1=\mathbf{0}$, $p_1=0$, $\mathbf{B}_1\neq\mathbf{0}$, $\mathbf{j}_1\neq\mathbf{0}$) for which $\bar{P}_{ex}$ disappears. P_{ex} can also touch its lower bound zero since there exist nontrivial states ($\mathbf{B}_1=\mathbf{0}$, $\mathbf{j}_1=\mathbf{0}$, $\mathbf{v}_1\neq\mathbf{0}$, $\nabla\times[\nabla\times(\mathbf{v}_1\times\mathbf{e}_z)]\neq\mathbf{0}$) for which $P_{ex}=0$ and $\partial_{tt}P_{ex}\neq0$. $P_{ex}+\bar{P}_{ex}$ decreases monotonically until

$$\nabla\times\mathbf{j}_1=\mathbf{0} \tag{3.22}$$

identically in t. Since also $\nabla\cdot\mathbf{j}_1\equiv0$, the only solution which agrees with $\mathbf{j}_1=\mathbf{0}$ at infinity is $\mathbf{j}_1\equiv\mathbf{0}$. In consequence, we have $\mathbf{B}_1\equiv\mathbf{0}$ and

$$\nabla\times(\mathbf{v}_1\times\mathbf{e}_z)\equiv\mathbf{0} \tag{3.23}$$

according to Eq. (3.9). The most general solution of this equation is

$$\mathbf{v}_1=v_1(x,y,z)\mathbf{e}_z+\mathbf{e}_z\times\nabla\phi(x,y)\ . \tag{3.24}$$

Since $\mathbf{v}_1$ must vanish for $z\to\pm\infty$, the $\nabla\phi$ contribution must be zero. Inserting $\mathbf{v}_1=v_1\mathbf{e}_z$ in Eqs. (3.6)—(3.8) one realizes that solutions may be only obtained if $\rho_1=\rho_1(z,t)$, $p_1=p_1(z,t)$, and $v_1=v_1(z,t)$. With this, however, from the boundary conditions at $x\to\pm\infty$ and $y\to\pm\infty$ we finally get $\rho_1\equiv0$, $p_1\equiv0$, and $\mathbf{v}_1\equiv\mathbf{0}$. Thus, all excess quantities must vanish in the asymptotic state which is therefore characterized by $P_{ex}=\bar{P}_{ex}=0$.

In summary one obtains the same picture as in the hydrodynamic case of Sec. II, Fig. 3: The kinetic potential $P_{ex}+\bar{P}_{ex}$ decreases monotonically until it disappears. When this happens a steady state is reached. P_{ex} is a measure of the dissipation of the system. It is an oscillatory quantity whose oscillations are bounded by the kinetic potential from above and by zero from below. The physical significance of $\bar{P}_{ex}$ is that of a dissipative potential which stores the dissipative capability of the system. (The reasoning which leads to this interpretation of $\bar{P}_{ex}$ is exactly the same as that given in Sec. II E.) Asymptotically both P_{ex} and $\bar{P}_{ex}$ disappear.

IV. SYSTEMS DESCRIBED BY LINEAR NETWORK EQUATIONS

Electrical circuits constitute another type of dissipative system in which inertial effects may play an important role. Although an accurate theory would have to use field equations, they can very effectively be described by network equations. A similar description has been developed for treating problems of irreversible thermodynamics.[10–12] Concerning electrical networks, under specific conditions (the network connects either current sources with resistors and inductors only or it connects voltage sources with resistors and capacitors only) there have been proven variational properties of the steady state which are closely related to the theorem of minimum entropy production.[13,14] On the other hand, even network theory provides situations in which a state characterization by minimum entropy production fails.[14] After all this is not very surprising since minimum entropy production is certainly not a universal principle of nature. Nevertheless it appears useful to look for possible extensions.

Let us consider (passive) linear electrical networks which are excited by voltage sources only. A network with N nodes and B branches can be described either by $Z=B-(N-1)$ independent equations for the mesh currents (loop currents) J_l or by $N-1$ independent equations for the node voltages. We shall employ a mesh analysis and choose a set of independent loops such that each loop contains at most one voltage generator, none of these being common to several loops. The loop equations are then[15]

$$\sum_{l=1}^{Z} L_{kl} d_t J_l + \sum_{l=1}^{z} R_{kl} J_l + \sum_{l=1}^{z} \mathscr{C}_{kl} \int J_l dt = E_k , \quad k=1,\ldots,Z . \tag{4.1}$$

Note that the loop currents J_l are subsidiary quantities which coincide with the actual branch currents I_v only in branches which do not belong to more than one loop. In branches which are common to several loops, I_v is given by the sum of the corresponding mesh currents. Kirchhoff's current law is satisfied automatically by the means of this definition. E_k is the electromotive force of the generator in the kth loop and is set to zero if there is no generator. The quantities L_{kl}, R_{kl}, and $\mathscr{C}_{kl}$ are the total series inductance, series resistance, and series combination of capacitances, respectively, common to the loops labeled k and l, multiplied by $+1$ or -1 corresponding to whether J_k and J_l flow in the same direction or in opposite directions. (Take all elements of the loop and set $J_k = J_e$ for $k=l$.) The matrices L_{kl}, R_{kl}, and $\mathscr{C}_{kl}$ are obviously symmetric; in addition they are positive semidefinite.[16]

In order to obtain the energy dissipation of the system in terms of the loop currents, we multiply Eq. (3.1) by J_k and sum up over all k. Using the symmetry of L_{kl} and $\mathscr{C}_{kl}$ we get

$$\tfrac{1}{2} d_t \sum_{k,l} \left[L_{kl} J_k J_l + \mathscr{C}_{kl} \int J_k dt \int J_l dt \right] + \sum_{k,l} R_{kl} J_k J_l = \sum_k E_k J_k \tag{4.2}$$

this way. Since in the generator branches $J_k = I_{v_k}$, we have $\sum_k E_k J_k = \sum_v E_v I_v$ which is the energy released from the generators. In a steady state (which requires time-independent voltages E_k) the time derivative on the left-hand side of Eq. (4.2) vanishes and we obtain

$$\sum_k E_k J_k = \sum_{k,l} R_{kl} J_k J_l . \tag{4.3}$$

Due to the balance of energy input and energy dissipation we can identify the right-hand side of this equation as the energy dissipation of the system. If the system would allow for a nondissipative steady state in which the J_l do not vanish simultaneously it would be short-circuited. We exclude this possibility by assuming that the matrix R_{kl} is positive definite.

If all resistors dissipate energy at the common temperature T_0, then the entropy production of the system is given by

$$P = \frac{1}{T_0} \sum_{k,l} R_{kl} J_k J_l . \tag{4.4}$$

If they dissipate at different temperatures and if T_0 is the smallest one of these, we have the obvious inequality

$$P < \frac{1}{T_0} \sum_{k,l} R_{kl} J_k J_l . \tag{4.5}$$

In the following we shall comprise both possibilities in a single inequality by using the $\leq$ sign. Thus, for the excess entropy which is due to perturbations J_k^1 of a steady state J_k^0, we obtain

$$P_{\text{ex}} \leq \frac{1}{T_0} \sum_{k,l} R_{kl} J_k^1 J_l^1 . \tag{4.6}$$

Steady states are characterized by time-independent voltages E_k^0, time-independent loop currents J_k^0 and time-independent loop charges Q_k^0 (due to the presence of capacitors). According to Eqs. (4.1), these quantities must be determined from

$$\sum_l R_{kl} J_l^0 + \sum_l \mathscr{C}_{kl} \int J_l^0 dt = E_k^0 , \quad k=1,\ldots,Z \tag{4.7}$$

with

$$J_l^0 = 0 , \quad \int J_l^0 dt = Q_l^0 \tag{4.8}$$

in all loops containing a capacitor. [Note that the second term on the left-hand side of Eq. (4.7) drops out in all loops which do not contain a capacitor.]

In the treatment of time-dependent states we shall keep the electromotive forces E_k fixed considering them as externally imposed like boundary conditions in continuous systems. Thus, putting

$$J_k = J_k^0 + J_k^1 , \quad E_k = E_k^0 \tag{4.9}$$

from Eqs. (4.1) and (4.7) we get

$$\sum_l L_{kl} d_t J_l^1 + \sum_l R_{kl} J_l^1 + \sum_l \mathscr{C}_{kl} \int J_l^1 dt = 0 , \quad k=1,\ldots,Z \tag{4.10}$$

for determining the excess loop currents J_l^1. From Eqs. (4.10), in analogy to Eq. (4.2) we obtain

$$\tfrac{1}{2} d_t \sum_{k,l} \left[L_{kl} J_k^1 J_l^1 + \mathscr{C}_{kl} \int J_k^1 dt \int J_l^1 dt \right] = -\sum_{k,l} R_{kl} J_k^1 J_l^1 . \tag{4.11}$$

The sum on the left-hand side is a kinetic potential which will now be brought into relation with the excess entropy production. In order to render this possible we assume that L_{kl} is a positive definite matrix. (In consequence the network should not contain any closed loop with zero series inductance.) It is clearly ascertained that this assumption implies some loss of generality. Nevertheless, a variety of interesting situations is left.

Now let L_0 be the smallest eigenvalue of the matrix L_{kl} and R_0 be the largest one of the matrix R_{kl}:

$$\sum_{k,l} L_{kl} J_n^1 J_l^1 \geq L_0 \sum_k (J_k^1)^2 , \qquad \sum_{k,l} R_{kl} J_k^1 J_l^1 \leq R_0 \sum_k (J_k^1)^2 . \tag{4.12}$$

According to the properties of the matrices L_{kl} and R_{kl} both R_0 and L_0 are positive quantities $\neq 0$. Hence, from the inequalities (4.6) and (4.12) we get

$$P_{\text{ex}} \leq \frac{R_0}{T_0 L_0} \sum_{k,l} L_{kl} J_k^1 J_l^1 \tag{4.13}$$

or

$$\overline{P}_{ex} \equiv \frac{R_0}{T_0 L_0} \sum_{k,l} L_{kl} J_k^1 J_l^1 - P_{ex} \geq 0 \ . \tag{4.14}$$

Since $\mathscr{C}_{kl}$ is positive semidefinite, we have in addition

$$\overline{\overline{P}}_{ex} \equiv \frac{R_0}{T_0 L_0} \sum_{k,l} \mathscr{C}_{kl} \int J_k^1 dt \int J_l^1 dt \geq 0 \ . \tag{4.15}$$

Inserting the definitions of $\overline{P}_{ex}$ and $\overline{\overline{P}}_{ex}$ in Eq. (4.11), we finally obtain

$$\tfrac{1}{2} d_t (P_{ex} + \overline{P}_{ex} + \overline{\overline{P}}_{ex}) = -\frac{R_0}{T_0 L_0} \sum_{k,l} R_{kl} J_k^1 J_l^1 \leq 0 \ . \tag{4.16}$$

$P_{ex} + \overline{P}_{ex} + \overline{\overline{P}}_{ex}$ is a kinetic potential which decreases monotonically until the excess dissipation of the system vanishes for good. Due to the positive definiteness of R_{kl} this is only possible if all J_k^1 vanish identically in t. When this is the case, also $P_{ex} + \overline{P}_{ex}$ and $\overline{\overline{P}}_{ex}$ vanish according to their definitions. Accordingly, the kinetic potential decreases until it disappears. Since the Eqs. (4.10) have oscillatory solutions $J_k^1(t)$, P_{ex} and $P_{ex} + \overline{P}_{ex}$ are oscillatory quantities. $P_{ex} + \overline{P}_{ex}$ is an upper bound of P_{ex} and is in turn bounded by the kinetic potential. Since the latter tends asymptotically towards zero, the same must be true for $P_{ex} + \overline{P}_{ex}$ and P_{ex}. The situation is as shown in Fig. 4, if one replaces $P_{ex}^G \to P_{ex}$, $P_{ex} \to P_{ex} + \overline{P}_{ex}$, and $P_{ex} + \overline{P}_{ex} \to P_{ex} + \overline{P}_{ex} + \overline{\overline{P}}_{ex}$ there. Note that for $L_{kl} \to m$ (mass), $R_{kl} \to r$ (coefficient of friction), and $\mathscr{C}_{kl} \to k_{spring}$ (spring constant) Eqs. (4.1) contain the harmonic oscillator as an especially simple example.

There exist special networks in which transient states with $\overline{P}_{ex} + \overline{\overline{P}}_{ex} = 0$ become possible. (A network which consists of just one single current loop provides an especially simple example.) In networks of this kind the situation of P_{ex} is exactly the same as in the continuous systems considered in Secs. II and III. Hence, in these systems the quantity $\overline{P}_{ex} + \overline{\overline{P}}_{ex}$ can be interpreted physically as a dissipative potential.

In more complicated networks this possibility may no longer exist. (We have employed several inequalities in which either the simultaneous appearance of the equals sign or its appearance at all may be excluded.) In those systems there is no simple interpretation of the quantity $\overline{P}_{ex} + \overline{\overline{P}}_{ex}$ in physical terms. Mathematically, $P_{ex} + \overline{P}_{ex} + \overline{\overline{P}}_{ex}$ constitutes a Liapunov function which has the additional property of yielding an upper bound for P_{ex}.

V. CONCLUSIONS

In this paper, the concept underlying the theorems of minimum entropy production has been extended to linear systems which involve inertial effects. This was made possible (1) by employing the excess entropy production P_{ex} instead of the full entropy production P, and (2) by constructing a kinetic potential which contains P_{ex} additively. This method turned out to be useful in several different physical situations. The first means of employing the excess entropy provides a decisive step on the way from the theorem of minimum entropy production and linear systems to the general evolution criterion for nonlinear systems.[2] It appears possible that the second of the means can also be utilized for the study of nonlinear systems.

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